Understanding the sensitivity of the stochastic radiometer analysis in terms of the strain tensor amplitude

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I. INTRODUCTION

The aim of this investigation is to construct the expected upper-limit sensitivity of the stochastic radiometer search in terms of h_0 the gravitational wave strain tensor amplitude commonly used parameter used within the LVC pulsar group. We focus only on the stochastic radiometer search as applied to the continuous wave targeted search for the low-mass X-ray binary Sco X-1 in the S4 [1] and S5 analyses. We do not refer to the optimised continuous wave radiometer analysis of [2].

II. THE CONTINUOUS WAVE SIGNAL

We start by defining a continuous wave signal assuming a rapidly rotating tri-axial neutron star as the source. The strain as measured at a detector can be written as

$$h(t) = F^{+}(t;\alpha,\delta)h^{+}(t;\vec{\theta}) + F^{\times}(t;\alpha,\delta)h^{\times}(t;\vec{\theta}) \tag{1}$$

where the antenna response functions to the "plus" and "cross" polarisations are defined as

$$F^{+}(\tau;\alpha,\delta) = a(t;\alpha,\delta)\cos 2\psi + b(t;\alpha,\delta)\sin 2\psi, \tag{2}$$
$$F^{\times}(t;\alpha,\delta) = b(t;\alpha,\delta)\cos 2\psi - a(t;\alpha,\delta)\sin 2\psi \tag{3}$$

and ψ is the polarisation angle of the signal. Note that the time and sky-position dependent functions $a(t;\alpha,\delta)$ and $b(t;\alpha,\delta)$ are exactly identical to the quantities labelled F^{+} and $F^{\times}$ as used in the stochastic search and defined in [3]. The contributions from the 2 signal polarisations are

$$h^{+}(t;\vec{\theta}) = A_{+}\cos\left[\Phi(\vec{\theta},t) + \phi_0\right] \tag{4}$$
$$h^{\times}(t;\vec{\theta}) = A_{\times}\sin\left[\Phi(\vec{\theta},t) + \phi_0\right] \tag{5}$$

with the polarisation amplitudes

$$A_{+} = \frac{1}{2}h_0(1+\cos^2\iota) \tag{6}$$
$$A_{\times} = h_0\cos\iota \tag{7}$$

where ι is the inclination angle of the pulsar.

The signal phase $\Phi(t;\vec{\theta})$ is a function of time t and of a multitude of signal parameters defined by the vector $\vec{\theta}$. This vector of parameters includes the source phase parameters $f_0,\ldots,f_0^n$, the spin epoch $t_{\rm ref}$, the sky position parameters α,δ that define the phase evolution due to the motion of the detector, the orbital parameters, semi-major axis a, orbital period P, time of periapsis $t_{\rm p}$, eccentricity e, argument of periapse ω, plus any number of other quantities.

In the stochastic radiometer analysis (it is my understanding that) the non-monochromatic nature of the target object is accounted for by choosing a frequency resolution that safely contains the signal's frequency variations in time throughout the coarse of the total observation. Within a single coherent segment the source is safely assumed to be mono-chromatic.

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III. THE CROSS-CORRELATION STATISTIC FOR A CONTINUOUS GW SIGNAL

The style of notation and description of the data analysis methods used in the stochastic radiometer search seems strongly rooted in it's original all-sky, all-frequency isotropic stochastic background analysis (based on [3]). For simplicity we have attempted to use a far simpler approach to the cross-correlation allowable by the fact that we fix the analysis to a single sky position and frequency. We also assume that the detector noise is stationary throughout the analysis. In addition we assume *co-located* but *non-aligned* detectors since the process of accounting for the separation of detectors is a trivial time shift which does not effect the sensitivity of the search when the sky position of the source is known.

We start by defining the cross-correlation statistic between 2 detector data-sets $s_1(t)$ and $s_2(t)$,

$$\begin{aligned} S &= \int_0^\infty dt\, s_1(t)s_2(t) \\ &= 2\Re\left\{\int_0^\infty df\, \tilde{s}_1^*(f)\tilde{s}_2(f)\right\} \\ &\equiv 2\Delta f\Re\left\{\tilde{s}_1^*(f)\tilde{s}_2(f)\right\} \end{aligned} \tag{8}$$

such that, given our assumptions that the detectors are co-aligned and the signal remains within a single discrete frequency bin Δf we arrive at the above simplistic frequency domain result.

The discrete Fourier transform definition we have used is

$$\tilde{s}(f_k) = \Delta t \sum_{j=0}^{N-1} s(t_j) e^{-2\pi i jk/N}. \tag{9}$$

where k indexes each of the N frequency bins, each of width $\Delta f = 1/T$ where T is the coherent observation time. Henceforth we will drop the frequency dependence from the data-sets since we assume that we are focused on a single frequency bin of interest.

For the pulsar signal defined in the previous section we obtain the following frequency domain representation if evaluated at a frequency bin containing the source instantaneous signal frequency,

$$\tilde{h}_i = \frac{1}{2} T e^{i\phi_0}\left(A^+ F_i^+ - iA^\times F_i^\times\right). \tag{10}$$

Therefore, for a common signal is additive Gaussian noise $\tilde{n}_i$ in each detector we compute the expectation value of the cross-correlation statistic S as

$$\begin{aligned} \langle S\rangle &= 2\Delta f\left\langle \Re\left\{\left(\tilde{n}_1^* + \tilde{h}_1^*\right)\left(\tilde{n}_2 + \tilde{h}_2\right)\right\}\right\rangle \\ &= 2\Delta f\Re\left\{\tilde{s}_1^*\tilde{s}_2\right\} \\ &= \frac{1}{2} T\left[(A^+)^2 F_1^+ F_2^+ + (A^\times)^2 F_1^\times F_2^\times\right]. \end{aligned} \tag{11}$$

It seems to be the intention in the stochastic radiometer analysis to normalise their filter function (which we have up to this point omitted) such that the cross-correlation statistic itself is an estimator of the quantity of interest. My understanding is that the quantity Y/T is chosen to be proportional to H where Y is equivalent to our S and H is defined as the one-sided spectrum of strain power.

In our case we notice that both A^+ and $A^\times$ are proportional to h_0 and we will therefore chose the following arbitrary choice of normalisation

$$\begin{aligned} S' &= \frac{2S}{T\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]} \\ &= \frac{4\Re\left\{\tilde{s}_1\tilde{s}_2\right\}}{T^2\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]} \end{aligned} \tag{12}$$

such that S' is proportional to h_0^2. Indeed we see that for a circularly polarised signal $|A^+| = |A^\times|$ our new normalised statistic is equal to h_0^2.

So for completeness, we define the expectation value of the new statistic as

$$\langle S'\rangle = \frac{\left[(A^+)^2F_1^+F_2^+ + (A^\times)^2F_1^\times F_2^\times\right]}{\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]} \tag{13}$$

and its variance as

$$
\begin{aligned}
\sigma^2 &= \langle S'^2\rangle - \langle S'\rangle^2 \\
&= \frac{16\left\langle \left[\Re(\tilde{n}_1^* + \tilde{h}_1^*)\Re(\tilde{n}_2 + \tilde{h}_2) + \Im(\tilde{n}_1^* + \tilde{h}_1^*)\Im(\tilde{n}_2 + \tilde{h}_2)\right]^2 \right\rangle}{T^4\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]^2} - \langle S'\rangle^2 \\
&= \frac{16\left\langle (\Re\tilde{n}_1\Re\tilde{n}_2)^2 + (\Im\tilde{n}_1\Im\tilde{n}_2)^2 + \left(\Re\tilde{n}_1\Re\tilde{h}_2\right)^2 + \left(\Re\tilde{h}_1\Re\tilde{n}_2\right)^2 + \left(\Im\tilde{n}_1\Im\tilde{h}_2\right)^2 + \left(\Im\tilde{h}_1\Im\tilde{n}_2\right)^2 \right\rangle}{T^4\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]^2} \\
&= \frac{2P_1P_2}{T^2\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]^2} + \left\{\frac{P_2\left[\left(A^+F_1^+\right)^2 + \left(A^\times F_1^\times\right)^2\right] + P_1\left[\left(A^+F_2^+\right)^2 + \left(A^\times F_2^\times\right)^2\right]}{T\left[F_1^+F_2^+ + F_1^\times F_2^\times\right]^2}\right\}
\end{aligned}
\tag{14}
$$

where we have used

$$
\langle \Re\tilde{n}_i^2\rangle = \langle \Im\tilde{n}_i^2\rangle = \frac{1}{2}TP_i \tag{15}
$$

where P_i is the single-sided noise power spectral density of the i'th detector estimated at our frequency of interest.

Note that in the weak signal case we can approximate the variance of S' by taking only the first term in Eq. 14 which seems to be an approximation also made in the stochastic radiometer analysis.

It appears that the reason behind dividing up the stochastic radiometer analysis in time is motivated by the requirement of noise stationarity. This lends itself usefully to the pulsar case where, if the signal frequency variation with time is ignored one must use broad frequency bins to "catch" the signal. We will therefore compute the cross-correlation statistic for each of M equal length segments of data and combine them to form a final optimal statistic.

Due to our choice of normalisation we will have M separate estimates of a quantity proportional to the constant h_0^2 and a theoretical estimate of the variance of each of these measurements. For the stochastic radiometer analysis the choice of how to combine these statistics is one whereby a weighted average of the statistic is computed where each measurement is weighted by it's theoretical variance. This is optimal in the maximum likelihood sense under the assumption that each summed statistic is Gaussian distributed with identical means. In the pulsar single-frequency-bin case this condition is not met.

We continue in the same fashion and generate our final statistic as

$$
S_{\rm opt} = \frac{\sum_{j=1}^M S'_j/\sigma_j^2}{\sum_{j=1}^M 1/\sigma_j^2} \tag{16}
$$

which, by the central limit theorem, will be distributed according to a Gaussian distribution with mean

$$
\langle S_{\rm opt}\rangle = \frac{\sum_{j=1}^M \left[(A^+)^2F_{1j}^+F_{2j}^+ + (A^\times)^2F_{1j}^\times F_{2j}^\times\right]\left[F_{1j}^+F_{2j}^+ + F_{1j}^\times F_{2j}^\times\right]}{\sum_{j=1}^M\left[F_{1j}^+F_{2j}^+ + F_{1j}^\times F_{2j}^\times\right]^2} \tag{17}
$$

where we note that since the antenna pattern functions are slowly varying and we have been able to approximate them as constant during a coherent segment length we now index them since they will be different for each segment. The variance of the final statistic is

$$
\begin{aligned}
\sigma_{\rm opt}^2 &= \left[\sum_{j=1}^M 1/\sigma_j^2\right]^{-1} \\
&= \frac{2P_1P_2}{T^2\sum_{j=1}^M\left[F_{1j}^+F_{2j}^+ + F_{1j}^\times F_{2j}^\times\right]^2}.
\end{aligned}
\tag{18}
$$

If we now define the signal-to-noise ratio as

$$
\rho = \frac{\langle S_{\rm opt}\rangle}{\sqrt{\sigma_{\rm opt}^2}} \tag{19}
$$

we see that in the special case of a circularly polarised signal we can write

$$\rho_{\text{circ}} = h_0^2 T \sqrt{\frac{\sum_{j=1}^{M} \left[F_{1j}^{+} F_{2j}^{+} + F_{1j}^{\times} F_{2j}^{\times}\right]^2}{2P_1 P_2}} \approx h_0^2 \sqrt{\frac{T T_{\text{obs}} \langle \left[F_1^{+} F_2^{+} + F_1^{\times} F_2^{\times}\right]^2 \rangle}{2P_1 P_2}} \quad (20)$$

from which it follows that for a fixed signal-to-noise ratio $h_0 \propto (T T_{\text{obs}})^{1/4} = \sqrt{T} M^{1/4}$. This is a standard result for an incoherent search.

IV. UPPER LIMITS ON h_0

The methods by which the upper-limits are obtained in the S4 and S5 stochastic radiometer analyses are not explained in either LSC/LVC results paper. It is mentioned however, that they are obtained using a Bayesian approach and a 90% confidence threshold is applied.

We will therefore go about applying a Bayesian upper limit on the parameter h_0 given a single measurement of the S_{opt} detection statistic. This statistic is drawn from a Gaussian distribution with mean and variance given by Eqs. 17 and 18 respectively. The associated likelihood is

$$L(S_{\text{opt}}|h_0, \cos\iota, \psi) = \frac{1}{\sqrt{2\pi\sigma_{\text{opt}}^2}} \exp\left\{-\frac{1}{2\sigma_{\text{opt}}^2}\left(S_{\text{opt}} - \mu(h_0, \cos\iota, \psi)\right)^2\right\} \quad (21)$$

where we have defined $\mu(h_0, \eta, \psi) = \langle S_{\text{opt}}(h_0, \eta, \psi) \rangle$.

The marginalised posterior probability density of h_0 is then

$$p(h_0) \propto \int_{-1}^{1} d\cos\iota \int_{-\pi/4}^{\pi/4} d\psi \exp\left\{-\frac{1}{2\sigma_{\text{opt}}^2}\left(S_{\text{opt}} - \mu(h_0, \eta, \psi)\right)^2\right\} \quad (22)$$

where we choose uniform priors consistent with the physical distributions one would attribute to $\cos\iota$ and ψ and for simplicity we adopt a non-physical uniform prior on h_0[1]. Also note that the variance of the statistic (in the weak signal regime) is independent of the signal parameters and is only a function of the antenna patterns meaning we can drop it from the posterior normalisation.

To compute an expected upper limit sensitivity for a given frequency bin we assume that we measure a value of S_{opt} consistent with its distribution in the absence of a signal i.e $S_{\text{opt}} = \mu(h_0 = 0) = 0$ (this is equivalent to a 50% false alarm rate). We then solve

$$0.9 = \int_0^{h_0^{90\%}} p(h_0)\, dh_0 \quad (23)$$

after correctly normalising the h_0 posterior obtained from Eq. 22.

Note that for the circular polarisation case where $\cos\iota = \pm 1$ the expectation value of the detection statistic μ becomes independent of ψ and the posterior on h_0 is simply

$$p(h_0)_{\text{circ}} \propto \frac{1}{\sigma_{\text{opt}}} \exp\left\{-\frac{\left(S_{\text{opt}} - h_0^2\right)^2}{2\sigma_{\text{opt}}^2}\right\}. \quad (24)$$

In [4] equation 3.51 equates a measurement result from the stochastic radiometer analysis to a quantity that is a function of the pulsar parameters. This quantity is identical to the quantity $\langle S_{\text{opt}} \rangle$ defined in Eq. 17 and it is stated that this is equal to $H(f)df$ which is what I assume (since it is not stated clearly) is what is referred to as "strain" and is therefore the quantity on which an upper-limit is set.

One can see from Eq. 17 that $\langle S_{\text{opt}} \rangle$ and hence $H(f)df$ is exactly equal to h_0^2 only for a circularly polarised continuous gravitational wave. This corresponds to the optimal configuration for a pulsar (spin axis (anti)parallel with the line of sight) and hence a null detection assuming a circularly polarised source leads to more stringent (lower) upper-limits than for any other polarisation.

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[1] This choice of prior is identical to that used in the known pulsar searches i.e. the Crab pulsar search.

V. CONCLUSIONS

The main point of this investigation is to understand what exactly the radiometer analysis actually sets it's upper-limit on and it appears that in terms of quantities familiar to the pulsar working group, it is equivalent to setting an upper limit on h_0 assuming a circularly polarised source. This does not mean that it is insensitive to any other polarisations, it simply means that the upper-limit always assumes circular polarisation and is hence always more stringent (less-conservative) than an angle-averaged upper-limit (such as those published within the pulsar group) by a sky position dependent factor of 2.43. This is the ratio in 90% confidence h_0 upper-limit values between the unknown nuisance parameter case and the circularly polarised case assuming that the detection statistic is equal to its expectation value. We find that for long total observation spans ($\gg$ 1 day) this ratio is dependent only upon the declination of the source on the sky and varies over the range 2.21–2.61.

In addition, we also note that for Sco X-1 the choice of a 0.25 Hz wide frequency bin leads to a potential loss in sensitivity at lower frequencies where the orbital Doppler modulation will be far smaller than this value. This orbital Doppler modulation is known to be

$$\Delta f_{\rm orb} = 0.133 \pm 0.017\,{\rm Hz}\left(\frac{f}{500\,{\rm Hz}}\right) \qquad (25)$$

which means that for frequencies $\gtrsim$ 500 Hz the sensitivity of the stochastic radiometer search becomes increasingly weakened. For example, at the maximum frequency searched $f = 1800$ Hz a single frequency bin will contain at most 50% of the signal. The upper-limits at these higher frequencies are therefore significantly under-estimated.

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