Evaluation of effect of blade internal modes on sensitivity of Advanced LIGO

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1. Introduction

The current model used to estimate the isolation achieved by the quadruple suspension system for the most sensitive mirrors (ETM and ITM), the results of which were presented in the suspension conceptual design, (LIGO T010103-00-D), does not include several features which will affect the isolation at high frequencies. These features include:

1) effects due to the finite mass of the blades
2) the violin modes of the wires

One of the actions raised in the report of the SUS design requirements review (LIGO L010161-00-D) was to evaluate the effects of blade internal modes. This comes under 1) above. 2) is not directly addressed in this document.

In this paper we present an estimate of the peak height of the lowest in frequency of the blade internal modes, and compare it to the expected sensitivity of Advanced LIGO at that frequency, which is limited by sapphire internal losses. We conclude that given the assumptions presented here, the internal modes of the blades will not require to be passively damped.

2. Isolation

The isolation curves presented in the conceptual design come from the MATLAB model of the quadruple suspension. This model currently consists of 4 uncoupled sets of dynamical equations, corresponding to vertical motion, yaw, longitudinal and pitch (together) and transverse and roll (together). In this paper we will focus on the vertical isolation as being the limiting effect. The lowest internal mode of a blade involves vertical motion. It will couple in to horizontal motion of the mirror through cross-coupling from vertical to horizontal. Higher modes could involve twisting, which could produce direct horizontal motion. However the horizontal isolation should be more than adequate at the frequencies of such internal modes, as demonstrated below.

2.1 Direct Horizontal Isolation

An estimate of the residual motion due to direct horizontal excitation can be made as follows. The horizontal transfer function of the quad falls off as f^7, taking into account eddy current damping between the first stage and its support. (If active control is used, the transfer function would fall as f^8 at higher frequencies where the gain is rolled off,).

The transfer function should continue to fall as a steep function of frequency at least up to the violin mode frequencies. By 75 Hz (the frequency of the lowest of the blade internal modes) it is estimated to have a value of ~ 1.5 $x10^{-13}$, assuming f^7 behaviour. This value can be combined with the residual noise from the isolation platform, specified to be not greater than $3x10^{-14}$ m/√Hz at these frequencies, giving an overall background level of horizontal displacement noise of ~$4.5x10^{-27}$ m/√Hz. A blade internal mode might have a loss corresponding to a Q of 10^4 (the value we assume for maraging steel). Thus a well coupled mode might produce a peak at a level of ~$4.5x10^{-23}$ m/√Hz. This is well below the expected internal thermal noise level for sapphire at 75 Hz, ~$8x10^{-21}$ m/√Hz.

2.2 Vertical Isolation

The MATLAB model for vertical motion assumes that the blades are ideal massless springs. The overall behaviour of a blade from whose tip a mass is suspended via a wire is treated as a simple mass/spring unit, where the spring constant is found by adding in series the spring constant of the blade and the wire. The behaviour is dominated by the spring constant of the blade, which is much less than that of the wire.

In reality a blade has finite mass. A more complete treatment of the blade/wire/mass system can be carried out, where the mass and moment of inertia of the blade are included. This has been done for example by Husman (1999), where he uses the Lagrangian technique for analysing the system. The potential energy of the system consists of two terms, corresponding to the energy stored in the blade and the wire. The kinetic energy terms correspond to the translational energy of the mass and the blade and the rotational energy of the blade. From the resulting equations of motion, the magnitude of the transfer function (transmissibility) from the base of the blade to that of the suspended mass can be found. Examples, using Husman's formula (see Appendix A), of the transmissibility for the three different blades in the current baseline design for Advanced LIGO are shown in Figure 1. Note that there is no damping assumed in this model.

Several features should be noted.

1) There are two resonances in each curve. The lower resonance is the familiar one corresponding to a simple spring/mass system. The upper resonance corresponds to the mode of the combined blade/wire/mass system where the blade tip and mass move out of phase as the wire stretches between them. These peaks are sufficiently high in frequency that they should not compromise the overall sensitivity (but note comment in conclusions below).
2) The transmissibility falls off as f^{-2} above the first resonance as expected. However the transmissibility flattens out around 100Hz for these blades. Note that the flattening out is not due to the presence of the second resonance. It would still be seen if the mass were rigidly attached to the end of the blade. It is due to the finite mass of the blade and the fact that the wire suspending the mass is attached at a point that is not the centre of percussion. The blade/wire/mass system is behaving in a manner analogous to a compound pendulum.

These curves can be used to give an estimate of the overall vertical transmissibility of the quadruple pendulum at a particular frequency. We note that above the coupled resonances of a system consisting of several stages, the overall transmissibility tends to the product of the individual transmissibilities of the uncoupled stages. Thus an estimate of the overall transmissibility for the quadruple pendulum consisting of these three blade

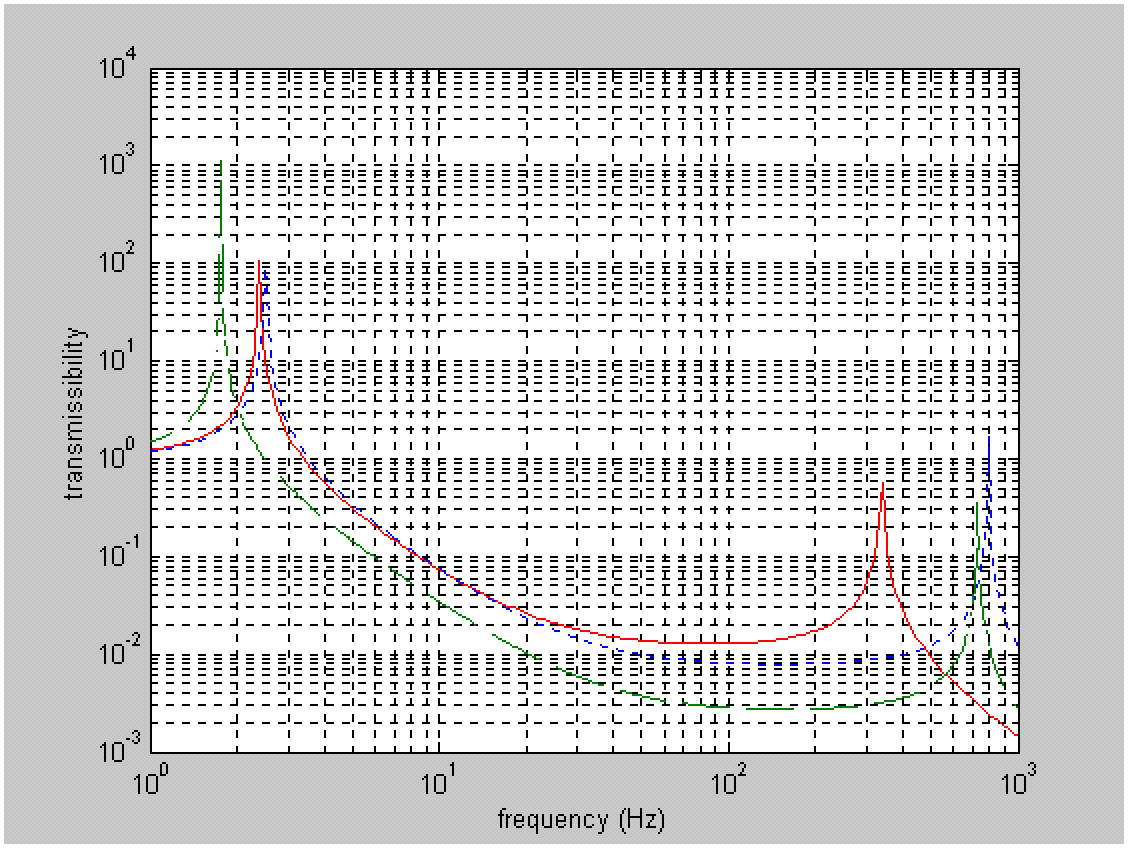


Figure 1: Vertical transmissibility of the three blade/wire/mass stages in the baseline Advanced LIGO quadruple suspension design. Each curve represents an uncoupled stage consisting of a particular blade, the wire or wires suspended from it and the mass it supports in that stage alone (since there are two blades per stage, the mass involved is half of the total mass of that stage). The blades are assumed to be triangular. Red, solid = top stage, blue, dotted = second stage, green, dashed = third stage. Note that the peak heights are limited by the number of data points displayed.

stages plus a final stage of fused silica fibres suspending the mirror can be found by multiplying the individual transmissibilities from figure 1, and an estimated transmissibility of the final stage.

2.3 Estimate of Vertical Transmissibility for Quadruple Pendulum

We require to calculate the transmissibility of the final stage. Since we want to consider the uncoupled behaviour, the vertical frequency can be found from the spring constant of a silica fibre supporting ¼ of the load (since there are 4 fibres). For the particular parameters used in the baseline design (fibre radius = 200 micron, length = 0.6 m and sapphire mass = 40 kg) this yields a frequency of 6.1 Hz. Note that this is less than the coupled frequency of ~8 Hz which is more familiar from noise curves.

At this point we note that the estimated frequencies of the first internal mode of the three different blades in the baseline design are 75, 98 and 118 Hz (from top to bottom). So we shall consider the situation at the lowest of these frequencies, namely 75 Hz, where in general the isolation is less. The product of the transmissibilities is thus given approximately by

$$3 \times 10^{-3} \times 9 \times 10^{-3} \times 1.5 \times 10^{-2} \times (6.1/75)^2 = 2.8 \times 10^{-9}$$

The residual noise on the isolation platform supporting the quadruple pendulum is taken as 3×10^{-14} m/√Hz. We further assume a cross-coupling of 0.1% from vertical to horizontal, thus giving an overall horizontal noise level of 8.4×10^{-26} m/√Hz.

To put this into context with respect to any internal modes, such peaks if fully coupled might appear at a level of Q above this background. For $Q=10^4$, this would give a peak height of 8.4×10^{-22} m/√Hz. This is a factor of ~10 below the estimated sapphire internal thermal noise level.

2.4 Consideration of Effect of Damping on Vertical Isolation.

The above estimation has been made by multiplying the transmissibilities of 4 stages, assuming no damping. However if eddy current damping is used the uppermost stage will be damped, and this will effect its transmissibility. To test the significance of damping on the conclusions above, a model of a blade/wire/mass system with damping was required. A slightly simpler model was used for this – one in which the effect of the wire stretching is not included. The resulting transfer function can be obtained by letting the spring constant of the wire go to infinity in the formula derived by Husman. It can also be derived directly from writing down the equation of motion for such a system, and it was checked that these two methods yielded the same relationship. It was then straightforward to incorporate a damping term in the latter formulation (see Appendix A). Figure 2 shows the results of this analysis carried out on the top blade i.e. in the uppermost stage, the stage in which damping will be applied.

Several features should be noted.

1) On comparing the original curve including the finite wire spring constant with the curve where the wire is assumed infinitely stiff, we see in the latter curve the flattening of transmissibility without the high frequency peak.
2) The effect of damping (here chosen to give a Q of the system of around 5) is to slightly modify the curve between the resonance and the region of flattening, but the additional effect of the damping by 75 Hz is not significant.

Thus we can conclude that our original estimate of noise level is valid in the presence of damping.

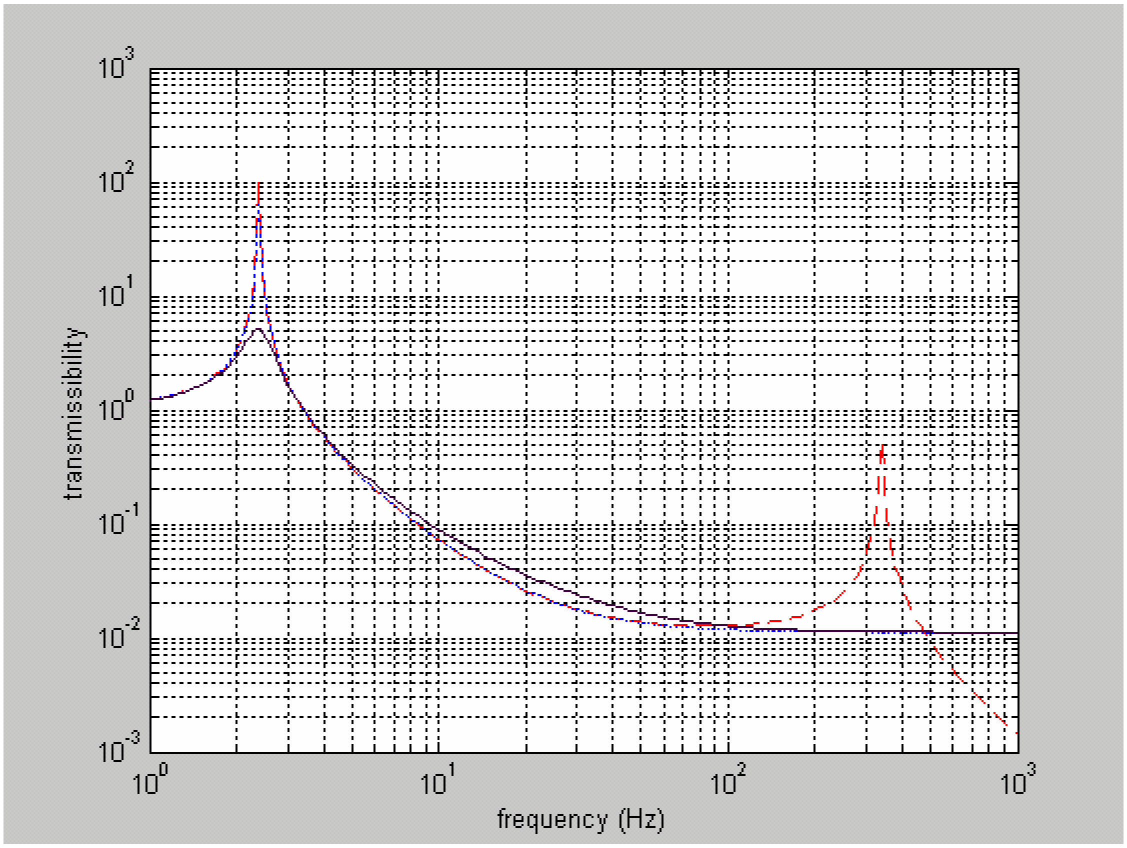


Figure 2: Vertical transmissibility of the top blade/wire/mass stage in the baseline Advanced LIGO quadruple suspension design. Red dashed line shows the original, undamped case as in figure 1. Blue dotted line shows the same system with an infinitely stiff wire. Black solid line is damped version of the latter, with a Q value of ~5.

3. Conclusions

From the arguments presented here, it appears that it should not be necessary to passively damp the internal modes of the blades. However one feature which this analysis reminds us of is that there is another family of peaks corresponding to the upper modes of the blade/wire/mass systems. And we have also not included the violin modes in this analysis. An undesirable situation could arise where there is overlap of mode frequencies – e.g. of an internal blade resonance and a wire stretching resonance. A check should be made on such a potential overlap once the baseline design parameters are more firmly chosen, and before committing to a final design.

It should be noted that in the analysis presented here several parameter values have been used which could take different values in a future design.

1) The so-called "shape factor", α, which is a geometric factor dependent on blade shape, and is used in the calculation of spring constant, has been taken to equal 1.38 here (α=1 for rectangle, α=1.5 for triangle). This value was experimentally found to

fit measurements of spring constants for the blades designed for GEO 600. Measurements on blades more recently acquired have suggested that a value of around 1.56 fits the new experimental data better. Further investigation of blade design using finite element analysis will help to more fully explore their behaviour under load and should lead to a better estimation of this factor. For the purposes of this document we have chosen to use the value which leads to a larger spring constant, and hence higher uncoupled frequencies, which gives a more conservative estimate of the isolation.

2) The blade design has been carried out assuming that the maximum stress in the blades should not exceed ~800 MPA, a value which is ~ half of the yield strength. Again this may be conservative, and one could push up the stress to say ~ 2/3 of the yield strength, and still be within the linear stress/strain region. By doing this, one could push up the internal mode frequencies of the blades, by a factor which for the first internal mode is equal to the ratio of the stresses, assuming the spring constant is not changed. An example is given in the Appendix.

A summary of the key blade equations can be found in Torrie (1999).

It should be noted that if better high frequency vertical isolation were required, one method of doing this could be to modify a blade such that the mass is suspended effectively at the blade's centre of percussion. For example Husman analysed and experimentally investigated this effect for a GEO blade, by extending the tip and adding a suitably chosen small mass to the new tip to move the centre of percussion. In practice however, adding more mass will reduce the internal mode resonant frequency, and so it is not simple to predict whether such a modification gives any significant reduction in the peak motion associated with the resonance.

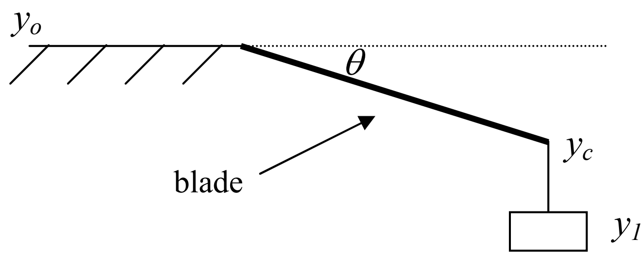
References

Husman, ME, 1999, PhD Thesis "Suspension and Control for Interferometric Gravitational Wave Detectors" (University of Glasgow)

Torrie, CIE, 1999, PhD Thesis "Development of Suspensions for the GEO 600 Gravitational Wave Detector" (University of Glasgow)

Appendix A: Summary of Equations and Parameters Used

A 1 Model Used by Husman (1999)



(looking from the side)

The variables y_o, y_c and y_1 measure the vertical displacement of the base to which the blade is rigidly attached, the tip of the blade, and the suspended mass respectively. Note that the suspended mass is assumed point-like in this analysis.

The transfer function Y_1/Y_o is given by (c.f. eqn. 4.43 in Husman)

$$\frac{Y_1}{Y_0} = \frac{-k_w\left(\left(2m_c l_c^2 - 9I_{cant}\right)s^2 + 9k_c l_c^2\right)}{\left(9I_{cant}^2 m_1 + m_c l_c^2 m_1\right)s^4 + \left(9I_{cant}k_w + m_c l_c^2 k_w + 9\left(k_w + k_c\right)l_c^2 m_1\right)s^2 + 9k_c k_w l_c^2} \qquad \text{eqn. 1}$$

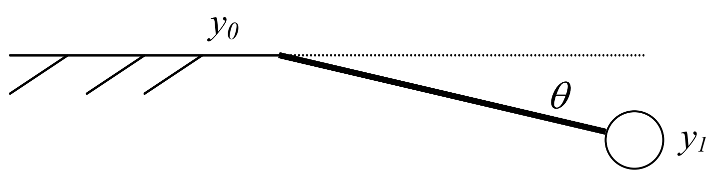
where k_w and k_c are the spring constants of the wire and the cantilever blade respectively, m_c and m_1 are the masses of the blade and suspended mass respectively, I_{cant} is the moment of inertia of the blade about a vertical axis through its centre and l_c is the length of the blade. For a triangular blade $I_{cant} = m_c l_c^2/18$.

A simplified model can be derived assuming $y_c = y_1$, which is equivalent to setting $k_w \to \infty$.
In that case, and for a triangular blade, the transfer function becomes

$$\frac{Y_1}{Y_0} = \frac{-\frac{1}{6}m_c l_c^2 s^2 + k_c l_c^2}{\left(\frac{1}{6}m_c l_c^2 + l_c^2 m_1\right)s^2 + k_c l_c^2} \qquad \text{eqn. 2}$$

A 2 Derivation of Simpler Model Including Damping.

The relationship given in eqn. 2 can be derived directly from the equation of motion of a blade clamped at one end with a (point) mass rigidly attached to its tip.



Consider the rotational equation for the motion of the blade

$$I_{tot,0}\ddot{\theta} = \sum \Gamma = -k_c(y_1 - y_0)l_c - (m_c + m_1)\ddot{y}_0 l_{cm} \qquad \text{eqn. 3}$$

Here $I_{tot,0}$ is the moment of inertia of the blade plus mass through a vertical axis at the wide end of the blade rigidly attached to the "ground".
The first term on the right hand side is the restoring torque due to the blade when its tip is deflected by θ, where $\theta = \left(\frac{y_1 - y_0}{l_c}\right)$. The second term is the torque introduced by the acceleration of the ground, $\ddot{y}_0$, where l_{cm} is the position of the centre of mass of the blade/mass assembly, measured from the wide end of the blade.
$I_{tot,0}$ is given by $I_{blade,0} + I_{mass,0}$.
Using the parallel axis theorem, the first term = $I_{cant} + m_c(l_c/3)^2 = m_c l_c^2/6$, where we have assumed a triangular blade, for which the centre of mass of the blade is at $l_c/3$ from the base. The second term is $m_1 l_c^2$.
It can be shown that l_{cm} is given by $l_{cm} = l_c \frac{(m_c + 3m_1)}{3(m_c + m_1)}$.

Combining all of the above, and using Laplace transforms, we can derive eqn 2.

To add damping, we include a factor in eqn. 3 on the right hand side of the form $b\dot{\theta}$. The resulting transfer function is

$$\frac{Y_1}{Y_0} = \frac{-\frac{1}{6}m_c l_c^2 s^2 + bs + k_c l_c^2}{\left(\frac{1}{6}m_c l_c^2 + l_c^2 m_1\right)s^2 + bs + k_c l_c^2} \qquad \text{eqn. 4}$$

The value of b can be suitably chosen to give a resonance with a certain Q value.

A 3 Parameters Used in Figures 1 and 2.

Blade 1 (top) kc=4.19*10^3;
kw=6.14*10^5;
mc=1.24;
lc=0.5;
m1=18.2;
for damping b=(m1*lc*lc/Q)*((kc/m1)^0.5)
and Q=5

Blade 2 kc=4.56*10^3;
kw=2.2*10^6;
mc=0.81;
lc=0.48;
m1=18.2;

Blade 3 kc=4.47*10^3;
kw=1.24*10^6;
mc=0.54;
lc=0.4;
m1=36.2;

A 4 Example of Two Blade Designs: same k_c, different first internal mode frequency

Blade 1 (as above)
spring constant 4.19*10^3 N/m
length = 0.5 m, base = 0.124 m, thickness = 0.005 m
max. stress = 880 MPa
first internal mode = 75 Hz

Blade 1 (alternative)
spring constant 4.19*10^3 N/m
length = 0.457 m, base = 0.095 m, thickness = 0.005 m
max. stress = 1050 MPa
first internal mode = 90 Hz
