Combining the LIGO H1-H2, H1-L1, and H2-L1 stochastic background measurements optimally during triple coincident operations

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This note derives an *efficient* estimator for the pseudo-detector strain for the Hanford Observatory pair of detectors by considering the possibility of the presence of instrumental correlations between two machines co-located at one site. An expression is given for the effective power spectral density of combined noise in the pseudo-detector. This is then introduced into the standard optimal Wiener filter used to cross-correlated detector data streams in order to obtain an estimate of the stochastic gravitational wave background.

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I. INTRODUCTION

The two LIGO interferometers at Hanford are known to exhibit instrumental cross-correlations arising from a number of sources:

- Low-frequency seismicity
- Common-mode acoustic coupling among the input electro-optics systems
- Electromagnetic susceptibilities the are manifested by the presence of 60 Hz mains lines in the spectra and cross-spectra.

This note derives an *efficient* estimator for the pseudo-detector strain for the Hanford Observatory pair of detectors by considering the possibility of the presence of instrumental correlations between two machines colocated at one site. An expression is given for the effective power spectral density of combined noise in the pseudo-detector. This is then introduced into the standard optimal Wiener filter used to cross-correlated detector data streams in order to obtain an estimate of the stochastic gravitational wave background.

II. THE S1 ANALYSIS

For the S1 analysis of the stochastic gravitational wave background, the final results showed that there was substantial cross-correlated noise between the two (4km and 2km) Hanford inteferometers. This observation led us to disregard these results. In addition, two separate upper limits were obtained for the two transcontinental pairs, H1-L1 and H2-L1. These were not combined because of the known common cross-correlation contaminating the H1-H2 pair.

It is possible to take into account such local instrumental correlations by *first* combining the two local measurements into a single, *pseudo-detector* estimate of GW strain from the Hanford site, and then cross-correlating this pseudo-signal with the remaining Livingston signal.

In doing this, it is possible to obtain a self-consistent utilization of the three measurements to obtain a *single* estimate of Ω_{GW}. In order for this to be valid, the following reasonable assumptions are made:

- There are no broadband transcontinental correlations. This has been empirically observed to be the case for both the S1 and S2 science runs when the coherences between H1,2 and L1 are calculated over long periods of time.
- The local H1-H2 correlations are dominated by instrumental effects and not GW. The spectral magnitude of the H1-H2 coherence is greater than either of the H1,2-L1 pairs; moreover the frequency dependence of the coherence for H1-H2 is qualitatively different from the transcontinental pairs.

III. OPTIMAL ESTIMATE OF STRAIN FROM THE HANFORD INSTRUMENTS

The derivation of a *efficient* estimator of strain at Hanford is derived in Appendix A. The results are quoted here. Assume the two instruments produce data streams

$$s_{H1}(t) = h(t) + n_{H1}(t) \quad (3.1)$$
$$s_{H2}(t) = h(t) + n_{H2}(t) \quad (3.2)$$

The Fourier domain representations of these signals are[1]

$$\widetilde{s}_{H1}(f) = \widetilde{h}(f) + \widetilde{n}_{H1}(f) \quad (3.3)$$
$$\widetilde{s}_{H2}(f) = \widetilde{h}(f) + \widetilde{n}_{H2}(f) \quad (3.4)$$

[1] $\widetilde{a}(f)$ denotes the Fourier transforms of $a(t)$—i.e., $\widetilde{a}(f) \equiv \int_{-\infty}^{\infty} dt\, e^{-i2\pi f t}\, a(t)$.

The cross-correlation between the two Hanford machines is characterized by the coherence function:

$$\rho_{H_1H_2}(f) := \frac{P_{H_1H_2}(f)}{\sqrt{P_{H_1}(f)P_{H_2}(f)}} \quad (3.5)$$

$$\Gamma_{H_1H_2}(f) = |\rho_{H_1H_2}(f)|^2 = \frac{|P_{H_1H_2}(f)|^2}{P_{H_1}(f)P_{H_2}(f)} \quad (3.6)$$

$\rho_{H_1H_2}(f)$ is inherently a complex quantity contained within the unit circle.

Assume we form an *unbiased* linear combination of the s_i: $\widetilde{s}_H(f) = \widetilde{\alpha}(f)\widetilde{s}_{H_1}(f) + (1-\widetilde{\alpha}(f))\widetilde{s}_{H_2}(f)$. If $s_H(f)$ is also to be a minimum variance estimator, then $\widetilde{\alpha}(f)$ takes the following value:

$$\widetilde{\alpha}(f) = \frac{P_{H_2}(f) - \rho^*_{H_1H_2}(f)\sqrt{P_{H_1}(f)P_{H_2}(f)}}{P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}} \quad (3.7)$$

The corresponding power of the pseudo-signal is,

$$P_H(f) = \frac{P_{H_1}(f)P_{H_2}(f)(1-\Gamma_{H_1H_2}(f))}{P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}} \quad (3.8)$$

IV. THE CROSS-CORRELATION STATISTIC USING A COMPOSITE PSEUDO DETECTOR STRAIN FOR HANFORD

Since the instrumental transcontinental cross-correlations are assumed to be negligible, the derivation of the optimal filter when using the pseudo-detector signal for Hanford proceeds exactly as has been presented in the literature [1, 2, 3] with $P_{H_1}(f), P_{H_2}(f) \rightarrow P_H(f)$. The cross-correlation statistic is given by,

$$Y \equiv \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\, s_{L_1}(t_1)\, Q(t_1 - t_2)\, s_H(t_2)\,, \quad (4.1)$$

The frequency domain expression is,

$$Y \approx \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\ \delta_T(f-f')\, \tilde{s}^*_{L_1}(f)\, \widetilde{Q}(f')\, \tilde{s}_H(f')\,, \quad (4.2)$$

The optimal filter becomes,

$$\widetilde{Q}(f) \propto \frac{\gamma(|f|)S_{\rm gw}(|f|)}{P_{L_1}(|f|)P_H(|f|)}\,. \quad (4.3)$$

As before, if we specialize to the case $\Omega_{\rm gw}({\rm f}) \equiv \Omega_0 =$ const. Then,

$$\widetilde{Q}(f) = \mathcal{N}\frac{\gamma(|f|)}{|f|^3 P_{L_1}(|f|)P_H(|f|)}\,, \quad (4.4)$$

where $\mathcal{N}$ is a (real) overall normalization constant. In practice we choose $\mathcal{N}$ so that the expected cross-correlation is $\mu_Y = \Omega_0\, h_{100}^2\, T$. For such a choice,

$$\mathcal{N} = \frac{20\pi^2}{3H_{100}^2}\left[\int_{-\infty}^{\infty} df\, \frac{\gamma^2(|f|)}{f^6 P_{L_1}(|f|)P_H(|f|)}\right]^{-1}, \quad (4.5)$$

(4.6)

A. Combining triple and double coincident measurements of Ω_{GW}

In order to make use of this methodology for the analysis of the S2 and S3 data, we will need to partition the data into three *non-overlapping* (hence statistically independent) data sets: the H1-H2-L1 triple coincident data, and the two H1-L1 and H2-L1 double coincident data sets. The triple coincidence data would be analyzed in the manner described in this note. Measurements from the three observations may be combined under the assumption of statistical independence.

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APPENDIX A: OPTIMALLY COMBINING SIGNALS FROM TWO (CO-LOCATED) INTERFEROMETERS WITH CORRELATED INSTRUMENTAL NOISE

This appendix derives the minimum variance unbiased estimator of local GW strain from two interferometer data streams co-located at one site (i.e., LIGO Hanford Observatory). It takes into account the possibility that the two measurements contain instrumentally correlated noise in addition to the GW signal.

Consider two interferometers, labeled by indices $1, 2$ (e.g., $1 \equiv H1; 2 \equiv H2$). Both see the same GW signal, h but have different noise floors, $n_{1,2}$:

$$
\begin{aligned}
s_1(t) &= h(t) + n_1(t) \qquad (A1)\\
s_2(t) &= h(t) + n_2(t) \qquad (A2)
\end{aligned}
$$

The Fourier domain representations of these signals are[2]

$$
\begin{aligned}
\widetilde{s}_1(f) &= \widetilde{h}(f) + \widetilde{n}_1(f) \qquad (A3)\\
\widetilde{s}_2(f) &= \widetilde{h}(f) + \widetilde{n}_2(f) \qquad (A4)
\end{aligned}
$$

Assume the process generating h,n_i to be stochastic with the following statistical properties of the signals and noise components[3]:

$$
\begin{aligned}
< \widetilde{n}_i(f) > = < \widetilde{h}(f) > &= 0 \qquad (A5)\\
< |\widetilde{n}_i(f)|^2 > &:= P_i(f) \qquad (A6)\\
< |\widetilde{h}(f)|^2 > &\ll < |\widetilde{n}_i(f)|^2 > \qquad (A7)\\
< \widetilde{n}_i(f)\ \widetilde{h}^*(f) > &= 0 \qquad (A8)\\
< \widetilde{n}_i(f)\ \widetilde{n}_j^*(f) > &:= P_{ij}(f)\\
&= \rho_{ij}(f)\sqrt{P_i(f)P_j(f)} \qquad (A9)\\
\Gamma_{ij}(f) &= |\rho_{ij}(f)|^2 \qquad (A10)
\end{aligned}
$$

$\Gamma_{ij}(f)$ is the coherence between the two signals. $\rho_{ij}(f)$ is a complex quantity of magnitude less than or equal to unity.

Assume we form a linear combination of the s_i: $\widetilde{s}'(f) = \widetilde{\alpha}(f)\widetilde{s}_1(f) + \widetilde{\beta}(f)\widetilde{s}_2(f)$. If s' is to be an *unbiased* estimator of h, then the following must be true:

$$
< \widetilde{h}(f)\ \widetilde{s}'(f) > = < \widetilde{h}(f)^2 > \rightarrow \widetilde{\alpha}(f) + \widetilde{\beta}(f) = 1 \qquad (A11)
$$

In order to determine $\widetilde{\alpha}(f)$, the other constraint that can be applied is to require the estimator s' to have a *minimum* variance:

$$
\begin{aligned}
Var(s') &:= P_{s'}(f) \qquad (A12)\\
P_{s'}(f) &= < |\widetilde{\alpha}(f)|^2(\widetilde{h}(f) + \widetilde{n}_1(f))^2 > +\\
&< |1 - \widetilde{\alpha}(f)|^2(\widetilde{h}(f) + \widetilde{n}_2(f))^2 > +\\
&< (\widetilde{\alpha}(f)(1 - \widetilde{\alpha}^*(f)) \times\\
&(\widetilde{h}(f) + \widetilde{n}_1(f))(\widetilde{h}^*(f) + \widetilde{n}_2^*(f)) > +\\
&< (\widetilde{\alpha}^*(f)(1 - \widetilde{\alpha}(f)) \times\\
&(\widetilde{h}^*(f) + \widetilde{n}_1^*(f))(\widetilde{h}(f) + \widetilde{n}_2(f)) > \qquad (A13)
\end{aligned}
$$

Where $Var(s') := P_{s'}(f)$, the noise power of the signals s'.

Ignoring the magnitude of h in favor of the noise terms and taking the correlations (and lack thereof) into account, the expression reduces to,

$$
\begin{aligned}
P_{s'}(f) &= |\widetilde{\alpha}(f)|^2 P_1(f) + |1 - \widetilde{\alpha}(f)|^2 P_2(f) +\\
&\Big(\widetilde{\alpha}(f)(1 - \widetilde{\alpha}^*(f))\rho_{12}(f) +\\
&\widetilde{\alpha}^*(f)(1 - \widetilde{\alpha}(f))\rho_{12}^*(f)\Big) \times\\
&\sqrt{P_1(f)P_2(f)} \qquad (A14)
\end{aligned}
$$

Minimizing $P_{s'}(f)$ leads to the following pair of equations:

$$
\begin{pmatrix} \frac{\partial P_{s'}(f)}{\partial \widetilde{\alpha}(f)} \\ \frac{\partial P_{s'}(f)}{\partial \widetilde{\alpha}^*(f)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (A15)
$$

[2] $\widetilde{a}(f)$ denotes the Fourier transforms of $a(t)$—i.e., $\widetilde{a}(f) \equiv \int_{-\infty}^{\infty} dt\, e^{-i2\pi f t}\, a(t)$.

[3] The brackets $< ... >$ denote ensemble or statistical averages of random processes

The resulting equations are complex conjugates of each other. One of them is:

$$\begin{aligned} 0 &= \widetilde{\alpha}(f)P_1(f) - (1-\widetilde{\alpha}(f))P_2(f) + \\ &\quad (1-\widetilde{\alpha}(f)(\rho_{12}(f)+\rho^*_{12}(f))) \times \\ &\quad \sqrt{P_1(f)P_2(f)} \end{aligned} \tag{A16}$$

$$\widetilde{\alpha}(f) = \frac{P_2(f) - \rho^*_{12}(f)\sqrt{P_1(f)P_2(f)}}{P_1(f)+P_2(f)-(\rho_{12}(f)+\rho^*_{12}(f))\sqrt{P_1(f)P_2(f)}} \tag{A17}$$

This expression for $\widetilde{\alpha}(f)$ results in an *efficient* estimator for $\widetilde{h}(f)$. Substituting for $\widetilde{\alpha}(f)$ in Eq. A14 , the noise power (variance) for s' becomes:

$$P_{s'}(f) = \frac{P_1(f)P_2(f)(1-\Gamma(f))}{P_1(f)+P_2(f)-(\rho_{12}(f)+\rho^*_{12}(f))\sqrt{P_1(f)P_2(f)}} \tag{A18}$$

Limiting cases:

I. If $\rho_{12}(f) \rightarrow 0$: Then $\widetilde{\alpha}(f)$ becomes,

$$\widetilde{\alpha}(f) \rightarrow \frac{P_2(f)}{P_1(f)+P_2(f)} \tag{A19}$$

IIa. If $P_1(f) \rightarrow P_2(f)$: Then $\widetilde{\alpha}(f)$ is independent of $P(f)$,

$$\widetilde{\alpha}(f) \rightarrow \frac{1-\rho^*_{12}(f)}{2-(\rho_{12}(f)+\rho^*_{12}(f))} \tag{A20}$$

IIb. If $\rho_{12}(f) \rightarrow 1$, then $\rho_{12}(f) = \rho^*_{12}(f) = \sqrt{\Gamma(f)}$ and $P_1(f) \rightarrow P_2(f)$. Then $P_{s'}(f) \rightarrow P_1(f)$:

$$P_{s'}(f) = \lim_{\Gamma(f)\rightarrow 1} \frac{P_1(f)}{2}\frac{1-\Gamma(f)}{1-\sqrt{\Gamma(f)}} = P_1(f) \tag{A21}$$

III. For H1 and H2 the limiting design performance will have $P_2(f) = 4P_1(f)$ due to the 1 : 2 arm length ratio,

$$\widetilde{\alpha}(f) \rightarrow \frac{2(2-\rho^*_{12}(f))}{5-2(\rho_{12}+\rho^*_{12})} \tag{A22}$$

If the noise were either completely correlated ($\rho_{12} \rightarrow 1, \widetilde{\alpha}(f) \rightarrow 2$) or anti-correlated ($\rho_{12} \rightarrow -1, \widetilde{\alpha}(f) \rightarrow \frac{2}{3}$), then it would be possible to exactly cancel the noise in the signals s_i.