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Detecting a Stochastic Background of Gravitational Radiation - Background Information

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1 Stochastic gravitational wave backgrounds

Here we briefly describe the standard optimally-filtered cross-correlation technique used to search for a stochastic background of gravitational radiation. Readers interested in more details should consult the original papers [1, 2, 3] or longer review articles (e.g., [4, 5, 6]) for a more indepth discussion.

1.1 Spectrum

A stochastic background of gravitational radiation is a *random* gravitational wave signal produced by a large number of weak, independent, unresolved gravitational wave sources. Its spectral properties are described by the dimensionless quantity

$$\Omega_{\rm gw}(f) := \frac{1}{\rho_{\rm critical}} \frac{d\rho_{\rm gw}}{d\,\ln\,f}\,, \tag{1}$$

which is the ratio of the energy density in gravitational waves contained in a bandwidth $\Delta f = f$ to the total energy density required (today) to close the universe:

$$\rho_{\rm critical} = \frac{3c^2 H_0^2}{8\pi G}\,. \tag{2}$$

H_0 is the Hubble expansion rate (today):

$$H_0 = h_{100}\cdot 100\,\frac{\rm km}{\rm sec\cdot Mpc} \approx 3.24\times 10^{-18}\,h_{100}\,\frac{1}{\rm sec}\,, \tag{3}$$

and h_{100} is a dimensionless factor, included to account for the different values of H_0 that are quoted in the literature.[1] Note that $\Omega_{\rm gw}(f)\,h_{100}^2$ is *independent* of the actual Hubble expansion rate, and for this reason we will often focus attention on this quantity, rather than $\Omega_{\rm gw}(f)$ alone. In addition, $\Omega_{\rm gw}(f)$ is related to the one-sided power spectral density $S_{\rm gw}(f)$ via[2]

$$S_{\rm gw}(f) = \frac{3H_0^2}{10\pi^2}\,f^{-3}\Omega_{\rm gw}(f)\,. \tag{4}$$

Thus, for a stochastic gravitational wave background with $\Omega_{\rm gw}(f) = {\rm const}$, the power in gravitational waves falls off like $1/f^3$.

1.2 Statistical assumptions

The spectrum $\Omega_{\rm gw}(f)$ completely specifies the statistical properties of a stochastic background of gravitational radiation provided we make enough additional assumptions. Here, we assume that the stochastic background is: (i) isotropic, (ii) unpolarized, (iii) stationary, and (iv) Gaussian. Anisotropic or non-Gaussian backgrounds (e.g., due to an incoherent superposition of gravitational waves from a large number of unresolved white dwarf binary star systems in our own galaxy, or a “pop-corn” stochastic signal produced by

[1] h_{100} almost certainly lies within the range $1/2 < h_{100} < 1$.

[2] $S_{\rm gw}(f)$ is defined by $\frac{1}{T}\int_0^T |h(t)|^2\,dt = \int_0^\infty S_{\rm gw}(f)\,df$, where $h(t)$ is the gravitational wave strain in a single detector due to the stochastic background signal.

gravitational waves from supernova explosions [7, 8, 9]) will require different data analysis techniques than the one we present here. (See, e.g. [10, 11] for a detailed discussion of these different techniques.)

In addition, we will assume that the intrinsic detector noise is: (i) stationary, (ii) Gaussian, (iii) uncorrelated between different detectors and with the stochastic gravitational wave signal, and (iv) much greater in power than the stochastic gravitational wave background.

1.3 Cross-correlation statistic

The standard method of detecting a stochastic gravitational wave signal is to *cross-correlate* the output of two gravitational wave detectors [1, 2, 3, 4, 5, 6]:

$$Y_Q = \int_0^T dt_1 \int_0^T dt_2 \, h_1(t_1) \, Q(t_1 - t_2) \, h_2(t_2) \tag{5}$$

$$= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \, \delta_T(f - f') \, \tilde{h}_1^*(f) \, \tilde{Q}(f') \, \tilde{h}_2(f') \,, \tag{6}$$

where T is the observation time and $\delta_T(f - f')$ is a finite-time approximation to the Dirac delta function $\delta(f - f')$.[3] Assuming that the detector noise is uncorrelated between the detectors, it follows that the expected value of Y_Q depends only on the cross-correlated stochastic signal:

$$\mu = \frac{T}{2} \int_{-\infty}^{\infty} df \, \gamma(|f|) S_{\rm gw}(|f|) \, \tilde{Q}(f) \,, \tag{7}$$

while the variance of Y_Q is dominated by the noise in the individual detectors:

$$\sigma^2 \approx \frac{T}{4} \int_{-\infty}^{\infty} df \, P_1(|f|) \, |\tilde{Q}(f)|^2 \, P_2(|f|) \,. \tag{8}$$

($P_1(|f|)$ and $P_2(|f|)$ are again one-sided power spectral densities.) The integrand of Eq. (7) contains a factor $\gamma(f)$, called the *overlap reduction function* [3], which characterizes the reduction in sensitivity to detecting a stochastic background due to: (i) the separation time delay, and (ii) the relative orientation of the two detectors. (For coincident and coaligned detectors, $\gamma(f) = 1$ for all frequencies.) Plots of the overlap reduction function for correlations between LIGO Livingston and the other major interferometers and ALLEGRO are shown in Fig. 1.

1.4 Optimal filter

Given Eqs. (7) and (8), it is relatively straightforward to show that the SNR ($= \mu/\sigma$) is maximized when

$$\tilde{Q}(f) = \lambda \frac{\gamma(|f|) S_{\rm gw}(|f|)}{P_1(|f|) P_2(|f|)} \propto \frac{\gamma(|f|) \Omega_{\rm gw}(|f|)}{|f|^3 P_1(|f|) P_2(|f|)} \,, \tag{9}$$

where λ is a (real) overall normalization constant. Such a $\tilde{Q}(f)$ is called the *optimal filter* for the cross-correlation statistic. For such a $\tilde{Q}(f)$, the expected SNR is

$$\mathrm{SNR} \approx \frac{3H_0^2}{10\pi^2} \sqrt{T} \left[\int_{-\infty}^{\infty} df \, \frac{\gamma^2(|f|) \Omega_{\rm gw}^2(|f|)}{f^6 P_1(|f|) P_2(|f|)} \right]^{1/2} \,, \tag{10}$$

which grows like the square-root of the observation time T.

[3] $\delta_T(f) := \int_{-T/2}^{T/2} dt \, e^{-i2\pi f t} = \sin(\pi f T)/\pi f$.

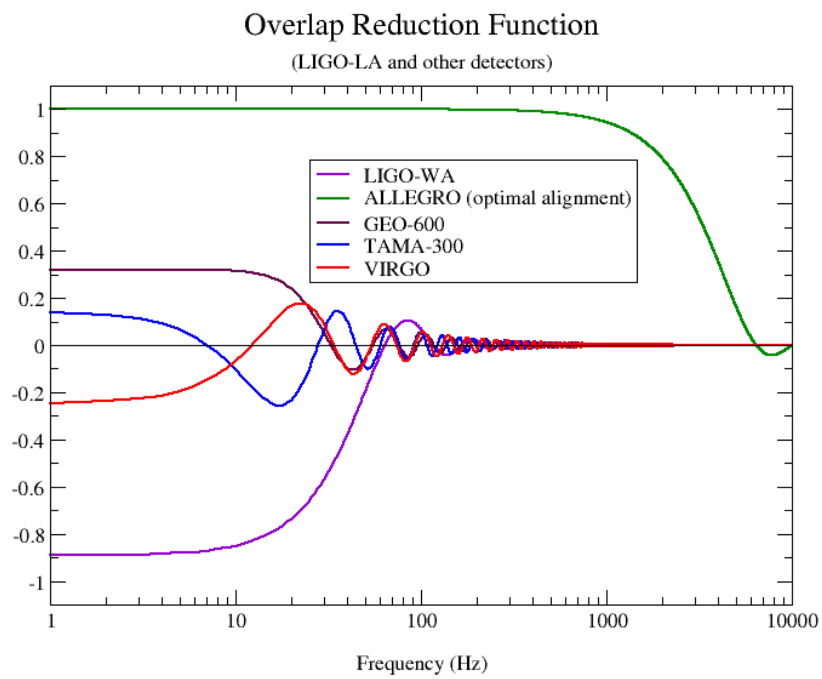

Figure 1: Overlap reduction function between LIGO Livingston and the other major interferometers plus ALLEGRO (in an optimal alignment of 72° East of North).

1.5 Time-shifted data

If the time-series data $h_1(t)$ and $h_2(t)$ are shifted in time relative to one another, the cross-correlation statistic Y_Q will depend on this shift according to:

$$Y_Q(\tau) = \int_0^T dt_1 \int_0^T dt_2\, h_1(t_1+\tau)\, Q(t_1-t_2)\, h_2(t_2)\ . \tag{11}$$

Making a change of variables $\bar{t}_1 = t_1 + \tau$, we have

$$\begin{aligned} Y_Q(\tau) &= \int_0^T d\bar{t}_1 \int_0^T dt_2\, h_1(\bar{t}_1)\, Q[(\bar{t}_1 - t_2) - \tau]\, h_2(t_2) && (12)\\ &= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\ \delta_T(f-f')\, \tilde{h}_1^*(f)\, \tilde{Q}(f')\, e^{i2\pi f'\tau}\, \tilde{h}_2(f')\ , && (13)\end{aligned}$$

which is the same as Eq. (6) with $\tilde{Q}(f)$ replaced by $\tilde{Q}(f)\, e^{i2\pi f\tau}$. The expected value is thus (c.f. Eq. (7))

$$\mu(\tau) = \frac{T}{2}\int_{-\infty}^{\infty} df\, \gamma(|f|) S_{\rm gw}(|f|)\, \tilde{Q}(f)\, e^{i2\pi f\tau}\ , \tag{14}$$

which is simply the inverse Fourier transform of

$$\tilde{\mu}(f) := \frac{T}{2}\gamma(|f|) S_{\rm gw}(|f|)\, \tilde{Q}(f) = T\,\frac{3H_0^2}{20\pi^2}\,\gamma(|f|)\,|f|^{-3}\Omega_{\rm gw}(|f|)\,\tilde{Q}(f)\ . \tag{15}$$

This is a useful result since $\mu(\tau)$ tells us how the mean value of the cross-correlation statistic changes with time lag.

1.6 Observational constraints

(i) The strongest observational constraint on $\Omega_{\rm gw}(f)$ comes from the high degree of isotropy observed in the CMBR. The one-year[12, 13], two-year[14], and four-year[15] data sets from the Cosmic Background Explorer (COBE) satellite place very strong restrictions on $\Omega_{\rm gw}(f)$ at very low frequencies:

$$\Omega_{\rm gw}(f)\ h_{100}^2 \le 7\times 10^{-11}\left(\frac{H_0}{f}\right)^2 \quad \text{for} \quad H_0 < f < 30 H_0\ . \tag{16}$$

Since $H_0 \approx 3.24\times 10^{-18}\ h_{100}$ Hz, this limit applies only over a narrow band of frequencies (10^{-18} Hz $< f < 10^{-16}$ Hz), which is far below any frequency band accessible to investigation by either earth-based (10 Hz $\lesssim f \lesssim 10^3$ Hz) or space-based (10^{-4} Hz $\lesssim f \lesssim 10^{-1}$ Hz) detectors.

(ii) Another observational constraint comes from roughly a decade of monitoring the radio pulses arriving from a number of stable millisecond pulsars[16]. These pulsars are remarkably stable clocks, and the regularity of their pulses places tight constraints on $\Omega_{\rm gw}(f)$ at frequencies on the order of the inverse of the observation time of the pulsars ($\sim 10^{-8}$ Hz):

$$\Omega_{\rm gw}(f = 10^{-8}\ \text{Hz})\ h_{100}^2 \le 10^{-8}\ . \tag{17}$$

Like the constraint on the stochastic gravitational wave background from the isotropy of the CMBR, the millisecond pulsar timing constraint is irrelevant for earth-based and space-based detectors.

(iii) The third and final observational constraint on $\Omega_{\rm gw}(f)$ comes from the standard model of big-bang nucleosynthesis[17]. This model provides remarkably accurate fits to the observed abundances of the light elements in the universe, tightly constraining a number of key cosmological parameters. One of the parameters constrained in this way is the expansion rate of the universe at the time of nucleosynthesis. This places a constraint on the energy density of the universe at that time, which in turn constrains the energy density in a cosmological background of gravitational radiation:

$$\int_{f>10^{-8}\ {\rm Hz}} d\ln f\ \Omega_{\rm gw}(f)\ h_{100}^2 \le 10^{-5}\ . \tag{18}$$

This constraint corresponds to a 95% confidence upper bound on $\Omega_{\rm gw}(f)$ of roughly 10^{-7} in the frequency band of earth-based interferometers.

1.7 Upper-limits

In addition to the above observational constraints, there are a couple of (much weaker) upper-limits on $\Omega_{\rm gw}(f)$ that have been set directly using gravitational wave data: (i) An upper-limit from a correlation measurement between the Garching and Glasgow prototype interferometers[18]:

$$\Omega_{\rm gw}(f)\, h_{100}^2 \le 3\times 10^5 \quad {\rm for} \quad 100 < f < 1000\ {\rm Hz}\ , \tag{19}$$

(ii) An upper-limit from data taken by a single resonant bar detector[19]:

$$\Omega_{\rm gw}(f=907\ {\rm Hz})\, h_{100}^2 \le 100\ . \tag{20}$$

(iii) An upper-limit from a correlation measurement between the EXPLORER and NAUTILUS resonant bar detectors[20, 21]:

$$\Omega_{\rm gw}(f=907\ {\rm Hz})\, h_{100}^2 \le 60\ . \tag{21}$$

Note that these last two upper-limits are for $\Omega_{\rm gw}(f)$ evaluated at a *single* frequency ($f = 907$ Hz), which is near the resonant frequency of the bar detectors.

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