Discussion about losses in the perpendicular and parallel directions

Kentaro Somiya

January 30, 2009

Purpose The goal of the calculation in this report is (i) to see if there is a difference between the results of the square-sum method with a monolithic coating and that with the actual multi-layer coatings, and (ii) to see if we can obtain the effective losses in the perpendicular and parallel directions ($\phi_\parallel$ and $\phi_\perp$) without approximating the Poisson ratio to be zero. We will derive the strain tensor and stress tensor elements E_{xx}^k, E_{zz}^k, T_{xx}^k, and T_{zz}^k of each layer, calculate the product $\sum_k (T_{xx}^k E_{xx}^k + T_{zz}^k E_{zz}^k)\phi_k$, and try to rewrite it by $\sum_k T_{xx}^k E_{xx}^k \phi_\parallel + \sum_k T_{zz}^k E_{zz}^k \phi_\perp$.

Multi-layer analysis Defining $A = \lambda + 2\mu$ and $B = \lambda$ with Lamme coefficients[1] λ and μ, we can write down the relation of the stress and strain tensor elements of k-th layer as[2]

$$\begin{pmatrix} T_{xx}^k \\ T_{zz}^k \end{pmatrix} = \begin{pmatrix} A_k & B_k \\ B_k & A_k \end{pmatrix} \begin{pmatrix} E_{xx}^k \\ E_{zz}^k \end{pmatrix} , \tag{1}$$

the inverse of which is

$$\begin{pmatrix} E_{xx}^k \\ E_{zz}^k \end{pmatrix} = \Gamma_k \begin{pmatrix} A_k & -B_k \\ -B_k & A_k \end{pmatrix} \begin{pmatrix} T_{xx}^k \\ T_{zz}^k \end{pmatrix} , \tag{2}$$

where $\Gamma_k = 1/(A_k^2 - B_k^2)$. The boundary condition of the k-th and $(k+1)$-th layers includes

$$E_{xx}^k = E_{xx}^{k+1} , \tag{3}$$

$$T_{zz}^k = T_{zz}^{k+1} . \tag{4}$$

Using these and Eqs. (1)(2), we get

$$\begin{aligned} \begin{pmatrix} E_{xx}^k \\ E_{zz}^k \end{pmatrix} &= \begin{pmatrix} E_{xx}^{k+1} \\ -B_k\Gamma_k(A_k E_{xx}^k + B_k E_{zz}^k) + A_k\Gamma_k T_{zz}^{k+1} \end{pmatrix} \\ &= \begin{pmatrix} E_{xx}^{k+1} \\ -B_k\Gamma_k(A_k E_{xx}^k + B_k E_{zz}^k) + A_k\Gamma_k(B_{k+1}E_{xx}^{k+1} + A_{k+1}E_{zz}^{k+1}) \end{pmatrix} , \end{aligned} \tag{5}$$

[1] Lamme coefficients can be written with Young's modulus Y and the Poisson ratio σ as

. $\lambda = \dfrac{Y\sigma}{(1+\sigma)(1-2\sigma)}$, $\mu = \dfrac{Y}{2(1+\sigma)}$.

[2] There is no special meaning of having k on top or bottom.

then

$$
\begin{aligned}
\begin{pmatrix} E_{xx}^k \\ E_{zz}^k \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ A_k B_k \Gamma_k & 1 + B_k^2 \Gamma_k \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ A_k B_{k+1} \Gamma_k & A_k A^{k+1} \Gamma_k \end{pmatrix} \begin{pmatrix} E_{xx}^{k+1} \\ E_{zz}^{k+1} \end{pmatrix} \\
&= \frac{1}{A_k} \begin{pmatrix} A_k & 0 \\ B_{k+1} - B_k & A_{k+1} \end{pmatrix} \begin{pmatrix} E_{xx}^{k+1} \\ E_{zz}^{k+1} \end{pmatrix} \ . \qquad (6)
\end{aligned}
$$

This relation of the neighboring layers yields the strain tensor elements of the k-th layer as a function of the strain tensor elements of the $2N$-th layer:

$$
\begin{aligned}
\begin{pmatrix} E_{xx}^k \\ E_{zz}^k \end{pmatrix} &= \left[\prod_{j=k}^{2N-1} \frac{1}{A_j} \begin{pmatrix} A_j & 0 \\ B_{j+1} - B_j & A_{j+1} \end{pmatrix} \right] \begin{pmatrix} E_{xx}^{2N} \\ E_{zz}^{2N} \end{pmatrix} \\
&= \frac{1}{A_k} \begin{pmatrix} A_k & 0 \\ B_{2N} - B_k & A_{2N} \end{pmatrix} \begin{pmatrix} E_{xx}^{2N} \\ E_{zz}^{2N} \end{pmatrix} \ . \qquad (7)
\end{aligned}
$$

Let us define that the first layer ($k = 1$) is the first tantala coating probed by the beam and there are $N - 1$ of silica-tantala doublets after that till the silica substrate, the surface of which is regarded as the $2N$-th layer. Since A_k and B_k just depend on whether k is even (silica) or odd (tantala); $A_k \to A$ (k : even), $A_k \to A'$ (k : odd), $B_k \to B$ (k : even), and $B_k \to B'$ (k : odd), the strain tensor elements can be simply described as

$$
\text{[silica]}: \quad \begin{pmatrix} E_{xx}^s \\ E_{zz}^s \end{pmatrix} = \begin{pmatrix} E_{xx}^{\text{sub}} \\ E_{zz}^{\text{sub}} \end{pmatrix} , \qquad (8)
$$

$$
\text{[tantala]}: \quad \begin{pmatrix} E_{xx}^t \\ E_{zz}^t \end{pmatrix} = \frac{1}{A'} \begin{pmatrix} A' & 0 \\ B - B' & A \end{pmatrix} \begin{pmatrix} E_{xx}^{\text{sub}} \\ E_{zz}^{\text{sub}} \end{pmatrix} , \qquad (9)
$$

Since each silica layer or tantala layer has the same strain tensor elements, thus the same stress tensor elements, adding up the product of the tensor elements (namely the elastic energy) of each layer is just same as calculating the product of a monolithic, thick silica (or tantala) layer as we usually do.

Losses in the perpendicular and parallel directions Let us first focus on the elastic energy in the parallel direction. Adding up the product of strain and stress tensor elements, we get (here, we omit to write "sub" on the shoulder of the tensor elements)

$$
\begin{aligned}
\sum_{j=1}^{2N-1} T_{xx}^j E_{xx}^j d_j \phi_j &= \sum_{j=\text{odd}} \left(A' E_{xx}^2 + B' E_{xx} E_{zz}^j \right) d_j \phi_t + \sum_{j=\text{even}} \left(A E_{xx}^2 + B E_{xx} E_{zz}^j \right) d_j \phi_s \\
&= \left(A' E_{xx}^2 + B' \frac{B - B'}{A'} E_{xx}^2 + B' \frac{A}{A'} E_{xx} E_{zz} \right) d_t \phi_t + \left(A E_{xx}^2 + B E_{xx} E_{zz} \right) d_s \phi_s \ . \qquad (10)
\end{aligned}
$$

If we want to rewrite this with the effective x-direction loss $\phi_{\|}$ as

$$
\sum_{j=1}^{2N-1} T_{xx}^j E_{xx}^j d_j \phi_j \to \sum_{j=1}^{2N-1} T_{xx}^j E_{xx}^j d_j \times \phi_{\|} \ , \qquad (11)
$$

then the loss would be given as

$$\phi_{\parallel} \quad = \quad \frac{A(A'^2 - B'^2 - \eta B')d_t\phi_t + A'(A^2 - B^2 - \eta B)d_s\phi_s}{A(A'^2 - B'^2 - \eta B')d_t + A'(A^2 - B^2 - \eta B)d_s} \tag{12}$$

with $\eta \equiv -T_{zz}/E_{xx}$, which **cannot** be obtained without solving the whole elastic equations including the substrate, unless we assume the Poisson ratios be zero ($\Rightarrow B = B' = 0$). With this assumption, the loss is written without η as

$$\phi_{\parallel}^{\rm app} \quad = \quad \frac{A'd_t\phi_t + Ad_s\phi_s}{A'd_t + Ad_s} \ , \tag{13}$$

or, with the Young's moduli,

$$\phi_{\parallel}^{\rm app} \quad = \quad \frac{Y'd_t\phi_t + Yd_s\phi_s}{Y'd_t + Yd_s} \ , \tag{14}$$

which is what has been used to calculate AdLIGO's coating thermal noise. The same thing can be done for the perpendicular direction. The product of strain and stress tensor elements is

$$\begin{aligned} \sum_{j=1}^{2N-1} T_{zz}^j E_{zz}^j d_j\phi_j \quad &= \quad \sum_{j=\rm odd} \left(A'{E_{zz}^j}^2 + B'E_{xx}E_{zz}^j\right) d_j\phi_t + \sum_{j=\rm even} \left(A{E_{zz}^j}^2 + BE_{xx}E_{zz}^j\right) d_j\phi_s \\ &= \quad A'\left(\frac{B-B'}{A'}E_{xx} + \frac{A}{A'}E_{zz}\right)^2 d_t\phi_t + B'\left(\frac{B-B'}{A'}E_{xx} + \frac{A}{A'}E_{zz}\right)E_{xx}d_t\phi_t \\ &\qquad + \left(AE_{zz}^2 + BE_{xx}E_{zz}\right)d_s\phi_s \ , \end{aligned} \tag{15}$$

and the perpendicular loss is

$$\phi_{\perp} \quad = \quad \frac{A(B'+\eta)d_t\phi_t + A'(B+\eta)d_s\phi_s}{A(B'+\eta)d_t + A'(B+\eta)d_s} \ . \tag{16}$$

If the Poisson ratios are zero, the loss becomes

$$\phi_{\perp}^{\rm app} \quad = \quad \frac{Ad_t\phi_t + A'd_s\phi_s}{Ad_t + A'd_s} \ , \tag{17}$$

or

$$\phi_{\perp}^{\rm app} \quad = \quad \frac{Yd_t\phi_t + Y'd_s\phi_s}{Yd_t + Y'd_s} \ , \tag{18}$$

which agrees to what has been used for AdLIGO.

Conclusion This report showed two facts on the calculation of coating thermal noise: (i) the square-sum method of monolithic coatings gives the same result as is obtained by square-summing each layer's Brownian motion in the multi-layer coatings, and (ii) the estimation of mechanical losses in the perpendicular and parallel directions cannot be done without the full calculation of substrate thermal noise unless the Poisson ratios can be approximated to be zero. Regarding these, as a conclusion, the calculation with the square-sum method would be the one to be used.

Let us put a couple of remarks here. First, the parallel-perpendicular method would be useful if we have a measurement result of the loss angles in the two directions instead of the loss angle of each material. Besides, it is certainly possible to calculate the noise level even more accurately with the square-sum method if we have the loss angles of the two directions for each material. Second remark is that the multi-layer analysis shown in this report does not include the fact that some unignorable fraction of the beam transmits the first few coating layers. A more accurate estimation would be obtained with this effect taken into account, and the analysis should be done in the near future.