LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Effect of the end mirror curvature error to the contrast of a Fabry-Perot based Michelson interferometer

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1 Introduction

We found that one of the ETMs of the 40m interferometer could be delivered with a curvature error of 5% from the nominal radius. In this document, the effect of this curvature error to the contrast of a FPMI is described. **This document does not mention how the power/signal recycling changes the contrast defect.**

For the parameter similar to the 40m interferometer with nominal end mirror curvature of 57 m, the contrast defect of the FPMI is given by:

$$dC = 3.61 \times 10^{-4} dR^2 + O^3(dR) \ , \tag{1}$$

and this is dominated by the higher-order modes. The result was confirmed with simulations on FINESSE [1].

2 Generation of the higher-order mode due to mode mismatch

First, we discuss the higher-order mode generation due to the mode-mismatched cavity.

The cavities of the 40m interferometer employ flat ITMs. This means that different ETM curvatures yield the same waist positions (i.e. on the ITMs) and the different waist sizes. Here we define the nominal waist radius as ω_0 while the deviated waist radius as ω_1.

Because of this mismatching, the reflected beam contains higher-order modes. The amplitude of the higher-order mode is calculated using modal decomposition of the laser modes. Here we use the higher-order modes up to 2nd order.

2.1 Higher-order modes in the arm reflection

Suppose that the input beam has the amplitude of E_0. The waist radii of the input beam and the mismatched cavity eigenmode are ω_0 and ω_1, respectively. The cavity feels that the incoming beam consists of TEM_{00} (for the arm) and other higher-order modes. Using the notation described in Appendix A, the input beam is decomposed as given by:

$$\begin{aligned} E_{\mathrm{in}}/E_0 &= U_{00}(\omega_0) \\ &= c_{00\leftarrow 00} U_{00}(\omega_1) + c_{02\leftarrow 00} U_{02}(\omega_1) + c_{20\leftarrow 00} U_{20}(\omega_1) \\ &= c_{00} U_{00}(\omega_1) - c_{02} U_{02}(\omega_1) - c_{02} U_{20}(\omega_1) \ . \end{aligned} \tag{2}$$

Here the first term is resonant in the cavity, and the second and third terms are not. Therefore the reflected beam has the field written as:

$$E_{\mathrm{ref}}/E_0 = r_{\mathrm{reso}} c_{00} U_{00}(\omega_1) - r_{\mathrm{anti}} c_{02} U_{02}(\omega_1) - r_{\mathrm{anti}} c_{02} U_{20}(\omega_1) \ . \tag{3}$$

Now we convert this reflected beam into the original bases [1] [2].

$$E_{\rm ref}/E_0 = (r_{\rm reso}c_{00}^2 + 2r_{\rm anti}c_{02}^2)U_{00}(\omega_0) + (r_{\rm reso}c_{00}c_{02} - r_{\rm anti}c_{02}c_{22})[U_{02}(\omega_0) + U_{20}(\omega_0)] \quad (4)$$
$$\equiv \alpha_{00}U_{00}(\omega_0) + \alpha_{02}[U_{02}(\omega_0) + U_{20}(\omega_0)] \quad (5)$$

Intuitive explanations of Eq.(4) are as follows:

1. Most of the original TEM_{00} is resonant to the cavity.
   (a) Most of this light returns to the original TEM_{00}. ($r_{\rm reso}c_{00}^2$ term of U_{00})
   (b) Small fraction of this light falls into the higher order of the original mode. ($r_{\rm reso}c_{00}c_{02}$ of U_{02}, U_{20})
2. Small fraction of the original TEM_{00} is not resonant in the cavity.
   (a) Most of this light stays at the higher order of the original mode. ($-r_{\rm anti}c_{02}c_{22}$ of U_{02}, U_{20})
   (b) Small fraction of this light comes back to the original TEM_{00}. ($2r_{\rm anti}c_{02}^2$ term of U_{00})

2.2 Electric field at the dark port

Now we consider the beam recombination at the beamsplitter. At the beamsplitter most of the TEM_{00} beams cancels each other. However, small fraction which does not match does leak out to the dark port. In addition, the higher-order modes leaks out without cancellation. Here we derive the amount of this contrast defect.

The Michelson phase is adjusted such that the electric field at the dark port is cancelled. This condition is given when the two encountering beams have opposite phases. Therefore the electric field at the dark port can be written as the following form. Here the beam from the laser source is assumed to have the amplitude of $E_{\rm L}$ and the mode matched to the primary arm.

$$E_{\rm dark}/E_{\rm L} = \frac{1}{2}U_{00}(\omega_0) - \frac{\alpha_{00}}{2}U_{00}(\omega_0) - \frac{\alpha_{02}}{2}[U_{02}(\omega_0) + U_{20}(\omega_0)] \quad (6)$$

The first term is the reflection from the primary (matched) arm. The second term is the TEM_{00} mode from the secondary (unmatched) arm. The third term is the higher order modes form the secondary arm. By substituting Eq.(4) we obtain the beam power at the dark condition.

$$P_{\rm dark} = |E_{\rm dark}|^2 = \frac{(1-\alpha_{00})^2}{4} + \frac{\alpha_{02}^2}{2} \quad (7)$$

[1] $U_{00}(\omega_1) = c_{00}U_{00}(\omega_0) + c_{02}U_{02}(\omega_0) + c_{02}U_{20}(\omega_0)$, $U_{02}(\omega_1) = -c_{02}U_{00}(\omega_0) + c_{22}U_{02}(\omega_0)$, etc.

[2] Sanity check: if $r_{\rm reso} = r_{\rm anti} = 1$ are substituted to Eq.(4), the original mode is reproduced. i.e. $r_{\rm reso}c_{00}^2 + 2r_{\rm anti}c_{02}^2 \sim 1$, $r_{\rm reso}c_{00}c_{02} - r_{\rm anti}c_{02}c_{22} \sim 1$

Similarly, the beam power at the bright fringe is also obtained.

$$P_{\rm bright} = |E_{\rm bright}|^2 = \frac{(1+\alpha_{00})^2}{4} + \frac{\alpha_{02}^2}{2} \quad . \tag{8}$$

The contrast defect is obtained as follows:

$$dC = 1 - C \tag{9}$$
$$= 1 - \frac{P_{\rm bright} - P_{\rm dark}}{P_{\rm bright} + P_{\rm dark}} \tag{10}$$
$$= 1 - \frac{2\alpha_{00}}{1+\alpha_{00}^2 + 2\alpha_{02}^2} \tag{11}$$

If we substitute $\alpha_{00} = 1 - d\alpha$, C can be expanded as

$$dC = \frac{d\alpha^2}{2} + \alpha_{02}^2 + O^3(d\alpha,\, \alpha_{02}) \tag{12}$$

2.3 Optical parameters

We can confirm the amount of the contrast defect by putting numerical values into the formulae. Here we use the simplified parameters for the 40m arm cavities.

- Laser wavelength: $\lambda = 1064\,{\rm nm}$
- Cavity length: $L = 38.4\,{\rm m}$
- ITMs: $T = 0.013846$, Curvature $R_{\rm ITM} = \infty$ (flat)
- ETMs: $T = 0$, Curvature $R_{\rm ETM} = 57\,{\rm m}$ (nominal), $R_{\rm ETM} = 57\,{\rm m} + dR$ (with mismatch)
- The mirrors have no loss. i.e. $r_{\rm reso} = 1$ and $r_{\rm anti} = -1$.
- Michelson length: 0 m (i.e. no Schnupp asymmetry)
- Waist size for the primary arm: $\omega_0 = 0.00300855\,{\rm m}$
- Waist size for the secondary arm: $\omega_1 = \omega_0(1 + dR/18.6)^{1/4}$

Under this condition the lowest order of the contrast defect is

$$dC = 3.61 \times 10^{-4} dR^2 + O^3(dR) \tag{13}$$

The mode decomposition coefficients are given by

$$\alpha_{00} = 1 - 3.61 \times 10^{-4} dR^2 + O^3(dR) \tag{14}$$
$$\alpha_{02} = 1.90 \times 10^{-3} dR + O^2(dR) \quad . \tag{15}$$

The contrast defect by the each TEM components are

$$\mathrm{TEM}_{00}: \; d\alpha^2/2 = 6.53 \times 10^{-8} dR^4 + O^5(dR) \tag{16}$$

$$\mathrm{TEM}_{02/20}: \; \alpha_{02}^2 = 3.61 \times 10^{-4} dR^2 + O^3(dR) \;. \tag{17}$$

This indicates that **the contrast defect is totally dominated by the higher-order modes**.

3 Confirmation by simulation

The above calculation was confirmed by the simulation on FINESSE. dR was scanned from -7 m to $+7$ m. The figure below shows that Eq.(13) agreed well with the simulation up to $|dR| \sim 1$m and still matched to the trend in the further range of dR.

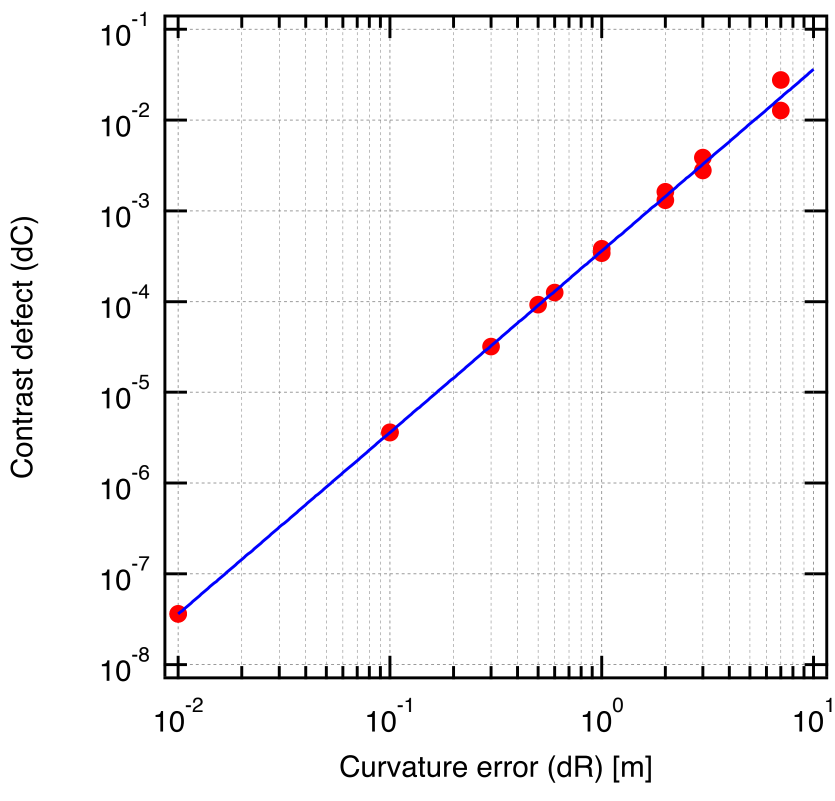

Figure 1: Relationship between the curvature error and the contrast defect. Red dots show the simulated results. The blue line is the curve derived from Eq.(13)

Appendices

A Expansion coefficients between the two TEM modes with different waist sizes

An electric field of the astigmatic $(n,\, m)$-th Hermite-gaussian beam propagating to the z direction is expressed as the following form [2]:

$$\begin{aligned} U_{nm}(x,\, y,\, z) &= u_n(x,\, z)\, u_m(y,\, z) \\ &= \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^n n!\, \omega_{0x}}\right)^{1/2} \left(\frac{\tilde{q}_{0x}}{\tilde{q}_x(z)}\right)^{1/2} \left[\frac{\tilde{q}_{0x}}{\tilde{q}^*_{0x}}\frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{n/2} H_n\left(\frac{\sqrt{2}x}{\omega_x(z)}\right) \exp\left[-\frac{\mathrm{i}k}{\tilde{q}_x(z)}\frac{x^2}{2}\right] \\ &\times \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^m m!\, \omega_{0y}}\right)^{1/2} \left(\frac{\tilde{q}_{0y}}{\tilde{q}_y(z)}\right)^{1/2} \left[\frac{\tilde{q}_{0y}}{\tilde{q}^*_{0y}}\frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{m/2} H_m\left(\frac{\sqrt{2}y}{\omega_y(z)}\right) \exp\left[-\frac{\mathrm{i}k}{\tilde{q}_y(z)}\frac{y^2}{2}\right] \quad . \end{aligned} \tag{18}$$

Here, $\tilde{q}_x(z)$ is the q-parameter in the horizntal direction, being defined by $\tilde{q}_x(z) \equiv z - z_{0x} + \tilde{q}_{0x}$, where $\tilde{q}_{0x} = \mathrm{i}z_{Rx}$. z_{Rx} is the Rayleigh range, being defined by $z_{Rx} = \pi\omega_{0x}^2/\lambda$, where ω_{0x} is the waist size in the horizontal direction. Same definitions for the vertical direction by replacing x to y. k is the wave number and defined by $k \equiv 2\pi/\lambda$, where λ is the wavelength of the laser beam. The symbols *tilde* ($\tilde{}$) express complex numbers.

In our context in this document, the beams are assumed to be non-astigmatic, andthe waists are always at the ITMs. Therefore it is simple to expand the modes at the waist where we can eliminate the curvature of the wave front (i.e. the real part of the q-parameter).

$$\begin{aligned} U_{nm}(\omega_0) &\equiv U_{nm}(x,\, y,\, 0) = u_n(x,\, 0)\, u_m(y,\, 0) \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2^{n+m} n! m!\, \omega_0^2}\right)^{1/2} H_n\left(\frac{\sqrt{2}x}{\omega_0}\right) H_m\left(\frac{\sqrt{2}y}{\omega_0}\right) \exp\left(-\frac{x^2+y^2}{\omega_0^2}\right) \quad . \end{aligned} \tag{19}$$

The U_{kl} mode with the waist size of ω_0 can be expressed by the sum of the modes with the waist size of ω_1 as shown below.

$$U_{kl}(\omega_1) = \sum_{n,\, m} c_{kl\to nm} U_{nm}(\omega_0) \tag{20}$$

The expansion coefficients are obtained by taking the integral because of the orthogonality of the U_{nm} modes.

$$c_{kl\to nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_{kl}(\omega_1) U_{nm}(\omega_0) \mathrm{d}x\mathrm{d}y \tag{21}$$

Here are some results of the coefficients.

$$c_{00\to00} = \frac{2\omega_0\omega_1}{\omega_0^2+\omega_1^2} \quad (\equiv c_{00}) \tag{22}$$
$$c_{00\to02} = -\frac{\sqrt{2}\omega_0\omega_1(\omega_0^2-\omega_1^2)}{(\omega_0^2+\omega_1^2)^2} \quad (\equiv c_{02}) \tag{23}$$
$$c_{00\to20} = c_{02} \tag{24}$$
$$c_{02\to00} = -c_{02} \tag{25}$$
$$c_{20\to00} = -c_{02} \tag{26}$$
$$c_{02\to02} = -\frac{\omega_0\omega_1(\omega_0^4-10\omega_0^2\omega_1^2+\omega_1^4)}{(\omega_0^2+\omega_1^2)^3} \quad (\equiv c_{22}) \tag{27}$$
$$c_{20\to20} = c_{22} \tag{28}$$

Note that the symmetry gives it clear that $c_{00\to01}$, $c_{00\to10}$, $c_{00\to11}$, ... are zero.

Similarly, the inverse matrix is also useful

$$c_{kl\leftarrow nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_{kl}(\omega_1)U_{nm}(\omega_0)\mathrm{d}x\mathrm{d}y \tag{29}$$

$$c_{00\leftarrow00} = c_{00} \tag{30}$$
$$c_{02\leftarrow00} = -c_{02} \tag{31}$$
$$c_{20\leftarrow00} = -c_{02} \tag{32}$$
$$c_{02\leftarrow02} = c_{22} \tag{33}$$
$$c_{20\leftarrow20} = c_{22} \tag{34}$$

B Simulation code for FINESSE

```
############################################################
##  40m FP model
##  Mirror curvature test
##  10 Sep, 2010 by K. Arai
############################################################

maxtem 6
trace 6

# modulation frequency
const fmod 15.0M

###############
# IOO section
```

```
###############

#    name Pow     f_ofs              node
l    L1   1.000   0                  nL          # Laser

tem  L1 0 0 1 0

#    name freq    m_depth  order mod   node
mod  EOM1 $fmod   0.01     1     pm    nL  n1 # EOM

s    s1   0       n1 nPRC6

###############
# Michelson section
###############

# Main BS
# bs name T   loss    phi     alpha     node1 node2 node3 node4
bs1  BS   0.5 0       0       45        nPRC6 nMIY  nMIX  nSRC6

# X Arm
s    sMIX  0                   nMIX    nPOX
#    name  T        loss phi    node
m1   ITMX  0.013846 0    90     nPOX    nARMX1
s    sARMX 38.4                 nARMX1  nARMX2
m1   ETMX  0        0    90     nARMX2  nTRX
attr ETMX Rc 57

# define working modes
cav cavX ITMX nARMX1 ETMX nARMX2

# Y Arm
s    sMIY  0                    nMIY   nPOY
m1   ITMY  0.013846 0   0       nPOY   nARMY1
s    sARMY 38.4                 nARMY1 nARMY2
m1   ETMY  0 0   0              nARMY2 nTRY
attr ETMY Rc 60
###########  ^^ change this number for the test ###########

#####################################
###  This is the end of the model ###
#####################################

## name f    node
xaxis BS phi lin -63.6396 63.6396 2
# This yields the scan of the BS phase
```

```
# from Bright through Dark and then to the bright again.
# This strange number -63.6396 is 90/Sqrt(2).
# This is caused by the incident angle of the beam to the BS is 45deg.

pd pddark nSRC6

# for servo locking
pd1 pd1dark $fmod 0 nPOY

# servo to lock the secondary arm
set err pd1dark re
lock z $err 10 1n
put* ETMY phi $z

yaxis lin abs

gnuterm pdf model_FPMI_RoC_test1.pdf
#gnuterm windows
#gnuterm x11
pause

GNUPLOT
set grid
END
```

References

[1] A. Freise, "Finesse - an interferometer simulation tool" (1999), http://www.gwoptics.org/finesse/.

[2] Eq.(54), Sec. 16.4, A. E Siegman, *Lasers*, University Science Books (1986).