LIGO Laboratory / LIGO Scientific Collaboration

LIGO-T030285-01-K	*Advanced LIGO UK*	October 2003

Investigation of blade spring design parameters

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This is an internal working note
of the Advanced LIGO Project, prepared by members of the UK team.

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1 Modification history

Revision 00 – early draft

Revision 01 – more rational ordering of topics, discussion of bend radius, effect of higher stress, etc.

2 Thanks

This note does not attempt to break any new ground. I have simply collected together the understanding I have gleaned from a variety of people and places, and in some cases expressed it in ways that I find easier to understand. Where I could, I have gone through the derivations in order to be sure that I understand them and in particular any limitations and so can apply the results with accuracy and confidence. Many people have helped, notably Mike Plissi, Calum Torrie, and Norna Robertson but there are others. Some of these I have cited, others I have not and I apologize to anyone who was left out.

3 Introduction

The principal free variables in the design of the spring are

Dimensions: the length, the width at the base and at the tip, the thickness

Material properties: Young's modulus and the maximum allowable stress

The principal performance criteria are

Stiffness – which governs the natural frequency of the blade-mass system. From this we find the deflection under load and the "uncoupled" frequency, meaning the frequency of the system formed by the spring and the mass it immediately supports (and ignoring, for the time being, the elasticity of the intervening wire).

Highest stress in the blade. This will not exceed the maximum allowable stress for an acceptable design. It will occur at the blade root if the shape is trapezoidal, and all along the blade if the shape is triangular.

Internal vibration modes; the key one being the lowest. This is hard to determine analytically because of the shape of the blade, but one can make a prediction based on measurements and extrapolation.

Another important parameter is the radius to which blade should be formed in order to be flat in use.

The purpose of this note is to explore the relationships between the free variables and the performance criteria. This is ground which has already been gone over by others – see for example references [1] [2] and [5], and the work is largely for my own understanding. I have tried to use the same symbols as in [5] which is the most recent exposition.

4 Triangular blade – design equations

4.1 Deflection and stiffness

I shall follow the engineer's approach and find the deflection from standard methods then infer the stiffness. Consider a simple encastered blade (built in at one end) whose width varies linearly to a point at the free end, and which point the load is applied. Take the x dimension along the blade with x =0 at the tip and the y dimension for deflection. The simple beam equation [ref 3] is

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}$$

Where y and x are as noted above, M is the bending moment, E the Young's Modulus and I is the second moment of area. In the simple case given, the bending moment simply varies with x. The second moment of area is given by

$$I = \frac{bd^3}{12}$$

Where b is the breadth of the blade and d is the thickness. Define a as the maximum blade width (at the root) and *l* and the length, then with a load P the moment is given by

$$M = Px$$

And the SMA is given by

$$I = \frac{x}{l} a \times \frac{h^3}{12}$$

(renaming the thickness as h).

Substituting into the original expression gives

$$\frac{d^2 y}{dx^2} = \frac{Px}{E} \times \frac{12l}{(xah^3)}$$

$$= \frac{12Pl}{Eah^3}$$

Integrating to find the slope and deflection

$$\frac{dy}{dx} = \frac{12Pl}{Eah^3} x + A$$

$$y = \frac{6Pl}{Eah^3} x^2 + Ax + B$$

The slope and deflection are both zero when x=*l*, so the constants of integration are

$$A = \frac{-12Pl^2}{Eah^3}$$

$$B = \frac{-6Pl^3}{Eah^3} + \frac{12Pl^3}{Eah^3} = \frac{6Pl^3}{Eah^3}$$

When x = 0, the deflection is given by B which, encouragingly, agrees with references such as [1], [2] and [5]. Comparing with [5]:

$$\lambda = \frac{6Pl^3}{Eah^3}$$

(this is exactly the same as eqn (3) in ref [5] with $\alpha = 1.5$ (see below) and with $P=m_t g$.

Note that m_t is the total mass supported by the spring – this is different from m which is the mass of the stage immediately below.

The stiffness is given from the standard

$$\lambda = \frac{P}{k} \text{ whence}$$

$$k = \frac{Eah^3}{6l^3}$$

4.2 Uncoupled natural frequency

From the standard equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

And noting that the relevant mass is the mass supported by the spring at this stage m, and not the total mass supported by the spring m_t we can substitute the stiffness from the expressions above to get

$$f = \sqrt{\frac{Eah^3}{24\pi^2 ml^3}}$$ which is the same as equation (5) in [5] when α=1.5

4.3 Stress

The stress at any point along the blade occurs at the top and bottom surfaces and is given by

$$\sigma = \frac{M}{I} \times \frac{h}{2}$$

Substituting expressions for M and I we find that the stress is independent of x and is given by

$$\sigma = \frac{6Pl}{ah^2}$$

as also reported in [1] and in [5].

4.4 Internal modes

See note in section 5.4 about internal modes

4.5 Bend radius

See note in section 5.5 on bend radius

5 Trapezoidal blade – design equations

5.1 Deflection

For the case of the trapezoidal blade the procedure outlined above becomes very messy and so I have not typeset it. Appendix 1 has the derivation in which I was much helped by already having the results in ref [2].

The result for maximum deflection given by [2] is

$$\lambda = \alpha \frac{4Pl^3}{Eah^3}$$

where the factor α varies between 1.5 for a triangular blade and 1 for a parallel-sided blade, and depends upon β (the ratio of the widths at the two ends, b/a) as shown in figure 4. This can be rewritten, as in [5] for example, as

$$\lambda = \alpha \frac{4m_t g l^3}{Eah^3}$$

This gives the same result as that derived above when α =1.5.

5.2 Frequency

The uncoupled natural frequency can be derived from the stiffness, which itself can be found from λ in the normal manner:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{Eah^3}{4ml^3\alpha}}$$

5.3 Stress

The maximum stress will occur at the root and will be same as that given for the triangular blade. The stress elsewhere along the blade will be lower than for a triangular blade of the size length, width, thickness, and loading.

5.4 Internal modes

It is hard to derive an expression for internal vibrational modes of a trapezoidal plate clamped at one end and pinned at the other, and no suitable expressions have been found in the literature (for the closest I found, see [6]). However, the standard vibration literature (see for example [7]) deals with plain beams (not tapered) and the results have been extrapolated to trapezoidal plates in a simple manner [1],[4] as follows:

$$f_i \propto \frac{h}{l^2}$$

Where *h* and *l* are the thickness and length of the blade, for a given root width and taper. Sadly this gives no insight in to how the internal modes vary with shape factor.

5.5 Bend radius

There are two approaches to finding the radius to which the blade should be bent in order to take a straight form when under load. The first is arrived at by considering the deflection and working out what shape the blade would have to be in order to deflect that amount. Some simple geometry gives

$$\lambda = R\left(1-\cos\left(\tfrac{l}{R}\right)\right)$$

from which R can be found. I see two problems with this approach. One is that the deflection has been derived using simple beam theory which assumes small deflections. Most of the blades will have deflections which significantly distort the blade geometry and cannot be considered small. (Note that this argument does not affect the stress and stiffness results, because the blade is in a flat, or near-flat state, when the stress and stiffness results are applied.) The second is that it is only for a triangular blade that the stress is constant along the blade and so it is only for a triangular that a constant bend radius will apply. If this is unclear, consider the following.

The standard beam bending result, from which the formulae above were derived, is

$$\frac{\sigma}{y} = \frac{E}{R} = \frac{M}{I}$$

(it is by setting $R = \frac{d^2 y}{dx}$ that the expression at the start of section 2 was arrived at).

The relevant result here is

$$R = \frac{EI}{M}$$

Note that *M* and *I* will vary along the blade. For a triangular blade (or for a portion of a blade which forms part of a triangle having the load application point at its apex), it turns out that *M* and *I* vary in the same way, giving a constant value of *R*. However, for a trapezoidal blade the radius near the tip will be lower than at the root.

It would be interesting to compare values of *R* derived from these two approaches with practical and FEA results.

6 Summary of equations

All of the equations can be summarised by considering the trapezoidal case; the triangular case is obtained by putting α=1.5

Parameter	Equation	Equation number
Deflection	$\lambda = \alpha \frac{4 m_t g l^3}{E a h^3}$	(1)
Uncoupled frequency	$f = \frac{1}{2\pi}\sqrt{\frac{E a h^3}{4 m l^3 \alpha}}$	(2)
Stress	$\sigma = \frac{6 P l}{a h^2}$	(3)
Lowest internal mode	$f_i \propto \frac{h}{l^2}$	(4)

The key equations for design are (2), (3) and (4).

7 Graphical interpretation

I find it helpful to visualize results graphically.

7.1 Varying length and thickness (fig 1)

A graph whose axes show length and thickness is a good place to start.

Take the top blades of the AdLIGO quads with parameters as currently given in the conceptual design document. The load (for stress purposes) is given by a half share of the weight of all the masses:

$$P = (22 + 22 + 40 + 40) * 9.81 / 2$$

The mass for uncoupled frequency is half the top mass

$$m = 22 / 2$$

I have taken Young's modulus as 176 Gpa (as reported in [5]). For want of somewhere to start I have used numbers current at the time I started writing, supplied by Norna (email at the end of this note). The blades have a root width of 95mm and alpha= 1.36. Allowing a maximum stress of 951 MPa we can plot *l* against h from equation (3). Requiring an uncoupled frequency of 2.41 Hz we can plot *l* against h from equation (2). See figure 1.

The dotted line is the one given by the stress equation – designs below the line are ruled out because the blade is too thin or too long and that makes the stress too high. The solid line is the one given by the uncoupled frequency equation – designs above that line are ruled out because the blades are too thick or too short and therefore too stiff. The "obvious" design is the one where the lines cross (h=~4.4mm, *l*=~480mm) but there is a range of possible designs which could be used if, for example, the design required a longer blade. Naturally the effect on the overall suspension system of changes in uncoupled frequency would need to be checked but making it lower is likely to be acceptable. But as will be seen shortly, the effect on the internal modes of moving away from the intersection point is likely to be undesirable.

7.2 Varying blade width as well as length and thickness (fig 2)

It is a simple matter to produce graphs with various values of blade width: see figure 2.

One counter-intuitive result is that the shortest possible blade (the "obvious" design where the lines cross) is shorter for a wider blade.

7.3 Internal modes of blades (fig 3)

Lines showing constant values of (h/l^2) are shown in figure 4. The higher modes are at the top left of the graph (short, thick blades) and so for the highest possible internal modes, one is driven towards the intersection point between the strength and stiffness lines.

7.4 Optimum combination of length, width and thickness

It will generally be the case that the higher the internal modes, the better. It is clear from looking at fig 4 that the optimum design from this point of view will be the point where the stress is the highest allowed and the uncoupled mode is the highest allowed, ie. the point where the lines in fig 1 cross. The locus of all such points for varying values of a is found by solving (2) and (3) simultaneously and eliminating *a*:

From (3): $h^2 = \dfrac{6P}{\sigma} \times \dfrac{l}{a}$ (5)

From (2): $2\pi f^2 = \dfrac{Eah^3}{4m\alpha l^3}$

Whence: $h^3 = \dfrac{2\pi f^2 \times 4m\alpha}{E} \times \dfrac{l^3}{a}$ (6)

Dividing (5) and (6): $h = l^2 \times \dfrac{2\pi f^2 \times 4m\alpha}{E} \times \dfrac{\sigma}{6P}$ (7)

The interesting point here is that the form of this equation (h/l^2=const) is the same as for the internal mode equation. In other words, the locus of the optimum blade designs on figure 2 or figure 3 is a line of constant f_i. (This is fairly obvious from careful study of figure 3.) Modifying the values of *h,l* and *a* will change the appearance of the blade but will not improve upon the lowest internal mode. What WILL have an effect is changing α, but we don't currently have an expression for how α affects the internal modes so we cannot say any more than that.

The final point to be made here, is that if one wanted to use a particular value of blade length, then there is a combination of thickness and width that will give the optimum internal mode.

7.5 Varying the allowable stress

If we allow the maximum stress to rise, can we design a blade with higher internal modes? There are two ways to look at this. One is to look at it graphically. Looking at figure 2, the dotted line will move down if we allow a higher stress value. This moves the intersect to the left. Turning now to figure 3, such a movement will indeed raise the internal mode. The other way is to do it algebraically – simply consider the effect on equation (7) (Previous section) of raising σ. The locus of optimum solutions (h vs *l*) will move up, giving a higher internal mode.

8 Other blade shapes

8.1 Non-trapezoidal shapes

All of the above, of course, assumes a trapezoidal blade with perfect fixings and ideal beam-like behaviour. It is unlikely that the Advanced LIGO blades will be entirely trapezoidal, they will not be ideally constrained, and they will behave like plates with Poisson edge-effects and all the rest. However, the exploration above gives at least some idea of the sorts of effects to expect.

8.2 Varying alpha

In all that has been considered above, α is a simple geometrical number found for a trapezoidal blade by measuring it, finding β, and applying the complicated formula. For non-trapezoidal blades, one can define *de facto* an equivalent α, by measuring one aspect of the blade performance and then inferring it from (1), (2) or (3). One can then go on to use it with some confidence in the other equations in order to predict the other behaviours of that blade and other blades of the same shape. Naturally this will depend upon knowing the value of, for example, E. Such measurements have resulted in values of α above 1.5 which, for the purely geometrical definition, would not be meaningful but which are perfectly sensible when applied to non-trapezoidal blades.

9 References

[1] Calum Torrie's Thesis Jan 2001

[2] VIRGO note of Cella and Vicere, "Super attenuator vertical performance beyond the low frequency range", June 1997, no number. Copy given to me by Mike Plissi.

[3] See any undergraduate solid mechanics book for example Roark & Young, "Equations for stress and strain" Sixth ed, chapter 7.

[4] Mike Plissi, private communication

[5] Mike Plissi, ALUKGLA0040a, Cantilever blade analysis for Advanced LIGO.

[6] NASA SP-160 "Vibration of plates" Arthur W Leissa. 1969.

[7] Weaver, Timoshenko and Young: "Vibration problems in Engineering" 5th ed.

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-----Original Message-----
From: Norna A Robertson [mailto:norna@fastloki.stanford.edu]
Sent: 26 August 2003 16:16
To: Greenhalgh, RJS (Justin)
Subject: info on wires etc


Justin

Here are the numbers you are looking for.

Upper wires: length = 0.54 m, but NOT vertical - vertical height is 0.517m
with wires sloping inwards from half separation at blade tips of 0.25 m to
half separation at attachments to top mass of 0.09 m
radius = 700 micron

Uppermost blades  length = 0.48 m, width at base =9.5 cm, thickness =4.4
mm, uncoupled freq.=2.41 Hz, max. stress=951 MPa (shape factor used=1.36)

Let me know if you need anything else.

Norna
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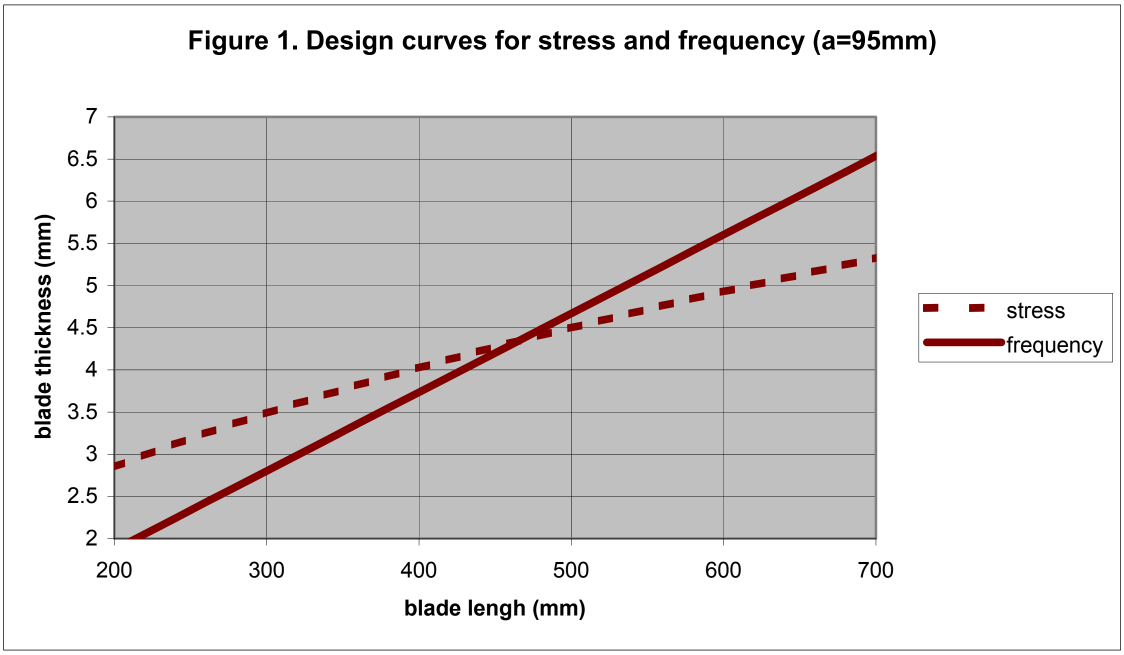


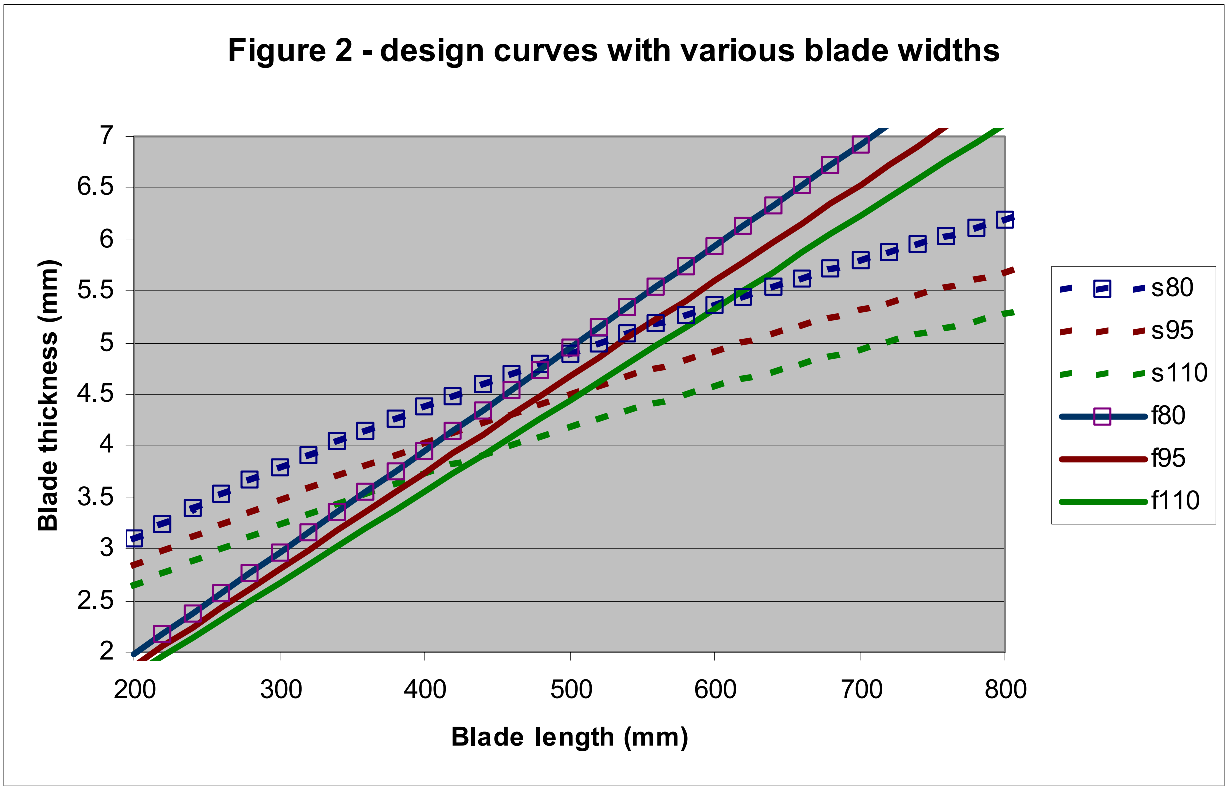


Figure 2. Design curves for various blade widths. The curves s80, s95, s110 refer to the stress constraint for widths of 80, 95 and 110 mm. The curves f80, f95, f110 are frequency constraints for the same widths.

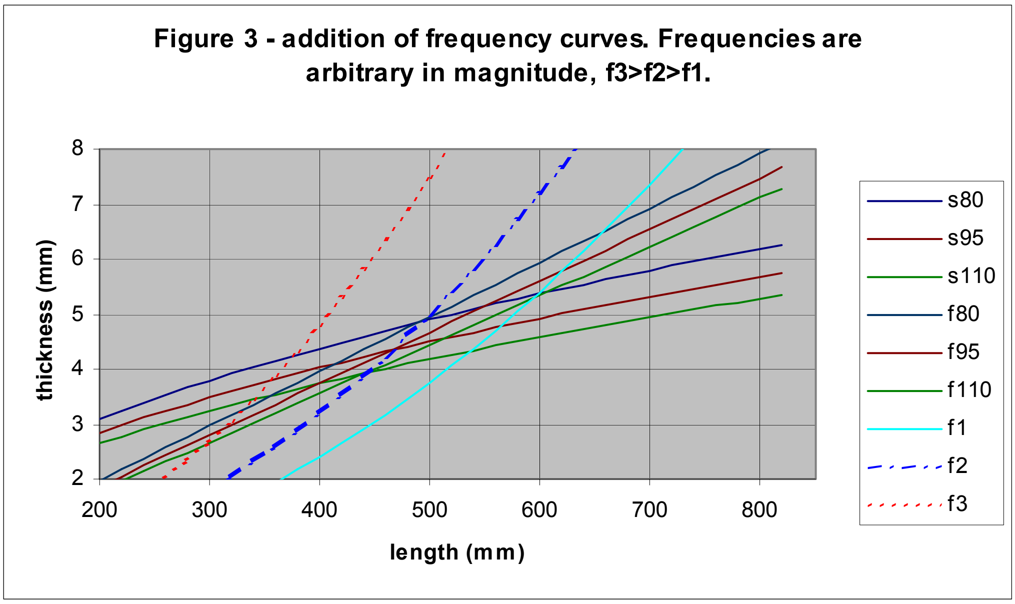
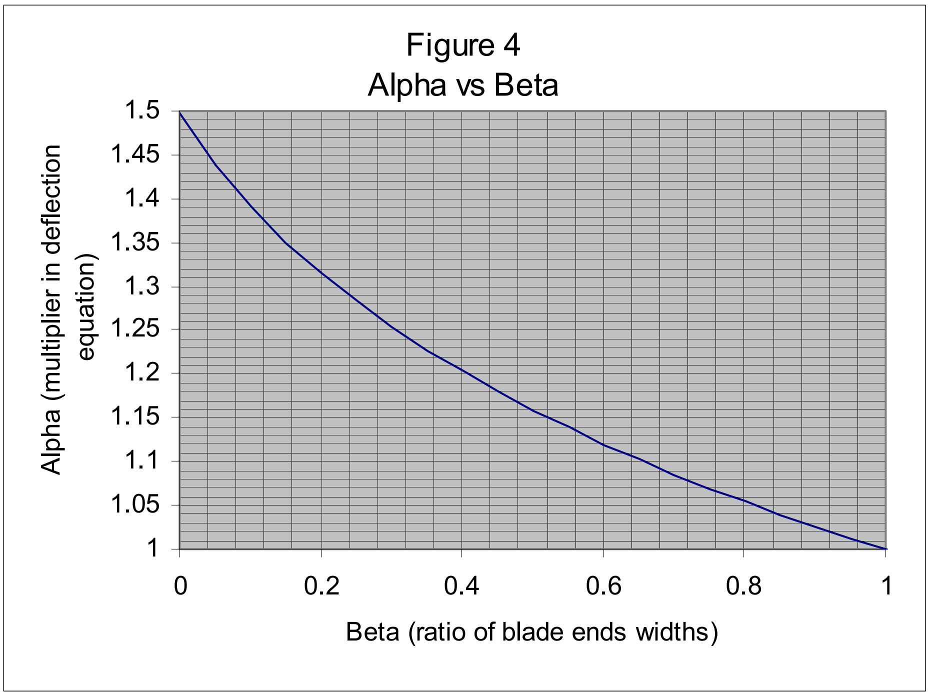
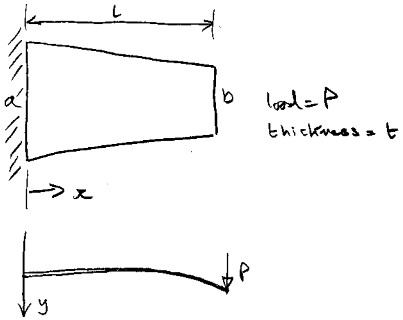


Figure 3. The lines s80, f80 etc are reproduced from figure2. The lines f1, f2 and f3 give constant values of internal mode frequency.



APPENDIX – derivation of the deflection formula fro a trapezoidal blade

①



$$I = \frac{wd^3}{12} = \frac{t^3}{12} \times \left(a - \frac{x}{L}(a-b)\right)$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{P(L-x)}{Et^3/12\,(a - x/L\,(a-b))}$$

$$= \frac{A - Bx}{C - Dx} \quad \begin{cases} A = PL & \text{[1a]} \\ B = P & \text{[1b]} \\ C = \frac{Et^3}{12} a & \text{[1c]} \\ D = \frac{Et^3}{12} \times \frac{a-b}{L} & \text{[1d]} \end{cases}$$

$$\frac{dy}{dx} = \frac{Bx}{D} + \frac{BC - AD \;\ln(C - Dx)}{D^2} + E \qquad [2]$$

$$y = \frac{Bx^2}{2D} + \frac{BC - AD}{D^2}\left(-x - \frac{C}{D}\ln(C - Dx) + x\ln(C - Dx)\right) + Ex + F \qquad [3]$$

② Check Integration:

$$y = \frac{Bx^2}{2D} + \frac{BC - AD}{D^2}\left(-x - \frac{C}{D}\ln(C - Dx) + x\ln(C - Dx)\right)$$

$$\frac{dy}{dx} = \frac{Bx}{D} + \frac{BC - AD}{D^2}\left(-1 + \frac{C}{\cancel{D}} \times \frac{+\cancel{D}}{C - Dx} + x \times \frac{-D}{C - Dx} + \ln(C - Dx)\right)$$

$$= \frac{Bx}{D} + \frac{BC - AD}{D^2}\left(\frac{-C + Dx + C - Dx}{C - Dx} + \ln(C - Dx)\right)$$

$$= \frac{Bx}{D} + \frac{BC - AD}{D^2}\left(\ln(C - Dx)\right)$$

$$y = \frac{B}{D} + \frac{BC - AD}{D^2} \times \frac{-D}{(C - Dx)}$$

$$= \frac{B\cancel{D}(C - Dx) - (BC - AD)\cancel{D}}{D^{\cancel{2}}\,(C - Dx)}$$

$$= \frac{B\cancel{C} - B\cancel{D}x - \cancel{B}C + A\cancel{D}}{\cancel{D}(C - Dx)}$$

$$= \frac{A - Bx}{C - Dx}$$

③ Boundary conditions.

$x=0,\ \frac{dy}{dx}=0$

$\Rightarrow$ from [2] $\quad 0 = 0 + \frac{Bc-AD}{D^2}\ln c + E$

$$E = -\frac{Bc-AD}{D^2}\ln c \qquad \text{[4]}$$

$x=0,\quad y=0$

$\Rightarrow$ from [3] $\quad 0 = 0 + \frac{Bc-AD}{D^2}\left(0 - \frac{c}{D}\ln c\right) + F$

$$F = \frac{Bc-AD}{D^2}\times\frac{c}{D}\ln c \qquad \text{[5]}$$

Deflection at the tip : Set $x=\ell$, use [4] & [5] in [3]

$$y_{tip} = \frac{B\ell^2}{2D} + \frac{Bc-AD}{D^2}\left(-\ell - \frac{c}{D}\ln(c-D\ell) + \ell\ln(c-D\ell)\right) - \frac{Bc-AD}{D^2}\ln(c)\ell + \frac{Bc-AD}{D^2}\frac{c}{D}\ln c$$

$$= \frac{B\ell^2}{2D} + \frac{Bc-AD}{D^2}\left[-\ell - \frac{c}{D}\ln(c-D\ell) + \ell\ln(c-D\ell) - \ell\ln(c) + \frac{c}{D}\ln(c)\right]$$

$$= \frac{B\ell^2}{2D} + \frac{Bc-AD}{D^2}\left[-\ell + \frac{c}{D}\left(\ln(c) - \ln(c-D\ell)\right) + \ell\left(\ln(c-D\ell) - \ln(c)\right)\right]$$

$$= \frac{B\ell^2}{2D} + \frac{Bc-AD}{D^2}\left[-\ell + \left(\ell - \frac{c}{D}\right)\left(\ln(c-D\ell) - \ln(c)\right)\right]$$

$$= \frac{B\ell^2}{2D} + \frac{Bc-AD}{D^2}\left[-\ell + \left(\ell - \frac{c}{D}\right)\ln\left(\frac{c-D\ell}{c}\right)\right] \qquad \text{[6]}$$

④ Substitute from [1a] etc.

$$\frac{B\ell^2}{2D} = \frac{P\ell^2}{(2Et^3/12)((a-b)/\ell)} = \frac{6P\ell^2\times\ell}{Et^3(a-b)} = \frac{6P\ell^3}{Et^3(a-b)} \qquad \text{[7]}$$

$$\frac{Bc-AD}{D^2} = \frac{P\times \frac{Et^3}{12}a - P\ell\times\frac{Et^3}{12}\times\frac{a-b}{\ell}}{\left(\frac{Et^3}{12}\right)^2\times\frac{(a-b)^2}{\ell^2}}$$

$$= \frac{Pa - P(a-b)}{\frac{Et^3}{12}\times\frac{a-b^2}{\ell^2}} = \frac{12Pb\ell^2}{Et^3(a-b)^2} \qquad \text{[8]}$$

$$\frac{c}{D} = \frac{a}{(a-b)/\ell} = \frac{a\ell}{a-b} \qquad \text{[9]}$$

$$\frac{c-D\ell}{c} = \frac{a-(a-b)}{a} = \frac{b}{a} \qquad \text{[10]}$$

Define $\beta = \frac{b}{a}$ [11]

Then $(a-b) = a(1-\beta)$ [12]

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⑤ substitute [7] – [12] in [6]

$$y_{tip} = \frac{6P\ell^3}{Et^3 a(1-\beta)} + \frac{12Pb\ell^2}{Et^3 a^2(1-\beta)^2}\left[-\ell + \left(\ell - \frac{aL}{a(1-\beta)}\right)\ln\beta\right]$$

$$= \frac{6P\ell^2}{Et^3 a}\left[\frac{L}{1-\beta} + \frac{2b}{a(1-\beta)^2}\left[-\ell + \ell\left(1-\frac{1}{1-\beta}\right)\ln\beta\right]\right]$$

$$= \frac{4P\ell^3}{Et^3 a}\,\alpha$$

where $\alpha = \frac{3}{2}\left[\frac{1}{1-\beta} + \frac{2\beta}{(1-\beta)^2}\left[-1 + \left(1-\frac{1}{(1-\beta)}\right)\ln\beta\right]\right]$

$$\downarrow$$

$$1 - \frac{1}{1-\beta} = \frac{1-\beta-1}{1-\beta} = \frac{-\beta}{1-\beta}$$

$$\alpha = \frac{3}{2(1-\beta)} - \frac{3\beta}{(1-\beta)^2}\left(1 + \frac{\beta\ln\beta}{1-\beta}\right)$$

$$= \frac{3}{2(1-\beta)}\left(1 - \frac{2\beta}{1-\beta}\left(1 + \frac{\beta\ln\beta}{1-\beta}\right)\right)$$

$$= \frac{3}{2(1-\beta)}\left(1 - \frac{2}{1-\beta}\left(\beta + \frac{\beta^2\ln\beta}{1-\beta}\right)\right)$$

or, for comparison with earlier results:

$$\alpha = \frac{3}{2(1-\beta)}\left(3 - 2\frac{(1-\beta)}{(1-\beta)} - \frac{2}{1-\beta}\left(\beta + \frac{\beta^2\ln\beta}{1-\beta}\right)\right)$$

$$= \frac{3}{2(1-\beta)}\left(3 - \frac{2}{1-\beta}\left(1 + \frac{\beta^2\ln\beta}{1-\beta}\right)\right) \quad \text{Q.E.D.}$$