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Comments on inspiral upper limit determination		
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Abstract

The traditional "most luminous event" analysis sets a bound on the inspiral event rate by making the assumption that the highest signal-to-noise event is of gravitational wave origin. Since this is a "worst case" scenario, the bound obtained is a "conservative" one. An alternative approach, entirely within the spirit of this analysis approach, is to assume that all foreground events, regardless of their number, are no more luminous than the most luminous observed event. The distinction allows that the most luminous event may be an accidental, which is inarguably correct. The resulting upper limit on the inspiral event rate is 40% tighter at 90% confidence. Neither analysis makes full use of the data and both are quite sensitive to statistical fluctuations.

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1 Upper limit determination

The "most luminous event" analysis (cf. [1]) sets a bound on the event rate by making the assumption that the most luminous (highest signal-to-noise event) event is of gravitational wave origin. Since this is a "worst case" scenario, the bound obtained is a "conservative" one.

An alternative approach, entirely within the spirit of this analysis approach, is to assume that all foreground events, regardless of their number, are no more luminous than the most luminous observed event. The distinction allows that the most luminous event may be an accidental, which is inarguably correct. While it may appear a subtle difference, this change in the interpretation of the observation of N events leads to a substantially tighter bound on the event rate then the "traditional" most luminous event approach. This note derives the bound under the traditional assumptions, and under the assumption just described.

2 The statistics of the most luminous event

2.1 Nomenclature

Let $\mathcal{I}$ denote a population model for the distribution of binary inspiral events, and $\mathcal{J}$ denote the analysis pipeline that identifies events (including thresholds, etc). From these one can determine the distribution of detected events of binary inspiral gravitational wave origin with signal-to-noise ρ (and chirp mass $\mathcal{M}$, etc). Hereafter refer to these as foreground events and write the distribution as $P_F(\rho, \mathcal{M}|\mathcal{I}\mathcal{J})$.

Denote the binary inspiral event rate by $\dot{n}_S$. Given the pipeline $\mathcal{J}$ and source distribution model $\mathcal{I}$ only some fraction of binary inspiral events will be observed. Denote this fraction $\epsilon(\mathcal{I}\mathcal{J})$. The rate of foreground events, $\dot{n}_F$ is related to the rate of binary inspiral events by $\epsilon(\mathcal{I}\mathcal{J})$:

$$\dot{n}_F = \epsilon(\mathcal{I}\mathcal{J})\dot{n}_S. \tag{1}$$

From the foreground event distribution P_F we can calculate the probability that a foreground event has signal-to-noise ρ_F less than or equal to some ρ_0. Denote this quantity $C_F(\rho_0|\mathcal{I}\mathcal{J})$:

$$C_F(\rho_0|\mathcal{I}\mathcal{J}) = \int_0^{\rho_0} d\rho \int_0^{\infty} d\mathcal{M}\, P_F(\rho, \mathcal{M}|\mathcal{I}\mathcal{J}), \tag{2}$$

Assume that gravitational wave events are Poisson distributed in time so that probability of N foreground events in an observation of duration T is

$$P(N|T\dot{n}_F) = \frac{(T\dot{n}_F)^N}{N!} e^{-T\dot{n}_F}. \tag{3}$$

The probability that all foreground events observed in an interval of duration T, regardless of their number, have signal-to-noise ρ_F less than or equal to ρ_0 is thus

$$\begin{aligned} P(\rho_F \leq \rho_0|T\dot{n}_F, \mathcal{I}\mathcal{J}) &= \sum_{n=0}^{\infty} P(N|T\dot{n}_F) C_F(\rho_0|\mathcal{I}\mathcal{J})^n & (4) \\ &= e^{-T\dot{n}_F[1-C_F(\rho_0|\mathcal{I}\mathcal{J})]}. & (5) \end{aligned}$$

(The absence of any foreground events of course means that no foreground events have signal-to-noise greater than ρ_0.) For convenience denote the combination $[1 - C_F(\rho_0|\mathcal{I}\mathcal{J})]$ by $\varepsilon(\rho_0|\mathcal{I}\mathcal{J})$:

$$\varepsilon(\rho_0|\mathcal{I}\mathcal{J}) = [1 - C_F(\rho_0|\mathcal{I}\mathcal{J})]. \tag{6}$$

2.2 Traditional most-luminous-event upper limit

Associated with probability $P(\rho_F \leq \rho_0|T\dot{n}_F, \mathcal{I}\mathcal{J})$ that all observed foreground events, regardless of number, have signal less than ρ_0 is the probability density that the most luminous event actually has signal-to-noise ρ_0:

$$\begin{aligned} p_F(\rho_0|T, \dot{n}_F, \mathcal{I}\mathcal{J}) &= \frac{d}{d\rho_0} e^{-T\dot{n}_F \varepsilon(\rho_0|\mathcal{I}\mathcal{J})} & (7) \\ &= T\dot{n}_F P_F(\rho_0|\mathcal{I}\mathcal{J}) e^{-T\dot{n}_F \varepsilon(\rho_0|\mathcal{I}\mathcal{J})}. & (8) \end{aligned}$$

This is thus likelihood for the observation that the most luminous foreground event has signal-to-noise ρ_0. In the most luminous event analysis one assumes that the most luminous event is a foreground event and then, from this likelihood, uses a Bayesian analysis with a uniform prior in $\dot{n}_F$ to determine a credible set, bounded below by $\dot{n}_F = 0$, associated with a probabiltiy p. The upper end of this credible set is taken to be the upper limit on $\dot{n}_F$ and, through equation 1, on $\dot{n}_S$. In particular, the posterior on $\dot{n}_F$ is

$$\begin{aligned} p(\dot{n}_F|\rho_0\mathcal{I}\mathcal{J}) &\propto p(\dot{n}_F)p_F(\rho_0|T\dot{n}_F,\mathcal{I}\mathcal{J}) & (9)\\ &= p_F(\rho_0|T\dot{n}_F,\mathcal{I}\mathcal{J}) & (10)\\ p(\dot{n}_F|\rho_0\mathcal{I}\mathcal{J}) &= \dot{n}_F T^2\varepsilon(\rho_0|\mathcal{I}\mathcal{J})^2 e^{-T\dot{n}_F\varepsilon(\rho_0|\mathcal{I}\mathcal{J})}. & (11)\end{aligned}$$

The simply-connected credible set of $\dot{n}_F$ containing a total probability p and bounded below by $\dot{n}_F = 0$ is bounded above by $\dot{n}_{F,UL}$ given by

$$\begin{aligned} p &= \int_0^{\dot{n}_{F,UL}} d\dot{n}_F p(\dot{n}_F|\rho_0\mathcal{I}\mathcal{J}) & (12)\\ &= 1 - e^{-T\dot{n}_{F,UL}\varepsilon(\rho_0|\mathcal{I}\mathcal{J})}\left[1 + T\dot{n}_{F,UL}\varepsilon(\rho_0|\mathcal{I}\mathcal{J})\right]. & (13)\end{aligned}$$

When p is 90% we find

$$\dot{n}_{F,UL} = \frac{3.890}{T\varepsilon(\rho_0|\mathcal{I}\mathcal{J})}. \tag{14}$$

2.3 An alternative approach

It is, of course, a stretch to assume that the most luminous event is a foreground event. One could take a different approach, within the spirit of the most luminous event analysis, and stay with the certainly true assumption that all foreground events, regardless of their number, have signal-to-noise less than ρ_0. The likelihood for this observation is equation 5. Using the same uniform prior on $\dot{n}_F$ the Bayesian posterior on $\dot{n}_F$ is

$$p(\dot{n}_F|\rho_F \leq \rho_0, \mathcal{I}\mathcal{J}) = T\varepsilon(\rho_0|\mathcal{I}\mathcal{J})e^{-T\dot{n}_F\varepsilon(\rho_0|\mathcal{I}\mathcal{J})}. \tag{15}$$

The corresponding probability p bound on $\dot{n}_F$, which is equivalent to a bound on $\dot{n}_S$ through equation 1, is

$$\dot{n}_{F,UL} = -\frac{\log(1-p)}{T\varepsilon(\rho_0|\mathcal{I}\mathcal{J})} \tag{16}$$

For p equal to 90% this is

$$\dot{n}_{F,UL} = \frac{2.302}{T\varepsilon(\rho_0|\mathcal{I}\mathcal{J})}. \tag{17}$$

This bound is nearly 40% tighter than the traditional most luminous event bound and is equally, if not more, defensible in its assumptions.

3 Discussion

If a set of event observations are to be conflated into the single characteristic, the maximum signal-to-noise of the set, then a the “traditional” most-luminous-event analysis is overly “conservative” in interpreting the event as an astrophysical event. A more defensible interpretation is that all inspiral events part of that set, regardless of number, are no more luminous then the most luminous observed event. This interpretation allows, but does not require, that the observed event is background. It makes no assumptions about the background, beyond that there may be a background (i.e., it is agnostic on whether a background exists). Under this interpretation the bound on the inspiral rate is more than 40% tighter than under the traditional analysis.

Neither of these event analysis can or should be considered definitive, or analyses of choice, however. Better analyses will use the information available in the distribution of events in signal-to-noise and other parameters (e.g., chirp mass) to separate a foreground from a background. In doing so they will improve upon the limits set by analyses like those described here (that conflate the observations to a single characteristic) by allowing us to be more sensitive to distant events, or distinguish between foreground and background events. A discussion of such analyses can be found in [2, 3]

4 References

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