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# Is the "Conservative" frequentist upper limit always worse than the "Optimistic" Bayesian upper limit ? No–A Simple Counterexample

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# 1 Model

Take a one dimensional Gaussian distribution for $X \geq 0$ with a parameter $A \geq 0$.

$$P(X|A) = N \exp^{-(X-A)^2} \tag{1}$$

$$N^{-1} = \frac{\sqrt{\pi}}{2}(1 + \mathrm{Erf}(A)) \tag{2}$$

Suppose a relation $A = HD$ where $D$ is not known but we know $D$ is bounded: $0 < W \leq D \leq B < \infty$. $W$ (worst) and $B$ (best) denote arbitrary finite real constants. (In this model, A is akin to gravitational-wave strain at the detector, projected onto the detector's antenna pattern, while H is akin to the source's intrinsic strength $h_0$, and D is the uncertain, angle-dependent factor that relates them.) Now suppose we want to set an upper limit on $H$, given a data $x$.

Given a significance level $p$ (and thus the confidence level 1-p), the frequentist upper limit on $A$, which we denote by $\hat{A}_f = \hat{A}_f(x,p)$, is given implicitly by

$$p = \int_0^x P(X|\hat{A}_f)dX = (\mathrm{Erf}(\hat{A}_f) - \mathrm{Erf}(\hat{A}_f - x))/(1 + \mathrm{Erf}(\hat{A}_f)). \tag{3}$$

The conservative frequentist upper limit on $H$, which we call $\hat{H}_{fw}$, is then $\hat{H}_{fw} = \hat{A}_f/W$.

Given the uniform priori probability $P(A)$ =const ($0 \leq A < \infty$) the Bayesian upper limit on $A$ denoted by $\hat{A}_b = \hat{A}_b(x,p)$ is given by[1]

$$\begin{aligned}
1 - p &= \int_0^{\hat{A}_b} P(A|x)\, dA = N' \int_0^{\hat{A}_b} P(x|A)P(A)\, dA = \\
&= N'' \int_0^{\hat{A}_b} P(x|A)\, dA = N''N \int_0^{\hat{A}_b} \exp(-(x-A)^2)\, dA = \\
&= (\mathrm{Erf}(\hat{A}_b - x) + \mathrm{Erf}(x))/(1 + \mathrm{Erf}(x)).
\end{aligned} \tag{4}$$

where we used

$$(N''N)^{-1} = \frac{\sqrt{\pi}}{2}(1 + \mathrm{Erf}(x)). \tag{5}$$

The optimistic Bayesian upper limit on $H$, which we call $\hat{H}_{bo}$, is then just given by $\hat{H}_{bo} = \hat{A}_b/B$.

# 2 Two examples

Suppose (W,B) = (0.5,1) and we take $p = 0.05$.

[1] This is not quite the procedure we followed in the Bayesian pulsar analysis. There we assumed the uniform priori probabilities for $h_0 > 0$, for the cosine of the inclination angle $\cos\iota$ that is the angle between the angular momentum of the pulsar and the line of the sight, the gravitational-wave polarization angle $\psi$, and the gravitational-wave initial phase $\Phi_0$. Then we marginalised the probability with respect to $\cos\iota, \psi$, and $\Phi_0$ to set an upper limit on $h_0$. Analogously, in this toy model, one may assume $P(H)$ =const. ($0 \leq H < \infty$), $P(D)$ =const. ($W \leq D \leq B$), and then marginalise with respect to $D$ to obtain the marginalised probability for $H$. Here, for simplicity, we do not marginalise the probability, but simply set D=B to yield the most optimistic limit as stated below by Eq. (5).

case 1.
Suppose x = 1.4, then $(\hat{A}_f, \hat{A}_b) = (2.6, 2.6)$ and correspondingly, $(\hat{H}_{fw}, \hat{H}_{bo}) = (5.2, 2.6)$.

case 2.
Suppose x = 0.1, then $(\hat{A}_f, \hat{A}_b) = (0.62, 1.5)$ and correspondingly, $(\hat{H}_{fw}, \hat{H}_{bo}) = (1.24, 1.5)$.

Obviously, $\hat{H}_{fw} < \hat{H}_{bo}$ in case 2. Note that the ratio $\hat{H}_{fw}/\hat{H}_{bo}$ depends on the data, not just on the methods.

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