Combining the LIGO H1-H2, H1-L1, and H2-L1 stochastic background measurements optimally during triple coincident operations

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This note derives an *efficient* estimator for the pseudo-detector strain for the Hanford Observatory pair of detectors by considering the possibility of the presence of instrumental correlations between two machines co-located at one site. An expression is given for the effective power spectral density of combined noise in the pseudo-detector. This is then introduced into the standard optimal Wiener filter used to cross-correlated detector data streams in order to obtain an estimate of the stochastic gravitational wave background. In addition. a *dual* to the efficient estimate of strain is derived. This dual is constructed to contain *no* gravitational wave signature and can thus be used as on "off-source" measurement to calibrate non-gravitational wave backgrounds in the "on-source" observation.

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I. INTRODUCTION

The two LIGO interferometers at Hanford are known to exhibit instrumental cross-correlations arising from a number of sources:

- Low-frequency seismicity
- Common-mode acoustic coupling among the input electro-optics systems
- Electromagnetic susceptibilities the are manifested by the presence of 60 Hz mains lines in the spectra and cross-spectra.

This note derives an *efficient* estimator for the pseudo-detector strain for the Hanford Observatory pair of detectors by considering the possibility of the presence of instrumental correlations between two machines co-located at one site. An expression is given for the effective power spectral density of combined noise in the pseudo-detector. This is then introduced into the standard optimal Wiener filter used to cross-correlated detector data streams in order to obtain an estimate of the stochastic gravitational wave background.

Once the efficient estimator is found, it is possible to subtract this quantity from the individual interferometer strain channels, producing a pair of *null* residual channels for the gravitational wave signature. The covariance matrix for these two null channels is Hermitian, therefore possesses two real eigenvalues and can be diagonalized by a unitary trasnformation (rotation). Because the covariance matrix is generated from a single vector, only one of the eigenvalues is nonzero. The corresponding eigenvector gives a single null channel that can be used as an "off-source" channel that can be prcoessed in the same manner as the efficient estimator of gravitational wave strain.

This technique is possible for the Hanford pair of detectors because to high accuracy the gravitational wave signature is guaranteed to be *identical* in both instruments. Coherent, time-domain mixing of the two interferometer strain channels can thus be used to optimal advantage by (i) providing the best possible estimate of gravitational wave strain and (ii) providing a null channel with which any gravitational wave analysis can be calibrated for backgrounds.

While the focus of this note is the search for stochastic gravitational waves, it appears to be the case that *any* analysis can exploit this approach.

II. THE S1 ANALYSIS

For the S1 analysis of the stochastic gravitational wave background, the final results showed that there was substantial cross-correlated noise between the two (4 km and 2 km) Hanford interferometers. This observation led us to disregard these results. In addition, two separate upper limits were obtained for the two transcontinental pairs, H1-L1 and H2-L1. These were not combined because of the known common cross-correlation contaminating the H1-H2 pair.

It is possible to take into account such local instrumental correlations by *first* combining the two local measurements into a single, *pseudo-detector* estimate of GW strain from the Hanford site, and then cross-correlating this pseudo-signal with the remaining Livingston signal.

In doing this, it is possible to obtain a self-consistent utilization of the three measurements to obtain a *single* estimate of Ω_{GW}. In order for this to be valid, the following reasonable assumptions are made:

- There are no broadband transcontinental correlations. This has been empirically observed to be the case for both the S1 and S2 science runs when the coherences between H1,2 and L1 are calculated over long periods of time.
- The local H1-H2 correlations are dominated by instrumental effects and not GW. The spectral magnitude of the H1-H2 coherence is greater than either of the H1,2-L1 pairs; moreover the frequency dependence of the coherence for H1-H2 is qualitatively different from the transcontinental pairs.

III. OPTIMAL ESTIMATE OF STRAIN FROM THE HANFORD INSTRUMENTS

The derivation of a *efficient* estimator of strain at Hanford is derived in Appendix A. The results are quoted here. Assume the two instruments produce data streams

$$s_{H_1}(t) = h(t) + n_{H_1}(t) \quad (3.1)$$
$$s_{H_2}(t) = h(t) + n_{H_2}(t) \quad (3.2)$$

The Fourier domain representations of these signals are[1]

$$\widetilde{s}_{H_1}(f) = \widetilde{h}(f) + \widetilde{n}_{H_1}(f) \quad (3.3)$$
$$\widetilde{s}_{H_2}(f) = \widetilde{h}(f) + \widetilde{n}_{H_2}(f) \quad (3.4)$$

The cross-correlation between the two Hanford machines is characterized by the coherence function:

$$\rho_{H_1H_2}(f) := \frac{P_{H_1H_2}(f)}{\sqrt{P_{H_1}(f)P_{H_2}(f)}} \quad (3.5)$$
$$\Gamma_{H_1H_2}(f) = |\rho_{H_1H_2}(f)|^2 = \frac{|P_{H_1H_2}(f)|^2}{P_{H_1}(f)P_{H_2}(f)} \quad (3.6)$$

$\rho_{H_1H_2}(f)$ is inherently a complex quantity contained within the unit circle.

Assume we form an *unbiased* linear combination of the s_i,

$$\widetilde{s}_H(f) = \widetilde{\alpha}(f)\widetilde{s}_{H_1}(f) + (1 - \widetilde{\alpha}(f))\widetilde{s}_{H_2}(f) \quad (3.7)$$

If $\widetilde{s}_H(f)$ is also to be a minimum variance estimator, then $\widetilde{\alpha}(f)$ takes the following value:

$$\widetilde{\alpha}(f) = \frac{P_{H_2}(f) - \rho^*_{H_1H_2}(f)\sqrt{P_{H_1}(f)P_{H_2}(f)}}{P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}} \quad (3.8)$$

The corresponding power of the pseudo-signal is,

$$P_H(f) = \frac{P_{H_1}(f)P_{H_2}(f)(1 - \Gamma_{H_1H_2}(f))}{P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}} \quad (3.9)$$

IV. USING THE TWO INTERFEROMETER SIGNALS TO GENERATE A DUAL TO s_H THAT CANCELS THE GRAVITATIONAL WAVE SIGNATURE: z_H

In the prior section an efficient estimator of h was derived by combining the signals from the two Hanford interferometers. It is also possible to form a *dual* to the h channel that explicitly cancels the gravitational wave signature. Call this dual channel z_H. We now proceed to determine z_H as follows. Starting with the earlier Eqns. 3.3,3.4, use the best estimate of h, s_H, to create h-subtracted residuals, z_{H_1,H_2}:

$$\widetilde{z}_{H_1}(f) = \widetilde{s}_{H_1}(f) - \widetilde{s}_H(f) \quad (4.1)$$
$$\widetilde{z}_{H_2}(f) = \widetilde{s}_{H_1}(f) - \widetilde{s}_H(f) \quad (4.2)$$

$$\widetilde{z}_{H_1}(f) = \big(1 - \widetilde{\alpha}(f)\big)[\widetilde{n}_1(f) - \widetilde{n}_2(f)] \quad (4.3)$$
$$\widetilde{z}_{H_2}(f) = \widetilde{\alpha}(f)[\widetilde{n}_1(f) - \widetilde{n}_2(f)] \quad (4.4)$$

$\widetilde{z}_{H_1}(f)$ and $\widetilde{z}_{H_2}(f)$ are both proportional to $[\widetilde{n}_1(f) - \widetilde{n}_2(f)]$. However, they have different frequency-dependent weighting functions depending on $\widetilde{\alpha}(f)$ (ref. Eqn 3.8).

In Appendix B the covariance matrix $\widetilde{\mathbf{C}}_z(f)\delta(f-f') := \langle \widetilde{z}^*_{H_i}(f)\widetilde{z}_{H_j}(f')\rangle$ is derived and diagonalized in order to determine the eigenvector, $\widetilde{z}_H(f)$. The resultant eigenvalues of $\widetilde{\mathbf{C}}_z(f)$ are found to be:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)} \\ &\quad \times(1-\widetilde{\alpha}^*(f) - \widetilde{\alpha}(f) + 2\widetilde{\alpha}^*(f)\widetilde{\alpha}(f)) \end{aligned} \tag{4.5}$$

The non-trivial solution corresponds to the desired "zero" pseudo-channel, z_H:

$$\widetilde{z}_H(f) \;=\; -\,(s_{H_1}(f) - s_{H_2}(f))\;\widetilde{\alpha}(f)\sqrt{\frac{1-\widetilde{\alpha}(f)-\widetilde{\alpha}^*(f)+2\,\widetilde{\alpha}(f)\,\widetilde{\alpha}^*(f)}{\widetilde{\alpha}(f)\,\widetilde{\alpha}^*(f)}} \tag{4.6}$$

The noise power of $\widetilde{z}_H(f)$ is given by the eigenvalue λ_2 above.

V. THE CROSS-CORRELATION STATISTICS USING COMPOSITE PSEUDO DETECTOR CHANNELS FOR STRAIN

Since the instrumental transcontinental cross-correlations are assumed to be negligible, the derivation of the optimal filter when using the pseudo-detector signal for Hanford proceeds exactly as has been presented in the literature [1, 2, 3] with $P_{H_1}(f), P_{H_2}(f) \;\rightarrow\; P_H(f), P_z(f)$ for the optimal estimate of h and the null measurement, respectively.

A. Cross-correlation statistic for the best estimate of h

The cross-correlation statistic is given by,

$$Y \equiv \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\, s_{L_1}(t_1)\, Q(t_1-t_2)\, s_H(t_2)\,, \tag{5.1}$$

The frequency domain expression is,

$$\frac{Y}{T_{obs}} \approx \int_{-\infty}^{\infty} df\, \tilde{s}^*_{L_1}(f)\, \widetilde{Q}(f)\, \tilde{s}_H(f)\,, \tag{5.2}$$

Specializing to the case of $\Omega_{\rm gw}(\mathrm{f}) \equiv \Omega_0 = \mathrm{const}$, the optimal filter becomes,

$$\widetilde{Q}(f) = \mathcal{N}\frac{\gamma(|f|)}{|f|^3 P_{L_1}(|f|)P_H(|f|)}\,, \tag{5.3}$$

where $\mathcal{N}$ is a (real) overall normalization constant. In practice we choose $\mathcal{N}$ so that the expected cross-correlation is $\mu_Y = \Omega_0\, h^2_{100}\, T$. For such a choice,

$$\mathcal{N} \;=\; \frac{20\pi^2}{3H^2_{100}}\left[\int_{-\infty}^{\infty} df\, \frac{\gamma^2(|f|)}{f^6 P_{L_1}(|f|)P_H(|f|)}\right]^{-1}, \tag{5.4}$$

The following expression holds between the variance of Y, σ^2_Y and $\mathcal{N}$:

$$\begin{aligned} T_{obs}\,\mathcal{N} &= \frac{3H^2_{100}}{5\pi^2}\sigma^2_Y && (5.5) \\ &= c_1\sigma^2_Y, \text{ with} && (5.6) \\ c_1 &:= \frac{3H^2_{100}}{5\pi^2}. \end{aligned}$$

1. *Limiting case for no H1-H2 correlations*

If $\rho_{12}(f) \rightarrow 0$, the two interferometer noise floors become uncorrelated. In this case it is possible to show that combining the point estimate measurements made separately afterwards results in the same point estimate one gets by performing a coherent analysis with the single signal $\widetilde{s}_H(f)$.

From Appendix A, Eqns. A29, A30, we have,

$$\widetilde{s}_H(f) \;=\; \frac{P_{H_2}(f)\widetilde{s}_{H_1}(f) + P_{H_1}(f)\widetilde{s}_{H_2}(f)}{P_{H_1}(f)+P_{H_2}(f)} \tag{5.7}$$

$$P_H(f) \;=\; \frac{P_{H_1}(f)P_{H_2}(f)}{P_{H_1}(f)+P_{H_2}(f)} \tag{5.8}$$

Denote as Y_{L_1H} as the correlation measured using the best estimate of h from the Hanford pair. Then Y_{L_1H} becomes,

$$\frac{Y_{L_1H}}{T_{obs}} = \mathcal{N}_{L_1H}\int_{-\infty}^{\infty} df\tilde{s}^*_{L_1}(f)\frac{\gamma(f)}{f^3P_{L_1}(f)}\frac{P_{H_1}(f)+P_{H_2}(f)}{P_{H_1}(f)P_{H_2}(f)}\Big(\frac{P_{H_2}(f)\widetilde{s}_{H_1}(f)+P_{H_1}(f)\widetilde{s}_{H_2}(f)}{P_{H_1}(f)+P_{H_2}(f)}\Big) \tag{5.9}$$

$$\frac{Y_{L_1H}}{T_{obs}\mathcal{N}_{L_1H}} = \Big[\int_{-\infty}^{\infty} df\frac{\gamma(f)\tilde{s}^*_{L_1}(f)\widetilde{s}_{H_1}(f)}{f^3P_{L_1}(f)P_{H_1}(f)} + \int_{-\infty}^{\infty} df\frac{\gamma(f)\tilde{s}^*_{L_1}(f)\widetilde{s}_{H_2}(f)}{f^3P_{L_1}(f)P_{H_2}(f)}\Big] \tag{5.10}$$

$$= \Big[\frac{Y_{L_1H_1}}{T_{obs}\mathcal{N}_{L_1H_1}} + \frac{Y_{L_1H_2}}{T_{obs}\mathcal{N}_{L_1H_2}}\Big] \tag{5.11}$$

$$\frac{Y_{L_1H}}{\sigma^2_{Y_{L_1H}}} = \Big[\frac{Y_{L_1H_1}}{\sigma^2_{Y_{L_1H_1}}} + \frac{Y_{L_1H_2}}{\sigma^2_{Y_{L_1H_2}}}\Big] \tag{5.12}$$

The expression for $\sigma^2_{Y_{L_1H}}$ is obtained as follows,

$$\mathcal{N}_{L_1H} = \frac{c_1\sigma^2_{L_1H}}{T_{obs}} = \frac{4}{c_1}\Big[\int_{-\infty}^{\infty} df\frac{\gamma^2(f)\,(P_{H_1}(f)+P_{H_2}(f))}{f^6P_{L_1}(f)P_{H_1}(f)P_{H_2}(f)}\Big]^{-1} \tag{5.13}$$

$$\frac{1}{\mathcal{N}_{L_1H}} = \frac{T_{obs}}{c_1\sigma^2_{L_1H}} = \frac{c_1}{4}\Big[\int_{-\infty}^{\infty} df\frac{\gamma^2(f)\,(P_{H_1}(f)+P_{H_2}(f))}{f^6P_{L_1}(f)P_{H_1}(f)P_{H_2}(f)}\Big] \tag{5.14}$$

$$= \frac{c_1}{4}\Big[\int_{-\infty}^{\infty} df\frac{\gamma^2(f)}{f^6P_{L_1}(f)P_{H_1}(f)} + \int_{-\infty}^{\infty} df\frac{\gamma^2(f)}{f^6P_{L_1}(f)P_{H_2}(f)}\Big] \tag{5.15}$$

$$\frac{T_{obs}}{c_1\sigma^2_{L_1H}} = \frac{c_1}{4}\Big[\frac{4}{c_1}\frac{T_{obs}}{c_1\sigma^2_{L_1H_1}} + \frac{4}{c_1}\frac{T_{obs}}{c_1\sigma^2_{L_1H_2}}\Big] \tag{5.16}$$

$$\frac{1}{\sigma^2_{L_1H}} = \Big[\frac{1}{\sigma^2_{L_1H_1}} + \frac{1}{\sigma^2_{L_1H_2}}\Big] \tag{5.17}$$

Thus if there are no correlations between the interferometers, the combined results from independent measurements are equivalent to the coherent measurement:

$$Y_{L_1H} = \frac{\sigma^{-2}_{Y_{L_1H_1}}Y_{L_1H_1} + \sigma^{-2}_{Y_{L_1H_2}}Y_{L_1H_2}}{\frac{1}{\sigma^2_{L_1H_1}} + \frac{1}{\sigma^2_{L_1H_2}}} \tag{5.18}$$

B. Cross-correlation statistic for the null measurement, z_H

Once again, cross-correlation statistic in the frequency domain is given by,

$$\frac{Y_z}{T_{obs}} \approx \int_{-\infty}^{\infty} df\,\tilde{s}^*_{L_1}(f)\,\widetilde{Q}_z(f)\,\tilde{s}_H(f)\,, \tag{5.19}$$

As before, for $\Omega_{\rm gw}(\rm f) \equiv \Omega_0 =$ const the optimal filter becomes,

$$\widetilde{Q}_z(f) = \mathcal{N}_z\frac{\gamma(|f|)}{|f|^3P_{L_1}(|f|)P_z(|f|)}\,, \tag{5.20}$$

where $\mathcal{N}_z$ is a (real) overall normalization constant. It is chosen as before,

$$\mathcal{N}_z = \frac{20\pi^2}{3H^2_{100}}\Big[\int_{-\infty}^{\infty} df\,\frac{\gamma^2(|f|)}{f^6P_{L_1}(|f|)P_z(|f|)}\Big]^{-1} \tag{5.21}$$

The following expression holds between the variance of Y_z, $\sigma^2_{Y_z}$ and $\mathcal{N}_z$:

$$T_{obs}\mathcal{N}_z = c_1\sigma^2_{Y_z} \tag{5.22}$$

1. *Limiting case for no H1-H2 correlations*

If $\rho_{12}(f) \to 0$, the two interferometer noise floors become uncorrelated. The cross-correlation statistic for the null measurement simplifes.

From Appendix B, Eqns. B10, B11, we have,

$$\tilde{z}_H(f) = (s_{H_2}(f) - s_{H_1}(f)) \frac{\sqrt{P^2_{H_1}(f) + P^2_{H_2}(f)}}{P_{H_1}(f) + P_{H_2}(f)} \tag{5.23}$$

$$P_{z_H}(f) = \frac{P^2_{H_1}(f) + P^2_{H_2}(f)}{P_{H_1}(f) + P_{H_2}(f)} \tag{5.24}$$

Denote as $Y_{L_1 z}$ as the correlation measured using the null channel of z_H from the Hanford pair. Then $Y_{L_1 z}$ becomes,

$$\frac{Y_{L_1 z}}{T_{obs}} = \mathcal{N}_{L_1 z} \int_{-\infty}^{\infty} df \frac{\gamma(f)\tilde{s}^*_{L_1}(f)}{f^3 P_{L_1}(f)} \frac{P_{H_1}(f) + P_{H_2}(f)}{P^2_{H_1}(f) + P^2_{H_2}(f)} \Big((s_{H_2}(f) - s_{H_1}(f)) \frac{\sqrt{P^2_{H_1}(f) + P^2_{H_2}(f)}}{P_{H_1}(f) + P_{H_2}(f)} \Big) \tag{5.25}$$

$$\frac{Y_{L_1 z}}{T_{obs}} = \mathcal{N}_{L_1 z} \int_{-\infty}^{\infty} df \frac{\gamma(f)\tilde{s}^*_{L_1}(f) \, (s_{H_2}(f) - s_{H_1}(f))}{f^3 P_{L_1}(f) \sqrt{P^2_{H_1}(f) + P^2_{H_2}(f)}} \tag{5.26}$$

The expression for $\mathcal{N}_{L_1 z}$ is given by,

$$\mathcal{N}_{L_1 z} = \frac{c_1 \sigma^2_{L_1 z}}{T_{obs}} = \frac{4}{c_1} \Big[\int_{-\infty}^{\infty} df \frac{\gamma^2(f) \, (P_{H_1}(f) + P_{H_2}(f))}{f^6 P_{L_1}(f) [P^2_{H_1}(f) + P^2_{H_2}(f)]} \Big]^{-1} \tag{5.27}$$

Unlike the prior case, because of the dependence on the quadrature sum, $[P^2_{H_1}(f) + P^2_{H_2}(f)]$, even when there are no correlations between the interferometers, the combined null result does not follow directly from individual independent measurements.

C. Combining triple and double coincident measurements of Ω_{GW}

In order to make use of this methodology for the analysis of the S2 and S3 data, we will need to partition the data into three *non-overlapping* (hence statistically independent) data sets: the H1-H2-L1 triple coincident data, and the two H1-L1 and H2-L1 double coincident data sets. The triple coincidence data would be analyzed in the manner described in this note. Measurements from the three observations may be combined under the assumption of statistical independence.

VI. CONCLUSION

The approach presented above is fundamentally different from how the analysis of S1 data was conducted and represents a manner to maximally exploit the feature of LIGO that has two co-located interferometers. This technique is possible for the Hanford pair of detectors because to high accuracy the gravitational wave signature is guaranteed to be *identically* imprinted on both data streams. Coherent, time-domain mixing of the two interferometer strain channels can thus be used to optimal advantage by (i) providing the best possible estimate of gravitational wave strain and (ii) providing a null channel with which any gravitational wave analysis can be calibrated for backgrounds.

The usefulness of $\tilde{z}_H(f)$ is that it may be used to analyze the cross-correlations for non-gravitational wave signals between the Livingston and Hanford sites. This would enable a *null* measurement to be made, i.e., one in which gravitational radiation had been effectively "turned off." In this sense, using $\tilde{z}_H(f)$ would be analogous to analyzing the ALLEGRO-L1 correlation when the orientation of the cryogenic resonant bar detector is such that the stochastic background does not contribute to the overall cross-correlations [4]. Under suitable analysis, the $\tilde{s}_{L1}(f)$-$\tilde{z}_H(f)$ could be used to establish a measurement (i.e., "off-source") background for the stochastic gravitational wave background.

Ultimately, the usefulness of such a null test will be related to how well the relative calibrations between H1 and H2 are known. If the contribution of $\tilde{h}(f)$ to $\tilde{s}_{H_1}(f)$ and $\tilde{s}_{H_2}(f)$ is not equal due to calibration uncertainties, then this error will propagate into the generation of $\tilde{s}_H(f)$, $\tilde{z}_H(f)$. It is possible to estimate this effect as follows. Due to the intended use of $\tilde{z}_H(f)$ in a null measurement, the leakage of h into this channel is the greater concern. Considering the structure of Eqns. 3.7 and 4.3, 4.4, it is clear that effects of *differential* calibration errors in $\tilde{s}_H(f)$ will tend to average out, whereas such errors will be *amplified* in $\tilde{z}_H(f)$. Assume a differential calibration error of $\pm\tilde{\epsilon}(f)$. Then $\tilde{z}_H(f)$ will contain a gravitational wave signature,

$$\delta\tilde{h}(f) = 2\tilde{\epsilon}(f)\tilde{h}(f) \tag{6.1}$$

$$\delta P_{gw}(f) = 4|\tilde{\epsilon}(f)|^2 P_{gw}(f) \tag{6.2}$$

The amplitude leakage affects single-interferometer based analyses; the power leakage affects multiple interferometer correlations (such as the stochastic background search). Assuming reasonably small values for $\pm\widetilde{\epsilon}(f)$, if a search sets a threshold ρ_* on putative gravitational wave events detected in channel $\widetilde{s}_H(f)$, then the corresponding contribution in $\widetilde{z}_H(f)$ would be $\approx 2|\overline{\epsilon}|\rho_*$, where $|\overline{\epsilon}|$ denotes the magnitude of the frequency integrated differential calibration errors. For any reasonable threshold (e.g., $\rho_* \approx 10$) above which one would claim a detection, and for typical differential calibration uncertainties of $2|\overline{\epsilon}| \lesssim 20\%$, then the same event would have signal to noise level of $\rho_* \approx 2$ in the null channel, well below what one would consider meaningful. A more careful analysis is needed to quantify these results, since calibration uncertainties also propagate into the $\widetilde{\alpha}(f)$.

While the focus of this note is the application of this technique to the search for stochastic gravitational waves, it appears to be the case that *any* analysis can exploit this approach. It should be straightforward to tune pipeline filters and cull spurious events by using the null channel to veto events seen in the h channel.

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APPENDIX A: OPTIMALLY COMBINING SIGNALS FROM TWO (CO-LOCATED) INTERFEROMETERS WITH CORRELATED INSTRUMENTAL NOISE

This appendix derives the minimum variance unbiased estimator of local GW strain from two interferometer data streams co-located at one site (i.e., LIGO Hanford Observatory). It takes into account the possibility that the two measurements contain instrumentally correlated noise in addition to the GW signal.

Consider two interferometers, labeled by indices H_1, H_2. Both see the same GW signal, h but have different noise floors, n_{H_1,H_2}:

$$s_{H_1}(t) = h(t) + n_{H_1}(t) \quad \text{(A1)}$$
$$s_{H_2}(t) = h(t) + n_{H_2}(t) \quad \text{(A2)}$$

The Fourier domain representations of these signals are[2]

$$\widetilde{s}_{H_1}(f) = \widetilde{h}(f) + \widetilde{n}_{H_1}(f) \quad \text{(A3)}$$
$$\widetilde{s}_{H_2}(f) = \widetilde{h}(f) + \widetilde{n}_{H_2}(f) \quad \text{(A4)}$$

Assume the process generating h,n_i to be stochastic with the following statistical properties of the signals and noise components[3]:

$$\langle \widetilde{n}_i(f) \rangle = \langle \widetilde{h}(f) \rangle = 0 \quad \text{(A5)}$$
$$\langle \widetilde{n}_i^*(f)\, \widetilde{h}(f) \rangle = 0 \quad \text{(A6)}$$
$$\langle \widetilde{n}_i^*(f) \widetilde{n}_j(f') \rangle = P_{ij}(f)\delta(f-f') \quad \text{(A7)}$$
$$= \rho_{ij}(f)\sqrt{P_i(f)P_j(f)} \times \delta(f-f') \quad \text{(A8)}$$
$$P_{ii}(f) := P_i(f) \quad \text{(A9)}$$
$$\langle \widetilde{h}^*(f)\widetilde{h}(f') \rangle = P_\Omega(f)\delta(f-f') \quad \text{(A10)}$$
$$P_\Omega(f) \ll P_i(f) \quad \text{(A11)}$$
$$\Gamma_{ij}(f) := |\rho_{ij}(f)|^2 \quad \text{(A12)}$$

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[2] $\widetilde{a}(f)$ denotes the Fourier transforms of $a(t)$—i.e., $\widetilde{a}(f) \equiv \int_{-\infty}^{\infty} dt\, e^{-i2\pi f t}\, a(t)$.

[3] The brackets $\langle ... \rangle$ denote ensemble or statistical averages of random processes

$\Gamma_{ij}(f)$ is the coherence between the two signals. $\rho_{ij}(f)$ is a complex quantity of magnitude less than or equal to unity. Using these results the covariance matrix, $\langle s^*_{H_i}(f) s_{H_j}(f'f)\rangle$, becomes,

$$\widetilde{\mathbf{C}}_s(f)\delta(f-f') = \begin{bmatrix} \langle \widetilde{s}^*_{H_1}(f)\widetilde{s}_{H_1}(f')\rangle & \langle \widetilde{s}^*_{H_1}(f)\widetilde{s}_{H_2}(f')\rangle \\ \langle \widetilde{s}^*_{H_2}(f)\widetilde{s}_{H_1}(f')\rangle & \langle \widetilde{s}^*_{H_2}(f)\widetilde{s}_{H_2}(f')\rangle \end{bmatrix} \tag{A13}$$

$$= \left(\begin{bmatrix} P_{H_1}(f) & P_{H_1H_2}(f) \\ P_{H_2H_1}(f) & P_{H_2}(f) \end{bmatrix} + P_\Omega(f) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \delta(f-f') \tag{A14}$$

The structure of $\widetilde{\mathbf{C}}_s(f)$ is such that the gravitational wave signature appears in all four matrix elements while the diagonal noise power terms, $P_{H_i}(f)$, dominate the covariance matrix. Because of this it is not possible to manipulate $\widetilde{\mathbf{C}}_s(f)$ at this point without first eliminating the contribution due to gravitational waves. While the covariance matrix is a $2x2$ object, the problem is inherently three dimensional, spanning the vector space, $\{\widetilde{n}_{H_1}(f), \widetilde{n}_{H_2}(f), \widetilde{h}(f)\}$ (see Fig. 1 for additional details).

It is possible to proceed as follows. Assume we form a linear combination of the s_i:

$$\widetilde{s}_H(f) = \widetilde{\alpha}(f)\widetilde{s}_{H_1}(f) + \widetilde{\beta}(f)\widetilde{s}_{H_2}(f) \tag{A15}$$

If s_H is to be an *unbiased* estimator of h, then the following must be true:

$$\langle \widetilde{h}^*(f)\ \widetilde{s}_H(f')\rangle = P_\Omega(f)\delta(f-f')$$
$$\rightarrow \widetilde{\alpha}(f) + \widetilde{\beta}(f) = 1 \tag{A16}$$

In order to determine $\widetilde{\alpha}(f)$, the other constraint that can be applied is to require the estimator s_H to have a *minimum* variance. $Var(s_H)$ is the noise power of the signal s_H:

$$Var(s_H) := P_H(f) \tag{A17}$$
$$\langle \widetilde{s}^*_H(f)\widetilde{s}_H(f')\rangle = P_H(f)\delta(f-f') \tag{A18}$$

$$P_H(f) = |\widetilde{\alpha}(f)|^2 P_{H_1}(f) + |1-\widetilde{\alpha}(f)|^2 P_{H_2}(f) + \left(\widetilde{\alpha}^*(f)(1-\widetilde{\alpha}(f))\rho_{H_1H_2}(f) + \widetilde{\alpha}(f)(1-\widetilde{\alpha}^*(f))\rho^*_{H_1H_2}(f)\right)\sqrt{P_{H_1}(f)P_{H_2}(f)} + P_\Omega(f) \tag{A19}$$

Equation A16 guarantees that the signature of h in s_H is independent of $\widetilde{\alpha}(f)$ and therefore does not come into play when determining the filter coefficient $\widetilde{\alpha}(f)$ (however, it is necessary to assume that Equation A11 holds in order to identify $P_{H_i}(f)$ with the measurable interferometer noise spectra). Minimizing $P_H(f)$ leads to the following pair of equations:

$$\begin{pmatrix} \frac{\partial P_H(f)}{\partial \widetilde{\alpha}(f)} \\ \frac{\partial P_H(f)}{\partial \widetilde{\alpha}^*(f)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A20}$$

(A21)

The resulting equations are complex conjugates of each other. One of them is:

$$0 = \widetilde{\alpha}(f)P_{H_1}(f) - (1-\widetilde{\alpha}(f))P_{H_2}(f) + \left[(1-\widetilde{\alpha}(f))\rho_{H_1H_2}(f) - \widetilde{\alpha}(f)\rho^*_{H_1H_2}(f)\right]\sqrt{P_{H_1}(f)P_{H_2}(f)} \tag{A22}$$

$$\widetilde{\alpha}(f) = \frac{P_{H_2}(f) - \rho_{H_1H_2}(f)\sqrt{P_{H_1}(f)P_{H_2}(f)}}{P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}} \tag{A23}$$

This expression for $\widetilde{\alpha}(f)$ results in an *efficient* estimator for $\widetilde{h}(f)$. Substituting for $\widetilde{\alpha}(f)$ in Eq. A19 , the noise power (variance) for $\widetilde{s}_H(f)$ becomes:

$$P_H(f) \;=\; \frac{P_{H_1}(f)P_{H_2}(f)(1-\Gamma(f))}{P_{H_1}(f)+P_{H_2}(f)-(\rho_{H_1H_2}(f)+\rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}} \tag{A24}$$

Limiting cases:

I. If $\rho_{H_1H_2}(f)\to 0$: Then $\widetilde{\alpha}(f)$ becomes,

$$\widetilde{\alpha}(f)\to\frac{P_{H_2}(f)}{P_{H_1}(f)+P_{H_2}(f)} \tag{A25}$$

IIa. If $P_{H_1}(f)\to P_{H_2}(f)$: Then $\widetilde{\alpha}(f)$ becomes,

$$\widetilde{\alpha}(f)\to\frac{1-\rho_{H_1H_2}(f)}{2-(\rho_{H_1H_2}(f)+\rho^*_{H_1H_2}(f))} \tag{A26}$$

IIb. If $\rho_{H_1H_2}(f)\to 1$, then $\rho_{H_1H_2}(f)=\rho^*_{H_1H_2}(f)=\sqrt{\Gamma(f)}$. If also $P_{H_1}(f)\to P_{H_2}(f)$, then $P_H(f)\to P_{H_1}(f)$:

$$P_H(f)= \lim_{\Gamma(f)\to 1}\frac{P_{H_1}(f)}{2}\frac{1-\Gamma(f)}{1-\sqrt{\Gamma(f)}}=P_{H_1}(f) \tag{A27}$$

III. For H1 and H2 the limiting design performance will have $P_{H_2}(f)=4P_{H_1}(f)$ due to the $1:2$ arm length ratio,

$$\widetilde{\alpha}(f)\to\frac{2(2-\rho_{H_1H_2}(f))}{5-2(\rho_{H_1H_2}(f)+\rho^*_{H_1H_2}(f))} \tag{A28}$$

If the noise were either completely correlated ($\rho_{H_1H_2}(f)\to 1, \widetilde{\alpha}(f)\to 2$) or anti-correlated ($\rho_{H_1H_2}(f)\to -1, \widetilde{\alpha}(f)\to \frac{2}{3}$), then it would be possible to exactly cancel the noise in the signals s_i.

For the case of uncorrelated noise between the two interferometers, the expressions for $\widetilde{s}_H(f)$, $P_H(f)$ reduce to:

$$\widetilde{s}_H(f) \;=\; \frac{P_{H_2}(f)\widetilde{s}_{H_1}(f)+P_{H_1}(f)\widetilde{s}_{H_2}(f)}{P_{H_1}(f)+P_{H_2}(f)} \tag{A29}$$

$$P_H(f) \;=\; \frac{P_{H_1}(f)P_{H_2}(f)}{P_{H_1}(f)+P_{H_2}(f)} \tag{A30}$$

APPENDIX B: COMBINING THE H_1 AND H_2 SIGNALS TO CANCEL h

In order to determine z_H, proceed as follows. Starting with the earlier Eqns. A3,A4, use the best estimate of h, s_H, to create h-subtracted residuals, z_{H_1,H_2}:

$$\widetilde{z}_{H_1}(f) \;=\; \widetilde{s}_{H_1}(f)-\widetilde{s}_H(f) \tag{B1}$$

$$\widetilde{z}_{H_2}(f) \;=\; \widetilde{s}_{H_1}(f)-\widetilde{s}_H(f) \tag{B2}$$

$$\widetilde{z}_{H_1}(f) \;=\; \big(1-\widetilde{\alpha}(f)\big)[\widetilde{n}_1(f)-\widetilde{n}_2(f)] \tag{B3}$$

$$\widetilde{z}_{H_2}(f) \;=\; \widetilde{\alpha}(f)[\widetilde{n}_1(f)-\widetilde{n}_2(f)] \tag{B4}$$

Figure 1 shows schematically the geometrical relationships of the signal vectors $\widetilde{s}_{H_i}(f)$ and $\widetilde{z}_{H_i}(f)$. Once the best estimate of h, $\widetilde{s}_H(f)$, is subtracted from the signals, the residuals lie in the $\widetilde{n}_{H_1}(f)-\widetilde{n}_{H_2}(f)$ plane. Their covariance matrix can then be diagonalized without affecting the gravitational wave signature contained in $\widetilde{s}_H(f)$ $\widetilde{z}_{H_1}(f)$ and $\widetilde{z}_{H_2}(f)$ are both proportional to $[\widetilde{n}_1(f)-\widetilde{n}_2(f)]$. However, they have different frequency-dependent weighting functions depending on $\widetilde{\alpha}(f)$ (ref. Eqn A23). Now consider the covariance matrix $\langle\widetilde{z}^*_{H_i}(f)\widetilde{z}_{H_j}(f')\rangle$:

$$\widetilde{\mathbf{C}}_z(f)\delta(f-f') = \begin{bmatrix} \langle \widetilde{z}^*_{H_1}(f)\widetilde{z}_{H_1}(f')\rangle & \langle \widetilde{z}^*_{H_1}(f)\widetilde{z}_{H_2}(f')\rangle \\ \langle \widetilde{z}^*_{H_2}(f)\widetilde{z}_{H_1}(f')\rangle & \langle \widetilde{z}^*_{H_2}(f)\widetilde{z}_{H_2}(f')\rangle \end{bmatrix} \tag{B5}$$

$$= \langle (\widetilde{n}^*_1(f) - \widetilde{n}^*_2(f))(\widetilde{n}_1(f') - \widetilde{n}_2(f'))\rangle \begin{bmatrix} (1-\widetilde{\alpha}(f))(1-\widetilde{\alpha}^*(f)) & -\widetilde{\alpha}(f)(1-\widetilde{\alpha}^*(f)) \\ -\widetilde{\alpha}^*(f)(1-\widetilde{\alpha}(f)) & \widetilde{\alpha}(f)\widetilde{\alpha}^*(f) \end{bmatrix} \tag{B6}$$

$$= \begin{bmatrix} (1-\widetilde{\alpha}(f))(1-\widetilde{\alpha}^*(f)) & -\widetilde{\alpha}(f)(1-\widetilde{\alpha}^*(f)) \\ -\widetilde{\alpha}^*(f)(1-\widetilde{\alpha}(f)) & \widetilde{\alpha}(f)\widetilde{\alpha}^*(f) \end{bmatrix} \times \Big(P_{H_1}(f) + P_{H_2}(f) - (P_{H_1H_2}(f) + P_{H_2H_1}(f)\Big)\delta(f-f') \tag{B7}$$

Comparing Eqns A14 and B7, note that $\widetilde{\mathbf{C}}_z(f)$, unlike $\widetilde{\mathbf{C}}_s(f)$, does not depend on $P_\Omega(f)$. It is possible to proceed further by diagonlizing $\widetilde{\mathbf{C}}_z(f)$. The eigenvalues of $\widetilde{\mathbf{C}}_z(f)$ are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \Big(P_{H_1}(f) + P_{H_2}(f) - (\rho_{H_1H_2}(f) + \rho^*_{H_1H_2}(f))\sqrt{P_{H_1}(f)P_{H_2}(f)}\Big) \\ &\quad \times \Big(1 - \widetilde{\alpha}^*(f) - \widetilde{\alpha}(f) + 2\widetilde{\alpha}^*(f)\widetilde{\alpha}(f)\Big) \end{aligned} \tag{B8}$$

The non-trivial solution corresponds to the desired "zero" pseudo-channel, z_H:

$$\begin{aligned} \widetilde{z}_H(f) &= -\left(n_{H_1}(f) - n_{H_2}(f)\right)\widetilde{\alpha}(f)\sqrt{\frac{1-\widetilde{\alpha}(f) - \widetilde{\alpha}^*(f) + 2\,\widetilde{\alpha}(f)\,\widetilde{\alpha}^*(f)}{\widetilde{\alpha}(f)\,\widetilde{\alpha}^*(f)}} \\ &= -\left(s_{H_1}(f) - s_{H_2}(f)\right)\widetilde{\alpha}(f)\sqrt{\frac{1-\widetilde{\alpha}(f) - \widetilde{\alpha}^*(f) + 2\,\widetilde{\alpha}(f)\,\widetilde{\alpha}^*(f)}{\widetilde{\alpha}(f)\,\widetilde{\alpha}^*(f)}} \end{aligned} \tag{B9}$$

The power spectrum of $\widetilde{z}_H(f)$ is given by the eigenvalue, B8.

In the limit that the two signals are uncorrelated, $\rho_{H_1H_2}(f) \rightarrow 0$, and the expression for $\widetilde{\alpha}(f)$ (ref. Eqn. A25) simplifies considerably. $\widetilde{z}_H(f)$ now becomes (refer to Eqns A29, A30 for the corresponding expressions for $\widetilde{s}_H(f)$ and $P_H(f)$):

$$\widetilde{z}_H(f) = \left(s_{H_2}(f) - s_{H_1}(f)\right)\frac{\sqrt{P^2_{H_1}(f) + P^2_{H_2}(f)}}{P_{H_1}(f) + P_{H_2}(f)} \tag{B10}$$

$$P_{z_H}(f) = \frac{P^2_{H_1}(f) + P^2_{H_2}(f)}{P_{H_1}(f) + P_{H_2}(f)} \leq P_{H_1}(f) + P_{H_2}(f) \tag{B11}$$

The last equation shows that the null channel, z_H, contains *less* noise power than the difference of the signals n_{H_i}. The filtering produced by $\widetilde{\alpha}(f)$ results in a less noisy null estimator than the quantity, $\widetilde{n}_{H_1}(f) - n_{H_2}(f)$. In fact, in the limit that either $P_{H_1}(f)$ or $P_{H_2}(f) \rightarrow \infty$, the noise power in always less than the larger of $P_{H_1,H_2}(f)$:

$$\lim_{P_{H_1,H_2}(f)\rightarrow\infty} \frac{P_{H_1}^2(f) + P_{H_2}^2(f)}{P_{H_1}(f) + P_{H_2}(f)} = max[P_{H_1}(f), P_{H_2}(f)] - min[P_{H_1}(f), P_{H_2}(f)] \tag{B12}$$

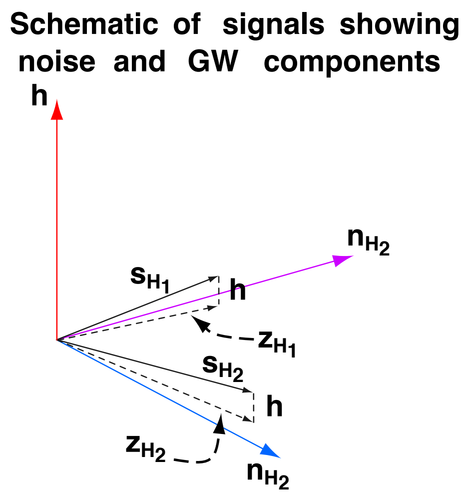

FIG. 1: Schematic showing how the H1 and H2 signals may be represented in a 3-dimensional space of noise components for the two detectors, $\widetilde{n}_{H_1,2}(f)$ and signal, $\widetilde{h}(f)$. First $\widetilde{h}(f)$ is estimated, then it is subtracted to produce the vectors $\widetilde{z}_{H1,2}(f)$ that lie in the $\widetilde{n}_{H_1,2}(f)$ plane. These vectors are not, in general, orthogonal if the coherence between the noise is not zero. The covriance matrix, $\widetilde{\mathbf{C}}_z(f)$ can be diagonlized. Then the dual to $\widetilde{s}_H(f)$, $\widetilde{z}_H(f)$ can be determined. It is necessary to first subtract out the contribution of h before analyzing the covariance matrix.