Thermal noise of GEO's beamsplitter

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May 4, 2009

1 Thermal noise spectra

This report shows that thermorefractive noise of the substrate is, as people have already known, the dominant noise source of various kinds of thermal noise in GEO's beamsplitter (BS). There exist a few noise sources to be considered for the BS, which are calculated in this report. The finite-size-mirror analysis is used to calculate conventional thermal noise, which could be important for the BS that is thinner than the core optics. After all, it seems none of them is larger than thermorefractive noise.

Figure 1 shows 11 different kinds of thermal noise in the BS, which can be categorized to three groups; one is coating thermal noise that is sensed by the light reflected by the BS, another is substrate thermal noise also sensed by the reflected light, and the other one is substrate thermal noise sensed by the light transmitting through the BS. Below, BR, TE, TR, TO, and PT stand for Brownian thermal noise, thermoelastic noise, thermorefractive noise, thermo-optic noise, and photo-thermal noise, respectively.

coating BR noise Brownian motion in the lossy coatings changes the location of the surface, on which 50 % of the incident light is reflected. This can be calculated with the finite-size analysis [1].

reflective substrate BR noise Brownian motion of the substrate also changes the location of the surface. This can be calculated with the finite-size analysis [2][3].

transmissive substrate BR noise Brownian motion changes the thickness of the substrate (expansion) and the optical path length for the transmitting light. This has been newly calculated (see Ref.[4]); the power spectrum is given by

$$S_{\rm BR}^{\rm tr} = \frac{32k_{\rm B}T}{\pi Y\Omega}\sum_m \frac{\cosh^2(k_m h/2)}{\sinh(k_m h)-k_m h}e^{-k_m^2 r_0^2/2}\cdot(n_s-1)^2\phi_s \tag{1}$$

where Y is Young's modulus, h is the thickness of the BS, r_0 is the beam radius[1], n_s is the refraction index of the substrate, ϕ_s is the loss angle of the substrate, and k_m is defined in

[1] r_0 is the beam radius in the definition of Braginsky it et al (the power becomes $1/e$ at the beam radius), which is $\sqrt{2}$ smaller than that in the definition of experimental groups (the power becomes $1/e^2$ at the beam radius).

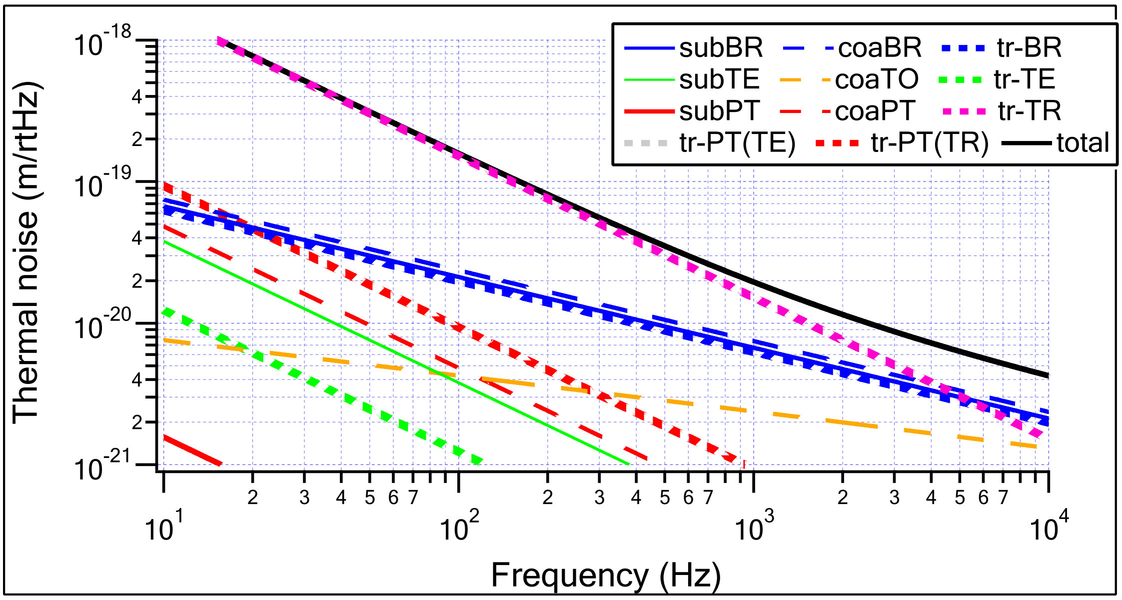

Figure 1: Spectra of various kinds of thermal noise in GEO's BS.

such a way that the first-order Bessel function $J_1[k_m a] = 0$ with a being the BS radius. At the derivation of Eq. (1), we used two approximations: one is to regard the Poisson ratio to be zero and the other is to regard the BS radius to be sufficiently larger than the beam radius.

coating TO noise Temperature fluctuation around the surface changes the location of the surface through the thermal expansion rate α, and also the fluctuation imposes the phase shift to the light through α and β $(= dn/dT)$. These two effects cancel each other [5]. The finite-size analysis is available [1]. Here we use the infinite-size result.

reflective substrate TE noise Temperature fluctuation in the BS substrate changes the location of the surface as well. This can be calculated with the finite-size analysis [3].

transmissive substrate TE noise Temperature fluctuation in the BS substrate changes the thickness of the substrate and the optical path length for the transmitting light. This has been newly calculated (see Ref.[4]); the power spectrum is given by

$$S_{\rm TE}^{\rm tr} = \frac{16\kappa\alpha^2 k_{\rm B} T^2}{\pi C_s \Omega^2} \sum_m k_m^2 \frac{\sinh(2k_m h) + \sinh(k_m h)}{(\sinh(k_m h) - k_m h)^2} e^{-k_m^2 r_0^2/2} \cdot (n_s - 1)^2 \qquad (2)$$

where C_s is specific heat, and κ is thermal diffusivity (i.e. thermal conductance divided by specific heat). The same approximation as in the derivation of transmissive BR noise is used.

transmissive substrate TR noise This is the dominant noise source. Temperature fluctuation in the BS substrate changes the refraction index of the substrate, thus the optical path length for the transmitting light [6]:

$$S_{\mathrm{TR}}^{\mathrm{tr}} = \frac{4\kappa\beta^2 k_{\mathrm{B}} T^2 \ell}{\pi C_s r_0^4 \Omega^2} . \tag{3}$$

Here ℓ is the physical length of the beam path in the BS. The BS thickness is $h = 8$ cm and $\ell = h/\cos\theta$ with $\theta = \mathrm{Arcsin}[(1/n_s)\sin\pi/4]$.

coating PT noise While TE noise and TR noise are caused by the thermo-dynamical fluctuation of the temperature, random absorption of photons in the incident light also causes TE noise and TR noise. This is called photo-thermal noise. Braginsky *et al* has derived PT noise due to the absorption at the coatings [7]:

$$S_{\mathrm{PT-TE}}^{\mathrm{coa}} = 2\alpha^2(1+\nu)^2 \frac{\hbar\omega_0 W_{\mathrm{coa}}}{C_s^2 \pi^2 r_0^4 \Omega^2} . \tag{4}$$

Here ν is Poisson's ratio, ω_0 is the laser angular frequency, and W_{coa} is the absorbed power. At the coatings of GEO's BS, the absorption will be about 1 ppm of 2 kW. The power spectrum of thermo-optic noise due to this photo-thermal fluctuation can be obtained as

$$S_{\mathrm{PT-TO}}^{\mathrm{coa}} = S_{\mathrm{PT-TE}}^{\mathrm{coa}} \times \frac{S_{\mathrm{TO}}^{\mathrm{coa}}}{S_{\mathrm{TE}}^{\mathrm{coa}}} , \tag{5}$$

using $S_{\mathrm{TO}}^{\mathrm{coa}}$ given in Ref. [5] and $S_{\mathrm{TE}}^{\mathrm{coa}}$ given in Ref. [8].

reflective substrate PT noise According to the picture of thermoelastic noise given by Braginsky *et al*, reflective substrate thermoelastic noise is caused by the temperature fluctuation in the volume $\sim r_0^3$ from the surface (We will explain more in Sec. 3.). Reflective substrate photo-thermal noise via thermoelastic effect is given by

$$S_{\mathrm{PT-TE}}^{\mathrm{sub-ref}} \simeq 2\alpha^2(1+\nu)^2 \frac{\hbar\omega_0 W_{\mathrm{sub}} \times r_0/\ell}{C_s^2 \pi^2 r_0^4 \Omega^2} . \tag{6}$$

Here W_{sub} is the absorbed power in the substrate, which in GEO is about 2 ppm/cm times ℓ times 1 kW.

transmissive substrate PT noise (TR) Photo-thermal fluctuation changes the refraction index of the substrate and the optical path length of the transmitting light changes. Temperature fluctuation of the entire substrate (where the light transmits) contributes, the noise spectrum is given by

$$S_{\mathrm{PT-TR}}^{\mathrm{sub-tr}} = 2\beta^2 \frac{\hbar\omega_0 W_{\mathrm{sub}}}{C_s^2 \pi^2 r_0^4 \Omega^2} . \tag{7}$$

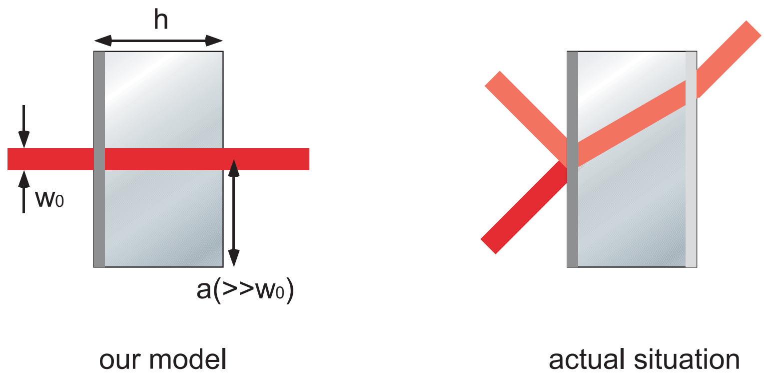

Figure 2: We calculate thermal noise probed by the beam at the transmission of the substrate. Our target is the beamsplitter of GEO600 but we simplify the situation to the normal-incident beam and the cylindrical substrate with its radius much larger than the beam radius.

transmissive substrate PT noise (TE) Photo-thermal fluctuation causes the random expansion of the substrate and the optical path length of the transmitting light changes. The noise spectrum is given as

$$\begin{aligned} S_{\mathrm{PT-TE}}^{\mathrm{sub-tr}} &= S_{\mathrm{PT-TR}}^{\mathrm{sub-tr}} \times \frac{S_{\mathrm{TE}}^{\mathrm{tr}}}{S_{\mathrm{TR}}^{\mathrm{tr}}} \\ &= \frac{8\alpha^2 \hbar \omega_0 W_{\mathrm{sub}}}{\pi^2 C_s^2 \Omega^2 h} \sum_m k_m^2 \frac{\sinh(2k_m h) + \sinh(k_m h)}{(\sinh(k_m h) - k_m h)^2} e^{-k_m^2 r_0^2/2} \cdot (n_s - 1)^2 \ . \end{aligned} \tag{8}$$

This noise is so small that the spectrum does not appear in the range of Fig. 1.

2 Our calculation model

What we can easily calculate today is thermal noise of a mirror as an infinite half plane or thermal noise of a mirror as a finite-size axissymmetric cylinder with the probe beam incident on its center. The situation here for the BS is a little different, so we need to make an approximation. See Fig. 2. The actual situation is the beam incident in 45 degrees, but out model is a mirror with the normally incident beam, 50 % of which is reflected by the first layer of the coatings (2 tantala and 1 silica) and the other 50 % transmits through the substrate.

The parameters we used are as follows. We used different β for coating and substrate, simply because previous analyses for GEO's BS uses 1.5×10^{-5} and Ref. [1][5] use 8×10^{-6}.

- mirror radius and thickness, and beam radius
  $a = 13$ cm, $h = 8$ cm, w_0 $(= \sqrt{2} r_0)$ $= 8.8$ mm
- Loss angle ϕ_{j}
  SiO_2 (sub) : 1.0×10^{-8}, SiO_2 (coa) : 1.0×10^{-4}, Ta_2O_5 : 4.0×10^{-4}

- Thermal conductivity k_j
  SiO_2 : 1.38 W/m·K, Ta_2O_5 : 33 W/m·K

- Thermal expansion α_j
  SiO_2 : 5.1×10^{-7}/K, Ta_2O_5 : 3.6×10^{-6}/K

- Specific heat per volume C_j
  SiO_2 : 1.64×10^6 J/K·m^3, Ta_2O_5 : 2.1×10^6 J/K·m^3

- Thermal diffusivity κ_j $(= k_\mathrm{j}/C_\mathrm{j})$

- Young's modulus Y_j
  SiO_2 : 7.2×10^{10} N/m^2, Ta_2O_5 : 1.4×10^{11} N/m^2

- Poisson ratio ν_j
  SiO_2 : 0.17, Ta_2O_5 : 0.23

- Refraction index n_j
  SiO_2 : 1.45, Ta_2O_5 : 2.06

- Temperature dependence of the refraction index β_j
  SiO_2 (sub) : 1.5×10^{-5}/K, SiO_2 (coa) : 8×10^{-6}/K, Ta_2O_5 : 1.4×10^{-5}/K

- Temperature T
  290 K

3 Intuitive understanding of TE/TR noise

In Ref. [8], Braginsky *et al* give us an intuitive understanding of thermoelastic noise and thermorefractive noise. The results obtained using their intuitive analysis mostly agree to the results given with Levin's method [9][10]. Let us see Braginsky's explanation; we will need this to derive the total noise level in a proper way.

First, we shall consider temperature fluctuation of a unit volume $V_\mathrm{unit} = \pi r_\mathrm{T}^3$, where $r_\mathrm{T} = \sqrt{\kappa/C_s\Omega}$. Temperature fluctuates coherently within one piece of this unit volume and the fluctuation can be regarded independent between different pieces. For thermo-dynamical fluctuation, the temperature variation of this small piece is

$$< u^2 > \; = \; \frac{k_\mathrm{B}T^2}{C_s V_\mathrm{unit}} \; . \tag{9}$$

Then we take the average of N pieces that contribute to the measurement. The averaged temperature fluctuation is $< \bar{u}^2 > \simeq < u^2 > /N$.

This N depends on which kind of noise we calculate. For coating TE noise, since the coating is thinner than r_T, N is given by $r_0^2 r_\mathrm{T}/r_\mathrm{T}^3$. Then, the noise spectrum is

$$S_\mathrm{TE}^\mathrm{coa} \; = \; [2\alpha(1+\nu)d]^2 \frac{k_\mathrm{B}T^2}{C_s \pi r_0^2 r_\mathrm{T}} \frac{2}{\Omega} \; . \tag{10}$$

This perfectly coincides to the result derived with Levin's method.

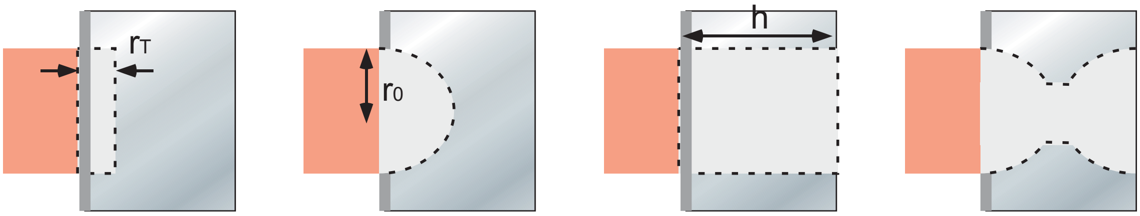

Figure 3: *Left Panel* (coating TE noise): temperature fluctuation is averaged within the volume $\pi r_0^2 r_{\rm T}$. *Second from the Left* (substrate TE noise): temperature fluctuation is averaged within the volume πr_0^3. *Second from the Right* (transmissive TR noise): temperature fluctuation is averaged within the volume $\pi r_0^2 h$. *Right Panel* (transmissive TE noise): temperature fluctuation is averaged within the volume $\pi r_0^2 r_{\rm T}$ from the front and back surface of the substrate.

For reflective substrate TE noise, since the elastic motion of a region further than r_0 does not reach to the surface, $N \simeq r_0^3/r_{\rm T}^3$, and the noise spectrum is

$$S_{\rm TE}^{\rm ref} \simeq [2\alpha(1+\nu)r_{\rm T}]^2 \frac{k_{\rm B}T^2}{C_s \pi r_0^3}\frac{2}{\Omega} . \tag{11}$$

Note that the length scale of each unit volume is $r_{\rm T}$. The result shown above is different from the result given with Levin's method by $\sqrt{2\pi}$.

It seems that transmissive substrate TR noise cannot be explained in the same way. The spectrum derived using Levin's method can be expressed as

$$S_{\rm TR}^{\rm tr} = [\beta r_{\rm T}]^2 \frac{k_{\rm B}T^2}{C_s \pi r_0^2 r_{\rm T}}\frac{2}{\Omega} \times \frac{h}{r_{\rm T}} \times \left(\frac{r_{\rm T}}{r_0}\right)^2 . \tag{12}$$

The second last multiple represents the summation in the longitudinal direction. The last multiple $r_{\rm T}^2/r_0^2$ would have some Physical meaning but unfortunately I could not explain it.

Figure 3 shows in which volume the temperature should be averaged. This picture helps us to properly sum up various kinds of thermal noise.

4 Summing up

There are a few things we should note in summing up those various kinds of thermal noise. First, we should multiply $1/\sqrt{2}$ to the reflective-noise levels derived above, since the incident angle of the light on the BS is 45 degrees. Second, we should further multiply 2 to the reflective noise levels, since the same fluctuation is probed twice; once by the reflecting light at the incident and once more when the other light returns from the inline arm. Third, we should NOT multiply 2 to the transmissive-noise levels even though the light transmits the substrate twice, since the phase shift at the reflection is already doubled (round-trip) while that during the transmission is for one way. At last, we should coherently add some kinds of thermal noise that come from the temperature fluctuation of the same part of the BS, while independent noise sources should be square-summed. Regarding the picture we showed in the last section, a proper way will be the following. For thermo-dynamical fluctuation,

$$
\begin{aligned}
S_{\rm TD}^{\rm tot} &= \left[-\sqrt{2S_{\rm TO}^{\rm coa}}+\sqrt{2S_{\rm TE}^{\rm ref}}\times\sqrt{\frac{r^{\rm T}}{r_0}}+\sqrt{S_{\rm TR}^{\rm tr}}\times\sqrt{\frac{r^{\rm T}}{h}}+\sqrt{S_{\rm TE}^{\rm tr}}\times\sqrt{\frac{r^{\rm T}}{2r_0}}\right]^2 \\
&+\left[\sqrt{2S_{\rm TE}^{\rm ref}}\times\sqrt{\frac{r_0-r^{\rm T}}{r_0}}+\sqrt{S_{\rm TR}^{\rm tr}}\times\sqrt{\frac{r_0-r^{\rm T}}{h}}+\sqrt{S_{\rm TE}^{\rm tr}}\times\sqrt{\frac{r_0-r^{\rm T}}{2r_0}}\right]^2 \\
&+\left[S_{\rm TR}^{\rm tr}\times\frac{h-r_0}{h}\right] \\
&+\left[\sqrt{S_{\rm TR}^{\rm tr}}\times\sqrt{\frac{r_0}{h}}+\sqrt{S_{\rm TE}^{\rm tr}}\times\sqrt{\frac{1}{2}}\right]^2 .
\end{aligned} \tag{13}
$$

In the same way for photo-thermal fluctuation,

$$
\begin{aligned}
S_{\rm tot}^{\rm PT} &= \left[-\sqrt{2S_{\rm PT-TO}^{\rm coa}}+\sqrt{2S_{\rm PT-TE}^{\rm sub-ref}}\times\sqrt{\frac{r^{\rm T}}{r_0}}+\sqrt{S_{\rm PT-TR}^{\rm sub-tr}}\times\sqrt{\frac{r^{\rm T}}{h}}+\sqrt{S_{\rm PT-TE}^{\rm sub-tr}}\times\sqrt{\frac{r^{\rm T}}{2r_0}}\right]^2 \\
&+\left[\sqrt{2S_{\rm PT-TE}^{\rm sub-ref}}\times\sqrt{\frac{r_0-r^{\rm T}}{r_0}}+\sqrt{S_{\rm PT-TR}^{\rm sub-tr}}\times\sqrt{\frac{r_0-r^{\rm T}}{h}}+\sqrt{S_{\rm PT-TE}^{\rm sub-tr}}\times\sqrt{\frac{r_0-r^{\rm T}}{2r_0}}\right]^2 \\
&+\left[S_{\rm PT-TR}^{\rm sub-tr}\times\frac{h-r_0}{h}\right] \\
&+\left[\sqrt{S_{\rm PT-TR}^{\rm sub-tr}}\times\sqrt{\frac{r_0}{h}}+\sqrt{S_{\rm PT-TE}^{\rm sub-tr}}\times\sqrt{\frac{1}{2}}\right]^2 .
\end{aligned} \tag{14}
$$

The total noise level in the power spectrum is, then,

$$
S_{\rm tot} \simeq S_{\rm tot}^{\rm TD}+S_{\rm tot}^{\rm PT}+2S_{\rm coa}^{\rm BR}+2S_{\rm ref}^{\rm BR}+S_{\rm tr}^{\rm BR} . \tag{15}
$$

Reflective substrate Brownian and transmissive substrate Brownian should be in some part added coherently, but it is not as simple as in the case with temperature fluctuations. A proper way to derive this will to use Levin's method with adding the imaginary force $(\sqrt{2}+n_s-1)F_0$ to the front surface and $(n_s-1)F_0$ to the back surface. Let us leave this for the near future.

Anyhow, transmissive TR noise due to the thermo-dynamical fluctuation is the dominant source and the others are rather negligible. The total noise level is only ~ 4 % higher than the TR-noise level alone.

5 Elliptic beam

Another tricky thing to calculate thermal noise in the actual situation, which we have ignored in the above calculations, is that the beam spot on the BS is not a circle but an ellipse. In the infinite-size analysis, the result with an elliptical beam (semi-axis of lengths r_a and r_b) is given by replacing r_0 to $\sqrt{r_a r_b}$. Assuming the same effect, we shall replace r_0 in the calculations above to $2^{1/4}r_0$. Figure 4 shows the result with the larger beam.

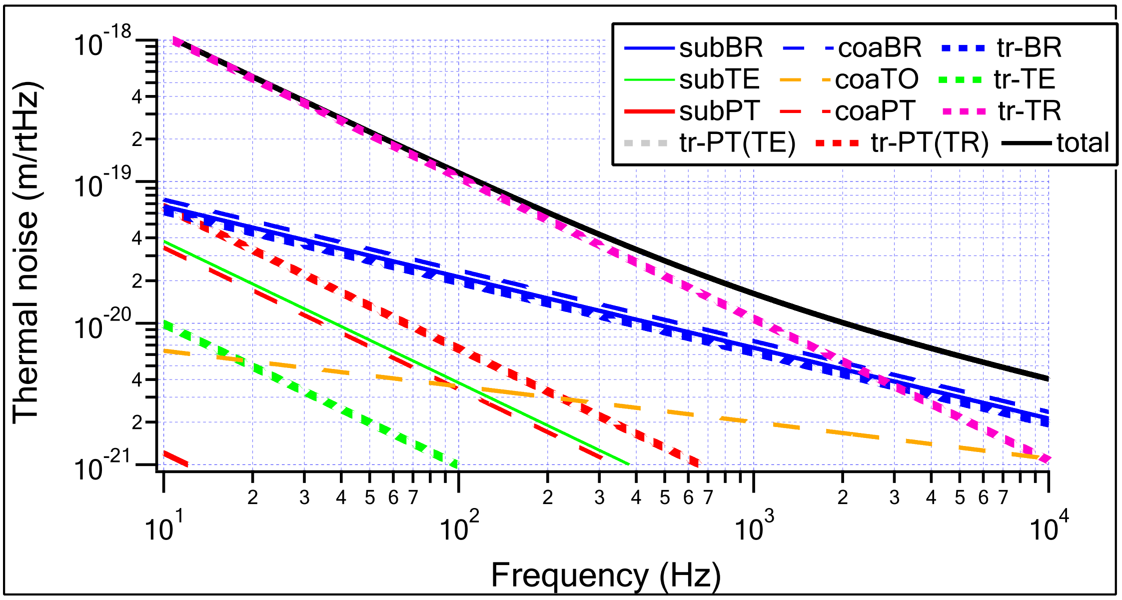

Figure 4: Spectra of thermal noise in GEO's BS with the beam radius $2^{1/4}$ larger than in Fig. 1.

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