LIGO-P080086-00-R

Creation of a quantum oscillator by classical control

Stefan Danilishin,[1,2] Helge Müller-Ebhardt,[3] Henning Rehbein,[3] Kentaro Somiya,[4,5] Roman Schnabel,[3] Karsten Danzmann,[3] Thomas Corbitt,[6] Chris Wipf,[6] Nergis Mavalvala,[6] and Yanbei Chen[4,5]

[1] *Max-Planck Institute for Gravitational Physics, Callinstr 38, 30167, Hannover, Germany*
[2] *Physics Faculty, Moscow State University, Moscow 119991, Russia*
[3] *Max-Planck Institute for Gravitational Physics and University of Hannover, Callinstr 38, 30167, Hannover, Germany*
[4] *Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125, USA*
[5] *Max-Planck Institute for Gravitational Physics, Am Mühlenberg 1, 14476 Golm, Germany*
[6] *LIGO Project, NW22-295, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

Quantum state preparation via linear feedback control, or cold damping, has been applied on test masses ranging from 10^{-11} kg to 1000 kg. As pure quantum state is being approached, thermal occupation number approaches and eventually goes below unity, optimal control becomes crucial: only a unique optimal controller will yield a state with minimum Heisenberg Uncertainty — yet in the literature feedback kernels have been chosen based on simplicity rather than optimality. We obtain theoretically the optimal feedback controller that minimizes uncertainty for a general linear measurement process, and show that even in absence of classical noise, a pure quantum state is not always achievable via feedback — in particular, for Markovian measurements, the deviation from minimum Heisenberg Uncertainty is closely related to the extent the device beats the free-mass Standard Quantum Limit for force measurement. We then specialize to a damped harmonic oscillator at non-zero temperature under conventional Markovian measurement (uncorrelated measurement and back-action noise), and deduce a critical temperature T_c, above which the minimum occupation number $N_{\rm eff}$ via control is always $1/\sqrt{2}$, while below T_c, $N_{\rm eff} \sim (T/T_c)^{1/2}$. We also consider gravitational-wave detectors, for which $T \gg T_c$, and show that a slightly unconventional measurement process (frequency independent input squeezing or homodyne detection at a non-phase quadrature) will allow controlled states with occupation number far below unity.

Motivation. LIGO detectors are currently operating at factor of 10 (in amplitude) above the Standard Quantum Limit (SQL) in their most sensitive band, with dominant noise sources arising from classical origin [1]. Such a classical noise budget has allowed a feedback control system to create a 2.5 kg oscillator with eigenfrequency of around 150 Hz (up shifted from the pendulum frequency of 1 Hz), and with occupation number below 300 [2]. Assuming only classical noise, one can obtain the *effective occupation number* of the controlled test mass,

$$N_{\rm eff} \approx Q_{\rm eff} S_x(\Omega_{\rm eff})/S_{\rm SQL}(\Omega_{\rm eff})\,, \tag{1}$$

where S_x the detector's position-referred noise budget, $S_{\rm SQL} = 2\hbar/(m\Omega^2)$ the free-mass SQL for position measurement, $\Omega_{\rm eff}$ the eigenfrequency of the controlled oscillator, and $Q_{\rm eff}$ its quality factor. In upgrades of LIGO detectors, classical noise will approach and eventually surpass the SQL, making quantum noise and Heisenberg Uncertainty a dominant effect, and Eq. (1) breaks down — although a naive extension of this equation indicates that ground state *might* be achieved with sub-SQL *total noise.* A major question arises as of *whether the measurement required for shifting oscillator eigenfrequency always produces significant decoherence to the oscillator's quantum state,* resulting in only semiclassical oscillators.

The cold-damping technique used by the above experiment was first proposed by Mancini et al. and Courty et al. [3]. In those works, simple feedback filters (constant plus derivative) have been chosen to illustrate the power of cold damping, yet it was not clear whether these filters are optimal. These works also focused mostly on configurations where the controller mainly damps the oscillator, instead of shifting its resonant frequency. The main aims of this paper are: (i) to deduce the *optimal* feedback controller that minimizes the Heisenberg Uncertainty of an oscillator under *a general linear measurement process,* and (ii) to apply our technique specifically to future gravitational-wave detectors, in which test-mass eigenfrequency must be upshifted significantly. Our results will also be relevant for cold-damping experiments performed on other mechanical structures [7].

Figure of merit. At a first yet close examination, occupation number is not a good measure of quantum-ness: squeezed states could have high occupation numbers, yet they are "more quantum" than the vacuum state. Moreover, the definition of occupation number requires a well-defined real-valued eigenfrequency, which can be ambiguous for two reasons: (i) the controller can modify the oscillator's original eigenfrequency, and (ii) for oscillators with a finite quality factor $Q_{\rm eff}$, the choice for an effective real eigenfrequency $\Omega_{\rm eff}$ would be ambiguous by $\sim \Omega_{\rm eff}/Q_{\rm eff}$. In this letter, we use the following quantity

$$U \equiv \sqrt{V_{xx}V_{pp} - V_{xp}^2} \geq \hbar/2 \tag{2}$$

which indicates the *purity* of Gaussian states. For steady states, which have $V_{\rm xp} = 0$, U can be converted *back* to an effective occupation number,

$$N_{\rm eff} \equiv U/\hbar - 1/2\,, \tag{3}$$

which is the *minimum* occupation number one could obtain when the same test-mass state is put into a quadratic potential well with an arbitrary eigenfrequency Ω_*, i.e.,

$$N_{\rm eff} = \min_{\Omega_*}\left[\frac{V_{\rm pp}/(2m) + m\Omega_*^2 V_{\rm xx}/2}{\hbar\Omega_*} - \frac{1}{2}\right] \tag{4}$$

Since Ω_* may be very different from the original eigenfrequency of the oscillator, the resulting quantum state tends to be *position squeezed* if $\Omega_* > \omega_p$, and *momentum squeezed* if $\Omega_* < \omega_p$. This $N_{\rm eff}$ is unambiguous regarding the choice of oscillator eigenfrequency[8] while still having a straightforward interpretation. In fact, $N_{\rm eff}$ directly determines the von Neumann entropy of the state [4]:

$$S = (N_{\rm eff}+1)\log(N_{\rm eff}+1) - N_{\rm eff}\log N_{\rm eff}\,. \tag{5}$$

General Optimal Controller. We now set off to obtain the optimal controller that yields a minimum U. We write the closed-loop position and momentum of the oscillator as (tilde denotes frequency-domain quantities)

$$\tilde x_{\rm ctrl} = \tilde x_0 - \tilde K_{\rm ctrl}\tilde y_0\,,\quad \tilde p_{\rm ctrl} = \tilde p_0 + im\Omega\tilde K_{\rm ctrl}\tilde y_0 \tag{6}$$

where x_0 and p_0 are the open-loop evolution of oscillator position, y_0 is the open-loop out-going field that we measure, and m is the oscillator's mass. The function $\tilde K_{\rm ctrl}$ is connected to the transfer function $\tilde H$ (from $\tilde x_0$ to $\tilde y_0$), control filter $\tilde C$ ($\tilde F_{\rm fb} = -\tilde C\tilde x$), and oscillator response

$$\tilde R_{xx} = -1/[(\Omega-\omega_p+i\gamma_p)(\Omega+\omega_p+i\gamma_p)], \tag{7}$$

where we have set the mass equal to one and ω_p and $\gamma_p = \omega_p/Q_p$ are frequency and relaxation rate, as

$$\tilde K_{\rm ctrl} = \tilde R_{xx}\tilde C/(1+\tilde R_{xx}\tilde C\tilde H)\,. \tag{8}$$

The closed-loop system is *stable* and the feedback $\tilde C$ is *proper, if and only if* $\tilde K_{\rm ctrl}$ is causal (i.e., no poles in the upper-half complex plane) and $\lim_{\Omega\to\infty}\Omega\tilde K_{\rm ctrl}(\Omega) = 0$ (which, together with causality, implies $K_{\rm ctrl}(0) = 0$ in the time domain). Closed-loop response function of the oscillator's location to external force is given by

$$\tilde R_{xx}^{\rm eff} = \tilde R_{xx}(1-\tilde H\tilde K_{\rm ctrl})\,. \tag{9}$$

Equation (6) has been put into a form in which the control system is viewed as subtracting the readout field from position and momentum of the oscillator. On the other hand, causal Wiener Filters $K_{x,p}$ can be constructed based on the cross spectral density between x_0, p_0 and y_0, to yield the best (least-mean-square) estimates of x_0 and p_0 based on past data of y_0,

$$\tilde K_a = \tilde G_a/\tilde\phi_+ \equiv \left[S_{ay}/\tilde\phi_-\right]_+/\tilde\phi_+. \tag{10}$$

Here $a = x, p$, S_{ay} is the cross spectrum between a and y (all for the open-loop system), $S_{yy} = \tilde\phi_+\tilde\phi_-$ is a spectral decomposition with $\tilde\phi_+$ only having zeros and poles in the lower-half complex plane and $\tilde\phi_+(\Omega) = \tilde\phi_-^*(\Omega^*)$, and $[\ldots]_+$ stands for taking the component with only poles in the lower-half complex plane. The conditional covariance matrix gives the minimum error covariance,

$$V_{ab}^{\rm c} = V_{ab} - \int_0^{+\infty} G_a(t)G_b(t)dt \tag{11}$$

with V_{ab} the open-loop unconditional variances ($V_{xp} = 0$). We also note that $G_p(t) = \dot G_x(t)$ (for $t > 0$), so $V_{xp}^{\rm c} = G_x^2(0)/2 > 0$. Now for a controlled system (6), we have $V_{xx}^{\rm ctrl} = V_{xx}^{\rm c} + \Delta_x$, $V_{pp}^{\rm ctrl} = V_{pp}^{\rm c} + \Delta_p$, where

$$\begin{bmatrix}\Delta_x\\ \Delta_p\end{bmatrix} \equiv \int_0^{+\infty}\begin{bmatrix}[G_x(t)-f(t)]^2\\ [G_p(t)-\dot f(t)]^2\end{bmatrix}dt \tag{12}$$

with

$$f(t) \equiv \int_0^t \phi_+(t-t')K_{\rm ctrl}(t')dt'\,. \tag{13}$$

Minimization of $\sqrt{V_{xx}^{\rm ctrl}V_{pp}^{\rm ctrl}}$ ($V_{xp}^{\rm ctrl} = 0$) over all possible f with $f(0) = 0$ gives

$$U_{\rm opt} \equiv \min_f\sqrt{V_{xx}^{\rm ctrl}V_{pp}^{\rm ctrl}} = \sqrt{V_{xx}^{\rm c}V_{pp}^{\rm c}} + V_{xp}^{\rm c}\,, \tag{14}$$

which is achieved by

$$\tilde\phi_+\tilde K_{\rm ctrl} = \tilde G_x - G_x(0)/(\rho - i\Omega),\ \rho = \sqrt{V_{pp}^{\rm c}/V_{xx}^{\rm c}}\,. \tag{15}$$

Here we see that the optimal controller is unique, and that $U_{\rm opt} > U_{\rm cond} \equiv \sqrt{V_{xx}^{\rm c}V_{pp}^{\rm c} - (V_{xp}^{\rm c})^2}$ unless $V_{xp}^{\rm c} = 0$. [9] Remarkably, even for a perfect measurement where conditional state is pure, it is *usually impossible* to apply a controller to create a perfectly pure stationary state.

General Markovian measurements. We next specialize to Markovian measurement systems, with

$$y = Z + x\,,\quad x = \tilde R_{xx}F \tag{16}$$

where y is the output field, Z is the total sensing noise, $\tilde R_{xx}$ is given by Eq. (9), and F is the total force noise, with Z and F characterized by *real and constant* (single-sided) cross spectral densities S_{ZZ}, S_{ZF} and S_{FF} satisfying Heisenberg Uncertainty Relation [6]

$$\sqrt{S_{ZZ}S_{FF} - S_{ZF}^2} \equiv (1+\mu)^{1/2}\hbar \geq \hbar\,. \tag{17}$$

This can describe measurement systems with white sensing and force noises, e.g., measurement of mirror location using a broadband Fabry-Perot cavity, with frequency independent input squeezing and homodyne detection.

If we write (assuming $\omega_p \gg \gamma_p$)

$$\begin{aligned} S_{yy} &= S_{ZZ} + 2{\rm Re}(\tilde R_{xx})S_{ZF} + S_{FF}|\tilde R_{xx}|^2 \\ &\equiv S_{ZZ}QQ^*/(PP^*) \end{aligned} \tag{18}$$

with $P \equiv -1/\tilde{R}_{xx}$ and $QQ^* \equiv \Omega^4 - 2A\omega_p^2\Omega^2 + B^2\omega_p^4$ and

$$A \equiv 1 + \frac{1}{\omega_p^2}\frac{S_{ZF}}{S_{ZZ}}\,, \quad B^2 \equiv 1 + \frac{2}{\omega_p^2}\frac{S_{ZF}}{S_{ZZ}} + \frac{1}{\omega_p^4}\frac{S_{FF}}{S_{ZZ}} \quad (19)$$

Using Eqs. (10), (11) and (18), (19), the conditional covariance matrix can be obtained:

$$\mathbf{V} = \frac{\hbar(1+\mu)^{1/2}}{2}\begin{bmatrix} \frac{1}{\omega_p}\sqrt{\frac{2}{A+B}} & \sqrt{\frac{B-A}{B+A}} \\ \sqrt{\frac{B-A}{B+A}} & \omega_p\sqrt{\frac{2B^2}{A+B}} \end{bmatrix} \quad (20)$$

The conditional purity is given by

$$U_c = \sqrt{|\mathbf{V}|} = (1+\mu)^{1/2}\hbar/2\,, \quad (21)$$

which is *identical* to the "purity" of the measurement process, see Eq. (17). In absence of classical noise, the conditional quantum state of the oscillator is always pure. With Eq. (14), we obtain

$$\frac{U_{\rm ctrl}}{\hbar/2} = (1+\mu)^{1/2}\frac{\sqrt{1-A/B}+\sqrt{2}}{\sqrt{1+A/B}} \quad (22)$$

which is obtained by the unique optimal filter [Cf. (8), (10) and (15)] and associated close-loop response function

$$\tilde{C} = C_0(\Omega - C_1)/(\Omega - C_2)\,, \quad (23)$$
$$\tilde{R}^{\rm eff}_{xx} = -(\Omega-\Omega_4)/[(\Omega-\Omega_1)(\Omega-\Omega_2)(\Omega-\Omega_3)], \quad (24)$$

where $C_0 = -(\omega_p^2 + \Omega_4\Omega_3)$, $C_1 = (\Omega_3^3 + \omega_p^2\Omega_4)/(\omega_p^2 + \Omega_4\Omega_3)$, $C_2 = \Omega_4$ and $\Omega_{1,2}$ are roots of Q, namely

$$\Omega_{1,2} = \pm\omega_p\sqrt{(B+A)/2} - i\omega_p\sqrt{(B-A)/2} \quad (25)$$

while $\Omega_{3,4}$ are purely imaginary:

$$\Omega_3 = -i\sqrt{B}\omega_p\,, \quad \Omega_4 = -i[\sqrt{B}+\sqrt{2(B-A)}]\omega_p\,. \quad (26)$$

We note the following features: (i) the poles of the resulting closed-loop dynamics, $\Omega_{1,2}$ are identical to zeros of S_{yy}, which indicates that the optimal controller "finds" the frequency of maximal sensitivity, and shifts the pole of the system to that frequency; (ii) even though we have employed substantial mathematics, for Markovian measurement processes, the optimal controller (23) can be motivated simply from constant feedback plus linear damping and simple band limiting (which is required if we would like $V_{pp}^{\rm ctrl}$ to be finite), therefore justifying previous choices by various authors [3]; (iii) pure state is only achievable when $\mu \approx 0$ *and* $A \approx B$ (note that $A \le B$), which means pure controlled state is not achievable even for many systems without classical noise $\mu = 0$.

Having $A \approx B$ corresponds to a high quality factor for the closed-loop dynamics [Cf. Eq. (25)], with

$$Q_{\rm eff} = \sqrt{B+A}/\sqrt{B-A}\,, \quad (27)$$

this is understandable because a low-Q oscillator cannot have a pure quantum state simply due to the Fluctuation Dissipation Theorem [6]. Moreover, $A \approx B$ also corresponds to S_{yy} having a very small minimum [in the limit of $A = B$, S_{yy} reaches 0, Cf. Eq. (18)]. Let us consider the force (G) - referred noise spectrum,

$$S_G = S_{yy}/|\tilde{R}_{xx}|^2 = S_{ZZ}QQ^*, \quad (28)$$

and compare it with free-mass SQL for force detection $S_G^{\rm SQL} = 2\Omega^2\hbar$. Taking minimum over all frequencies, we obtain the SQL-beating factor η^2 as

$$\eta^2 \equiv [S_G/S_G^{\rm SQL}]_{\rm min} = \sqrt{1+\mu}/Q_{\rm eff} \quad (29)$$

which leads to [Cf. Eq. (22)]

$$\frac{U_{\rm ctrl}}{\hbar/2} = \eta^2 + \sqrt{\frac{2(1+\mu)}{1+A/B}} \ge \eta^2 + 1,\ N_{\rm eff} \ge \eta^2/2. \quad (30)$$

This means an oscillator under measurement can only be converted into a quantum oscillator via control if it can beat the free-mass SQL in a force measurement, in which case the optimally-controlled closed-loop quality factor $Q_{\rm eff}$ would also far exceed unity.

Position measurement with light. In the realistic case with suspension and internal thermal noises and optical loss, we have (as in Ref. [5])

$$x = \tilde{R}_{xx}[\alpha a_1 + \xi_F + G] \quad (31a)$$
$$y = a_1\sin\phi + \cos\phi[a_2 + \alpha/\hbar(x+\xi_x)] + \sqrt{\epsilon}n \quad (31b)$$

where $a_{1,2}$ are input quadrature fields, α is the measurement strength ($\alpha = 4\sqrt{\hbar\omega_0 I_c/(\tau c^2)}$ for a Michelson interferometer with arm cavities, with ω_0 the carrier frequency, I_c circulating power in the arms, τ input-mirror power transmissivity and c the speed of light), ξ_x and ξ_F are position and force noises, ϵ is the optical loss, and ϕ the readout quadrature. In the notation of Eq. (16):

$$F = \alpha a_1 + \xi_F,\ Z = \xi_x + \frac{a_1\sin\phi + a_2\cos\phi + \sqrt{\epsilon}n}{\alpha/\hbar\cos\phi}, \quad (32)$$

and

$$S_{ZZ} = \hbar/\Omega_q^2\big[\epsilon/\cos^2\phi + 2\zeta_x^2 + \sigma_{11}\tan^2\phi + 2\sigma_{12}\tan\phi + \sigma_{22}\big] \quad (33a)$$
$$S_{FF} = \hbar\Omega_q^2(\sigma_{11} + 2\zeta_F^2), \quad (33b)$$
$$S_{ZF} = \hbar(\sigma_{11}\tan\phi + \sigma_{12}). \quad (33c)$$

where $\sigma_{ij} = S_{a_i a_j}$ are covariances of the input squeezed state, $\Omega_q \equiv \alpha/\sqrt{\hbar}$ is a characteristic measurement frequency, $\zeta_F\Omega_q$ and Ω_q/ζ_x are frequencies at which noise spectra ξ_F and ξ_x intersect the free-mass force- and position-SQL (same as defined in Ref. [5]).

Viscosity noise alone and phase-quadrature readout [$\epsilon = \zeta_x = \phi = 0$, $\sigma_{ij} = \delta_{ij}$, $\omega_p \neq 0$, $\zeta_F^2 = 4\gamma_p k_B T/(\Omega_q^2\hbar)$].

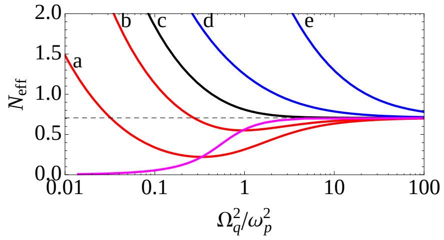

FIG. 1: $N_{\rm eff}$ as a function of measurement strength Ω_q^2/ω_p^2. Curves (a-e) represent $N_{\rm eff}$ at different temperatures θ: a) 0.1, b) 0.5, c) 1 (critical), d) 2, e) 10.

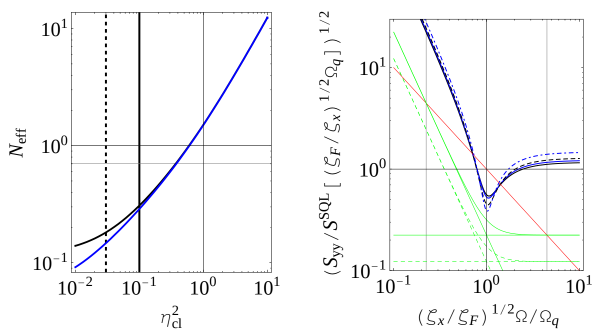

FIG. 2: Left panel: minimal $N_{\rm eff}$ as a function of factor $\eta_{\rm cl}^2$ for 10 dB squeezing (blue) and vacuum input (black). Right panel: total (blue for 10 dB squeezed and black for vacuum input) and classical (green) noise spectra of interferometers with minimum achievable occupation number when $\eta_{\rm cl}^2 = 0.1$ (solid) and 0.03 (dashed).

This includes conventional cold-damping, as well as the latest LIGO cooling experiment — ignoring certain noises. Since a free mass under such a measurement never beats the free-mass SQL, according to Eq. (30), it will always have $N_{\rm eff} \geq 1/2$. The only possibility for an oscillator to have a low $N_{\rm eff}$ is because an oscillator can beat the free-mass SQL around its (original) resonant frequency — yet this is subject to thermal noise. Analytic results can be obtained, analogous to a phase transition. For temperatures below a *critical temperature*, with

$$\theta \equiv T/T_c < 1, \quad T_c \equiv \hbar\omega_p Q_p/(2\sqrt{2}k_B)\,, \tag{34}$$

the minimum occupation number

$$N_{\rm opt}(\theta) \;=\; \frac{\sqrt{2-\theta^2}+\sqrt{2\theta\sqrt{2-\theta^2}}+\theta-\sqrt{2}}{2\sqrt{2}}\,, \tag{35}$$

$N_{\rm opt} \sim 2^{-3/4}\theta^{1/2}$ as $\theta \to 0$, can be achieved with

$$\frac{\Omega_q}{\omega_p} = \sqrt{\frac{\theta^{1/2}(2-\theta^2)^{3/4}}{\sqrt{2-\theta^2}-\theta} - \frac{\theta}{\sqrt{2}}} \sim (\sqrt{2}\theta)^{1/4}\,,\ (\theta \to 0) \tag{36}$$

(plus optimal control). For $T > T_c$, the temperature-independent minimum of $N_{\rm eff} = 1/\sqrt{2}$ is achieved with infinite measurement strength plus optimal control.

For LIGO, $\omega_p = 2\pi\times 1\,$Hz and $Q_p \approx 10^{10}$, we have $T_c \approx 0.2\,$K, which is beyond reach: a conventional LIGO would always have $N_{\rm eff} \geq 1/\sqrt{2}$ even when viscous damping alone is considered. For general devices [7]:

$$T_c = 45\,{\rm K} \times (Q_p/10^6) \times [\omega_p/(2\pi \times 1\,{\rm MHz})]\,. \tag{37}$$

Prospect of LIGO cooling ($\omega_p = 0$). In order to evade the limitation of $N_{\rm eff} \geq 1/\sqrt{2}$, we consider arbitrary readout quadrature with squeezed light input. Given a noise budget, with a fixed squeezing factor within σ_{ij} we optimize Ω_q, ϕ and the squeezing angle for a minimum $N_{\rm eff}$. We carry out numerical optimizations fixing $\epsilon = 1\%$, and for vacuum and 10 dB squeezing respectively, varying the product of $\zeta_F\zeta_x$ (their absolute values set the scale for Ω_q but does not affect $N_{\rm eff}$). In left panel of Fig. 2, we plot $N_{\rm eff}$ as a function of the factor $\eta_{\rm cl}^2 \equiv [S^{\rm cl}/S^{\rm SQL}]_{\rm min} = 2\zeta_F\zeta_x$ that the total classical noise beats the SQL. In right panel of Fig. 2, we plot noise spectra of interferometers that offers minimum $N_{\rm eff}$ for $\eta_{\rm cl}^2 = 0.1$ (solid curves) and 0.03 (dashed curves) with vacuum and 10 dB input squeezing respectively. For vacuum input, we have $\zeta_F = 0.138$, $\phi = 1.18$, while for 10 dB squeezing, we have $\zeta_F = 0.131$, $\phi = 0.35$, and $(\sigma_{11}, \sigma_{12}, \sigma_{22}) = (0.99, 2.33, 6.54)$. Note that here squeezed-input at $\eta_{\rm cl} = 0.1$ is nearly of no use because SQL-beating at this level is also possible with non-phase quadrature detection, while it is still useful at smaller classical noises (*e.g.* $\eta_{\rm cl}^2 = 0.03$).

Acknowledgment We thank the AEI-Caltech-MIT-MSU MQM discussions group for many discussions. Research of S.D., H.M-E., K.S. and Y.C. are supported by the Alexander von Humboldt Foundation. S.D., Y.C. and K.S. are also supported in part by NSF grants PHY-0653653 and PHY-0601459, as well as the David and Barbara Groce startup fund at Caltech. K.S. is supported by Japan Society for the Promotion of Science (JSPS).

[1] A. Abramovici et al., Science **256**, 325 (1992).
[2] LIGO Scientific Collaboration, B. Abbot et al., *Observation of a kilogram-scale oscillator near its quantum ground state*, submitted to Nature (2008).
[3] S. Mancini et al., Phys. Rev. Lett. **80**, 688 (1998); P.F. Cohadon, et al., Phys. Rev. Lett. **83**, 3174 (1999); J.M. Courty et al., Eur. Phys. J. D **17** 399 (2001); D. Vitali et al., J. Opt. Soc. Am. B **20**, 1054 (2003).
[4] W.H. Zurek et al., Phys. Rev. Lett. **70**, 1187 (1993).
[5] H. Müller-Ebhardt et al., Phys. Rev. Lett. **100**, 013601 (2008).
[6] V.B. Braginsky and F.Ya. Khalili, *Quantum Measurement*, Cambridge University Press, 1992.
[7] P.F. Cohadon et al., Phys. Rev. Lett. **83** 3174 (1999); A. Hopkins et al., Phys. Rev. B **68**, 235328 (2003); D. Kleckner and D. Bouwmeester, Nature **444**, 75 (2006); T. Corbitt et al., Phys. Rev. Lett. 99, 160801 (2007); A. Vinante et al., Phys. Rev. Lett. **101**, 033601 (2008).

[8] In Refs. [3], either kinetic or total energy is compared with quanta of the original oscillator; this will become ambiguous when $Q_{\rm eff}$ becomes low, and irrelevant when $\Omega_{\rm eff}$ is shifted significantly

[9] $V_{xp} = 0$ corresponds to $G_x(0) = 0$, which in turn requires that no information about $x(t)$ be collected at time $t - 0$. This point will be elaborated in a further publication.