LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Document Type Technical Note	LIGO-T040128-02-E	19 April 2006

Bias from power spectrum measurement in parameter estimation for the stochastic gravitational wave background

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Theoretical bias of parameters estimates for the stochastic background search

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revised : 2006.04 .19
LIGO - T040128 - 02 - E

Statement of the problem

In estimating the stochastic background parameter, Ω_{GW}, estimates of power spectra and cross-power spectra are calculated using finite stretches of data. Consequently, these estimates necessarily have some level of measurement uncertainty or error associated with them. In the literature of stochastic background estimation with gravitational wave data, it is assumed that noise power spectra and cross-correlation spectra are known *a priori*. In practice, these quantities are estimated by suitably averaging periodograms. In considering the effect of these measurement uncertainties on estimation of parameters, it is useful to start from the definitions of spectral estimates and their statistical properties and then to apply these results to the case of stochastic background estimation.

The effect of bias in the point estimate of $\Omega_{\text{measured}} \equiv \frac{Y}{T}$ is derived in detail, providing an example of the technique that can be used to determine bias in any experimental quantity derived in an analysis of cross-correlations looking for evidence of a stochastic GW signal. APPENDIX A provides a compilation of bias effects in other quantities of interest.

Power spectrum estimation and it statistical properties

Consider time series data streams x[i],y[i] containing NM samples and partitioned into N groups of M samples each. For each partition of M points (of duration T), the discrete Fourier transform (DFT) of the time series data is given by,

$$X_i[\ell] = \sum_{k=1}^{M} x[iM + k]\, e^{-2\pi i \frac{k\ell}{M}} \;;\; i = \{0, \ldots, N-1\};\; \ell = \{1, \ldots, M\}$$

where i is an index over epoch;
ℓ is an index over frequency. Periodogram estimates of power, cross – power, etc. follow.

$$\tilde{P}_{x_i}[\ell] = \frac{2}{M}\left| X_i[\ell] \right|^2;\; E[\tilde{P}_{x_i}[\ell]] \equiv P_x[\ell]$$

$$\text{Var}[\tilde{P}_{x_i}[\ell]] = E\left[\tilde{P}_{x_i}[\ell]^2\right] - P_x[\ell]^2 = P_x[\ell]^2 \;;$$

For a single instance of a power spectrum estimate or periodogram, the standard deviation equals the mean. In order to obtain suitably precise estimates of power, it is necessary to average a large number of periodograms taken over different epochs.

$$\hat{P}_x[\ell] \equiv \frac{1}{N}\sum_{i=1}^{N}\tilde{P}_{x_i}[\ell] = \frac{2}{NM}\sum_{i=1}^{N}\left|X_i[\ell]\right|^2, \text{ and}$$

$$\mathrm{Var}\left[\hat{P}_x[\ell]\right] = \frac{1}{N} P_x[\ell]^2,$$

Similar expressions hold for $Y_i[\ell]$.

Cross-power spectrum estimation and it statistical properties

Similarly, the cross-power spectrum is given by,

$$\tilde{P}_{ixy}[\ell, \ell'] = \frac{2}{M} X_i^*[\ell]\, Y_i[\ell'];\; E[\tilde{P}_{ixy}[\ell, \ell']] \equiv \delta(\ell - \ell')\, P_{xy}[\ell]$$

$$\mathrm{Var}[\tilde{P}_{ixy}[\ell, \ell']] = E[\,|\tilde{P}_{ixy}[\ell, \ell']\,|^2] - |\,P_{xy}[\ell]\,|^2 = \delta(\ell - \ell')\,(P_x[\ell]\, P_y[\ell] + |\,P_{xy}[\ell]\,|^2).$$

The $\delta(x)$ functions introduced above follow from the fact that different frequency bins are not correlated for stationary Gaussian noise. To simplify the presentation, this fact will be implictly used from now on by imposing the $\delta(x)$ without showing the step. By averaging over many epochs, an estimate of cross-power is attained,

$$\hat{P}_{xy}[\ell] = \frac{2}{NM}\sum_{i=1}^{N} X_i^*[\ell]\, Y_i[\ell], \text{ and}$$

$$\mathrm{Var}\left[\hat{P}_{xy}[\ell]\right] = \frac{1}{N}(P_x[\ell]\, P_y[\ell] + |\,P_{xy}[\ell]\,|^2).$$

Covariance between power and cross power estimates

Clearly, cross-power spectrum estimates and power spectrum estimates made from the same data set are correlated. The correlation is quantified by considering covariance of variance pairs of measurements,

$$\mathrm{Cov}\left[\hat{P}_{xy}[\ell]\, \hat{P}_x[\ell]\right] = E\left[\hat{P}_{xy}[\ell]\, \hat{P}_x[\ell]\right] - P_{xy}[\ell]\, P_x[\ell] = \frac{1}{N}(P_{xy}[\ell]\, P_x[\ell]) \text{ and}$$

$$\mathrm{Cov}\left[\hat{P}_{xy}[\ell]\, \hat{P}_y[\ell]\right] = \frac{1}{N}(P_{xy}[\ell]\, P_y[\ell]).$$

Covariance between power estimates may be written as,

$$\mathrm{Cov}[\tilde{P}_{ix}[\ell]\, \tilde{P}_{jy}[\ell]] = E[\tilde{P}_{ix}[\ell]\, \tilde{P}_{jy}[\ell]] - P_x[\ell]\, P_y[\ell] = E[\tilde{P}_{ix}[\ell]\, \tilde{P}_{jy}[\ell]] - |\,P_{xy}[\ell]\,|^2 + (|\,P_{xy}[\ell]\,|^2 - P_x[\ell]\, P_y[\ell])$$

$$= \delta_{ij}\,(E[\,|\,\tilde{P}_{ixy}[\ell]\,|^2] - |\,P_{xy}[\ell]\,|^2 + (|\,P_{xy}[\ell]\,|^2 - P_x[\ell]\, P_y[\ell]))$$

$$= \delta_{ij}\,(\mathrm{Var}[\tilde{P}_{ixy}[\ell]] + (|\,P_{xy}[\ell]\,|^2 - P_x[\ell]\, P_y[\ell]))$$

$$= \delta_{ij}\, 2\,|\,P_{xy}[\ell]\,|^2 \quad \text{and,}$$

$$\mathrm{Cov}\left[\hat{P}_x[\ell]\, \hat{P}_y[\ell]\right] = E\left[\hat{P}_x[\ell]\, \hat{P}_y[\ell]\right] - P_x[\ell]\, P_y[\ell]$$

$$= E\Big[\frac{2}{NM}\sum_{i=1}^{N}\left|X_i[\ell]\right|^2 \frac{2}{NM}\sum_{i=1}^{N}\left|Y_i[\ell]\right|^2\Big] - \left|P_{xy}[\ell]\right|^2 + (|\,P_{xy}[\ell]\,|^2 - P_x[\ell]\, P_y[\ell])$$

$$= E\Big[\frac{1}{N^2}\sum_{i=1}^{N}\hat{P}_{x_i}[\ell]\sum_{j=1}^{N}\hat{P}_{y_j}[\ell]\Big] - P_x[\ell]\, P_y[\ell]$$

$$= \frac{1}{N^2}\left(\sum_{i=1}^{N} E\left[\hat{P}_{x_i}[\ell]\,\hat{P}_{y_i}[\ell]\right] + \sum_{i\neq j=1}^{N} E\left[\hat{P}_{x_i}[\ell]\,\hat{P}_{y_j}[\ell]\right]\right) - P_x[\ell]\,P_y[\ell]$$

$$= \frac{1}{N^2}\left(2N\,|P_{xy}[\ell]|^2 + N\,P_x[\ell]\,P_y[\ell] + N(N-1)\,P_x[\ell]\,P_y[\ell]\right) - P_x[\ell]\,P_y[\ell]$$

$$\mathrm{Cov}\left[\hat{P}_x[\ell]\,\hat{P}_y[\ell]\right] = \frac{2}{N}\left|P_{xy}[\ell]\right|^2$$

Application to estimation of bias in E[$\frac{Y_i}{T}$]

CASE A:

Consider first a simplified analysis in which PSDs are estimated from N discrete segments, including the one for which the cross-correlation statistic, (CC), is also calculated. For simplicity in tracking the statistics in the analysis, assume further that no coarse graining of overlapping Hann PSD estimation is applied. The inclusion of such details will change the coefficients of the expansion results, but not the conclusions.

Now consider the statistic used in the stochastic background search,

$$Y_i/T = \frac{\sum_{\ell=1}^{N} \frac{\mathcal{D}[\ell]\,\tilde{s}_1^*[\ell]\,\tilde{s}_2[\ell]}{\hat{P}_1[\ell]\,\hat{P}_2[\ell]}}{\sum_{\ell=1}^{N} \frac{\mathcal{G}[\ell]}{\hat{P}_1[\ell]\,\hat{P}_2[\ell]}},$$

where $\mathcal{D}[\ell] = \frac{\gamma[\ell]}{f_\ell^{\,3}}$ and $\mathcal{G}[\ell] = \mathcal{D}[\ell]^2 = \frac{\gamma^2[\ell]}{f_\ell^{\,6}}$ are filter functions. T is the duration of the epoch i, $T \propto M$.

The quantity being estimate is normalized so that theoretically its mean value is exactly the quantity of interest, Ω_{GW}. However, we want to determine the effect of measurement uncertainty in the estimates of power and cross-power. We want to determine $E[Y_i/T]$ given that the estimators $\tilde{s}_1^*[\ell]\,\tilde{s}_2[\ell] = \frac{M}{2}\,\tilde{P}_{i12}[\ell]$, $\hat{P}_1[\ell]$ and $\hat{P}_2[\ell]$ each have statistical fluctuations and correlations among themselves.

To proceed, let $M\,\mathcal{C}_i \equiv \sum_{\ell=1}^{N} \frac{\mathcal{D}[\ell]\,\tilde{s}_{i1}^*[\ell]\,\tilde{s}_{i2}[\ell]}{\hat{P}_1[\ell]\,\hat{P}_2[\ell]}$, $\mathfrak{n} \equiv \frac{1}{\sum_{\ell=1}^{N} \frac{\mathcal{D}[\ell]^2}{\hat{P}_1[\ell]\,\hat{P}_2[\ell]}}$, then

$Y_i/T = \mathfrak{n}\,\mathcal{C}_i$. Assume that with sufficient number of averages the fluctuations in the denominators are sufficiently small to allow Taylor series expansion. Now consider first $\mathfrak{n}$,

$$\mathfrak{n} \equiv \frac{1}{\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2}{\hat{P}_1[\ell]\,\hat{P}_2[\ell]}} = \frac{1}{\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2}{P_1[\ell]\left(1+\frac{(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]}\right)P_2[\ell]\left(1+\frac{(\hat{P}_2[\ell]-P_2[\ell])}{P_2[\ell]}\right)}}$$

$$= 1\Bigg/\left(\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]} \times \left(1 - \frac{(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]} + \left(\frac{\hat{P}_1[\ell]-P_1[\ell]}{P_1[\ell]}\right)^2 + \ldots\right) \times \left(1 - \frac{(\hat{P}_2[\ell]-P_2[\ell])}{P_2[\ell]} + \left(\frac{\hat{P}_2[\ell]-P_2[\ell]}{P_2[\ell]}\right)^2 + \ldots\right)\right)$$

$$\approx 1 \Big/ \left(\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]} \times \left(1 - \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} - \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]\,P_2[\ell]^2}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} + \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]^2}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} + \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)^2}{P_1[\ell]^3\,P_2[\ell]}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} + \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)^2}{P_1[\ell]\,P_2[\ell]^3}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} \right) \right)$$

$$\approx \left(1 \Big/ \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]} \right) \left(1 + \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} + \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]\,P_2[\ell]^2}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} - \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]^2}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} + \frac{\sum_{\ell'=1}^{M} \frac{\mathfrak{D}[\ell']^2 \left(\hat{P}_1[\ell']-P_1[\ell']\right)}{P_1[\ell']^2\,P_2[\ell']} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]}}{\left(\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}\right)^2} + \frac{2 \sum_{\ell'=1}^{M} \frac{\mathfrak{D}[\ell']^2 \left(\hat{P}_1[\ell']-P_1[\ell']\right)}{P_1[\ell']^2\,P_2[\ell']} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]\,P_2[\ell]^2}}{\left(\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}\right)^2} + \frac{\sum_{\ell'=1}^{M} \frac{\mathfrak{D}[\ell']^2 \left(\hat{P}_2[\ell']-P_2[\ell']\right)}{P_1[\ell']\,P_2[\ell']^2} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]\,P_2[\ell]^2}}{\left(\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}\right)^2} - \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)^2}{P_1[\ell]^3\,P_2[\ell]}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} - \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)^2}{P_1[\ell]\,P_2[\ell]^3}}{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}} \right)$$

Defining $\bar{\mathfrak{n}} \equiv 1 \Big/ \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2}{P_1[\ell]\,P_2[\ell]}$

$$\mathfrak{n} \approx \bar{\mathfrak{n}} \left(1 + \bar{\mathfrak{n}} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]} + \bar{\mathfrak{n}} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]\,P_2[\ell]^2} - \bar{\mathfrak{n}} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]^2 \left(\hat{P}_1[\ell]-P_1[\ell]\right)\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_1[\ell]^2\,P_2[\ell]^2} + \right.$$

$$\overline{\pi}^2 \sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2 \left(\hat{P}_1[\ell'] - P_1[\ell']\right)}{P_1[\ell']^2 \, P_2[\ell']} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2 \left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]^2 \, P_2[\ell]} +$$

$$2\,\overline{\pi}^2 \sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2 \left(\hat{P}_1[\ell'] - P_1[\ell']\right)}{P_1[\ell']^2 \, P_2[\ell']} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2 \left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_1[\ell] \, P_2[\ell]^2} +$$

$$\overline{\pi}^2 \sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2 \left(\hat{P}_2[\ell'] - P_2[\ell']\right)}{P_1[\ell'] \, P_2[\ell']^2} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2 \left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_1[\ell] \, P_2[\ell]^2} -$$

$$\left. \overline{\pi} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2 \left(\hat{P}_1[\ell] - P_1[\ell]\right)^2}{P_1[\ell]^3 \, P_2[\ell]} - \overline{\pi} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^2 \left(\hat{P}_2[\ell] - P_2[\ell]\right)^2}{P_1[\ell] \, P_2[\ell]^3} \right) + O\left[\left(\hat{P}_i[\ell] - P_i[\ell]\right)^3\right]$$

The last equation above is the expansion to second order
in power spectrum fluctuations. Next consider the cross − correlation term,

$$M\,\mathcal{C}_i \equiv \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell] \, \tilde{s}_{i1}^*[\ell] \, \tilde{s}_{i2}[\ell]}{\hat{P}_1[\ell] \, \hat{P}_2[\ell]}.$$

Following the approach above, define $\overline{\mathcal{C}_i} \equiv \frac{1}{2} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell] \, P_{12}[\ell]}{P_1[\ell] \, P_2[\ell]}$, then express $\mathcal{C}_i$ as an expansion,

$$\mathcal{C}_i \equiv \frac{1}{2} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell] \; P_{12}[\ell] \left(1 + \frac{(\tilde{P}_{i12}[\ell] - P_{12}[\ell])}{P_{i12}[\ell]}\right)}{P_1[\ell] \left(1 + \frac{(\hat{P}_1[\ell] - P_1[\ell])}{P_1[\ell]}\right) P_2[\ell] \left(1 + \frac{(\hat{P}_2[\ell] - P_2[\ell])}{P_2[\ell]}\right)}$$

$$\mathcal{C}_i = \frac{1}{2} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell] \; P_{12}[\ell]}{P_1[\ell] \, P_2[\ell]} \times \left(1 + \frac{(\tilde{P}_{i12}[\ell] - P_{12}[\ell])}{P_{12}[\ell]}\right) \times \left(1 - \frac{\left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]} + \left(\frac{\left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]}\right)^2 + \ldots\right) \times$$

$$\left(1 - \frac{\left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_2[\ell]} + \left(\frac{\left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_2[\ell]}\right)^2 + \ldots\right)$$

$$\mathcal{C}_i \approx \frac{1}{2} \sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell] \; P_{12}[\ell]}{P_1[\ell] \, P_2[\ell]} \times \left(1 + \frac{(\tilde{P}_{i12}[\ell] - P_{12}[\ell])}{P_{i12}[\ell]} - \frac{\left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]} - \frac{\left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_2[\ell]} + \right.$$

$$\left(\frac{\left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]}\right)\left(\frac{\left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_2[\ell]}\right) + \left(\frac{\left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]}\right)^2 + \left(\frac{\left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_2[\ell]}\right)^2 -$$

$$\left. \frac{(\tilde{P}_{i12}[\ell] - P_{12}[\ell])}{P_{12}[\ell]} \frac{\left(\hat{P}_1[\ell] - P_1[\ell]\right)}{P_1[\ell]} - \frac{(\tilde{P}_{i12}[\ell] - P_{12}[\ell])}{P_{12}[\ell]} \frac{\left(\hat{P}_2[\ell] - P_2[\ell]\right)}{P_2[\ell]} \right) + O\left[\left(\hat{P}_i[\ell] - P_i[\ell]\right)^3\right]$$

Combining the expansions leads to,
$Y_i / T \approx$

$$\bar{n}\left(\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]}{P_1[\ell]\,P_2[\ell]}\times\left(1+\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}-\frac{(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]}-\frac{(\hat{P}_2[\ell]-P_2[\ell])}{P_2[\ell]}+\left(\frac{(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]}\right)\left(\frac{(\hat{P}_2[\ell]-P_2[\ell])}{P_2[\ell]}\right)+\left(\frac{(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]}\right)^2+\left(\frac{(\hat{P}_2[\ell]-P_2[\ell])}{P_2[\ell]}\right)^2-\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}\frac{(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]}-\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}\frac{(\hat{P}_2[\ell]-P_2[\ell])}{P_2[\ell]}\right)\right)\times$$

$$\left(1+\bar{n}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]^2\,P_2[\ell]}+\bar{n}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_2[\ell]-P_2[\ell])}{P_1[\ell]\,P_2[\ell]^2}-\bar{n}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_1[\ell]-P_1[\ell])(\hat{P}_2[\ell]-P_2[\ell])}{P_1[\ell]^2\,P_2[\ell]^2}+\bar{n}^2\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_1[\ell']-P_1[\ell'])}{P_1[\ell']^2\,P_2[\ell']}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]^2\,P_2[\ell]}+2\bar{n}^2\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_1[\ell']-P_1[\ell'])}{P_1[\ell']^2\,P_2[\ell']}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_2[\ell]-P_2[\ell])}{P_1[\ell]\,P_2[\ell]^2}+\bar{n}^2\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_2[\ell']-P_2[\ell'])}{P_1[\ell']\,P_2[\ell']^2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_2[\ell]-P_2[\ell])}{P_1[\ell]\,P_2[\ell]^2}-\bar{n}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_1[\ell]-P_1[\ell])^2}{P_1[\ell]^3\,P_2[\ell]}-\bar{n}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_2[\ell]-P_2[\ell])^2}{P_1[\ell]\,P_2[\ell]^3}\right)$$

Keeping terms to second order in products and cross – products of $(\hat{P}_i[\ell']-P_i[\ell'])(\hat{P}_j[\ell]-P_j[\ell])$, we get,

$$Y_i/T=\bar{n}\,\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]}{P_1[\ell]\,P_2[\ell]}\times\left(1+\bar{n}^2\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_1[\ell']-P_1[\ell'])}{P_1[\ell']^2\,P_2[\ell']}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_1[\ell]-P_1[\ell])}{P_1[\ell]^2\,P_2[\ell]}+2\bar{n}^2\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_1[\ell']-P_1[\ell'])}{P_1[\ell']^2\,P_2[\ell']}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_2[\ell]-P_2[\ell])}{P_1[\ell]\,P_2[\ell]^2}+\bar{n}^2\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_2[\ell']-P_2[\ell'])}{P_1[\ell']\,P_2[\ell']^2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^2(\hat{P}_2[\ell]-P_2[\ell])}{P_1[\ell]\,P_2[\ell]^2}+\bar{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_1[\ell']-P_1[\ell'])}{P_1[\ell']^2\,P_2[\ell']}-\bar{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2(\hat{P}_1[\ell']-P_1[\ell'])^2}{P_1[\ell']^3\,P_2[\ell']}-\right.$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_1[\ell']-P_1[\ell']\right)\left(\hat{P}_2[\ell']-P_2[\ell']\right)}{P_1[\ell']^2\,P_2[\ell']^2}+\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_2[\ell']-P_2[\ell']\right)}{P_1[\ell']\,P_2[\ell']^2}-$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_2[\ell']-P_2[\ell']\right)^2}{P_1[\ell']\,P_2[\ell']^3}+\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}-\frac{\left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]}-\frac{\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_2[\ell]}+$$

$$\frac{\left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]}\frac{\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_2[\ell]}+\left(\frac{\left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]}\right)^2+\left(\frac{\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_2[\ell]}\right)^2-$$

$$\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}\frac{\left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]}-\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}\frac{\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_2[\ell]}+$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_1[\ell']-P_1[\ell']\right)}{P_1[\ell']^2\,P_2[\ell']}\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}+$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_2[\ell']-P_2[\ell']\right)}{P_1[\ell']\,P_2[\ell']^2}\frac{(\tilde{P}_{i12}[\ell]-P_{12}[\ell])}{P_{12}[\ell]}-$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_1[\ell']-P_1[\ell']\right)}{P_1[\ell']^2\,P_2[\ell']}\frac{\left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]}-$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_1[\ell']-P_1[\ell']\right)}{P_1[\ell']^2\,P_2[\ell']}\frac{\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_2[\ell]}-$$

$$\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_2[\ell']-P_2[\ell']\right)}{P_1[\ell']\,P_2[\ell']^2}\frac{\left(\hat{P}_1[\ell]-P_1[\ell]\right)}{P_1[\ell]}-$$

$$\left.\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\left(\hat{P}_2[\ell']-P_2[\ell']\right)}{P_1[\ell']\,P_2[\ell']^2}\frac{\left(\hat{P}_2[\ell]-P_2[\ell]\right)}{P_2[\ell]}\right)$$

Next apply the expectation operator, E[...], to the fluctuating quantities,

$$E[Y_i/T]\;=\overline{n}\,\overline{C_i}\times\left(1+\overline{n}^2\sum_{\ell',\ell''=1}^{M}\frac{\mathcal{D}[\ell']^2\,\mathcal{D}[\ell'']^2\,E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)\left(\hat{P}_1[\ell'']-P_1[\ell'']\right)\right]}{P_1[\ell']^2\,P_2[\ell']\,P_2[\ell'']\,P_1[\ell'']^2}+\right.$$

$$2\,\overline{n}^2\sum_{\ell',\ell''=1}^{M}\frac{\mathcal{D}[\ell']^2\,\mathcal{D}[\ell'']^2\,E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)\left(\hat{P}_2[\ell'']-P_2[\ell'']\right)\right]}{P_1[\ell']^2\,P_2[\ell']\,P_1[\ell'']\,P_2[\ell'']^2}+$$

$$\overline{n}^2\sum_{\ell',\ell''=1}^{M}\frac{\mathcal{D}[\ell']^2\,\mathcal{D}[\ell'']^2\,E\left[\left(\hat{P}_2[\ell']-P_2[\ell']\right)\left(\hat{P}_2[\ell'']-P_2[\ell'']\right)\right]}{P_2[\ell']^2\,P_1[\ell']\,P_1[\ell'']\,P_2[\ell'']^2}+\overline{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\,E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)\right]}{P_1[\ell']^2\,P_2[\ell']}-$$

$$\bar{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)^2\right]}{P_1[\ell']^3\, P_2[\ell']}-\bar{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)\left(\hat{P}_2[\ell']-P_2[\ell']\right)\right]}{P_1[\ell']^2\, P_2[\ell']^2}+$$

$$\left.\bar{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, E\left[\left(\hat{P}_2[\ell']-P_2[\ell']\right)\right]}{P_1[\ell']\, P_2[\ell']^2}-\bar{n}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, E\left[\left(\hat{P}_2[\ell']-P_2[\ell']\right)^2\right]}{P_1[\ell']\, P_2[\ell']^3}\right)+$$

$$\bar{n}\left(\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E[(\tilde{P}_{i12}[\ell]-P_{12}[\ell])]}{P_1[\ell]\, P_2[\ell]\, P_{12}[\ell]}-\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_1[\ell]-P_1[\ell]\right)\right]}{P_1[\ell]^2\, P_2[\ell]}-\right.$$

$$\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_2[\ell]-P_2[\ell]\right)\right]}{P_1[\ell]\, P_2[\ell]^2}+$$

$$\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_1[\ell]-P_1[\ell]\right)\left(\hat{P}_2[\ell]-P_2[\ell]\right)\right]}{P_1[\ell]^2\, P_2[\ell]^2}+$$

$$\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_1[\ell]-P_1[\ell]\right)^2\right]}{P_1[\ell]^3\, P_2[\ell]}+\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_2[\ell]-P_2[\ell]\right)^2\right]}{P_1[\ell]\, P_2[\ell]^3}-$$

$$\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[(\tilde{P}_{i12}[\ell]-P_{12}[\ell])\left(\hat{P}_1[\ell]-P_1[\ell]\right)\right]}{P_1[\ell]^2\, P_2[\ell]\, P_{12}[\ell]}-$$

$$\frac{1}{2}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[(\tilde{P}_{i12}[\ell]-P_{12}[\ell])\left(\hat{P}_2[\ell]-P_2[\ell]\right)\right]}{P_1[\ell]\, P_2[\ell]^2\, P_{12}[\ell]}+$$

$$\bar{n}\,\frac{1}{2}\sum_{\ell,\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, \mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)(\tilde{P}_{i12}[\ell]-P_{12}[\ell])\right]}{P_1[\ell']^2\, P_2[\ell']\, P_1[\ell]\, P_2[\ell]\, P_{12}[\ell]}+$$

$$\bar{n}\,\frac{1}{2}\sum_{\ell,\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, \mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_2[\ell']-P_2[\ell']\right)(\tilde{P}_{i12}[\ell]-P_{12}[\ell])\right]}{P_1[\ell']\, P_2[\ell']^2\, P_1[\ell]\, P_2[\ell]\, P_{12}[\ell]}-$$

$$\bar{n}\,\frac{1}{2}\sum_{\ell,\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, \mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)\left(\hat{P}_1[\ell]-P_1[\ell]\right)\right]}{P_1[\ell']^2\, P_2[\ell']\, P_1[\ell]^2\, P_2[\ell]}-$$

$$\bar{n}\,\frac{1}{2}\sum_{\ell,\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, \mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_1[\ell']-P_1[\ell']\right)\left(\hat{P}_2[\ell]-P_2[\ell]\right)\right]}{P_1[\ell']^2\, P_2[\ell']\, P_1[\ell]\, P_2[\ell]^2}-$$

$$\bar{n}\,\frac{1}{2}\sum_{\ell,\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\, \mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_2[\ell']-P_2[\ell']\right)\left(\hat{P}_1[\ell]-P_1[\ell]\right)\right]}{P_1[\ell']\, P_2[\ell']^2\, P_1[\ell]^2\, P_2[\ell]}-$$

$$\bar{n}\,\frac{1}{2}\sum^{M}\frac{\mathcal{D}[\ell']^2\, \mathcal{D}[\ell]\; P_{12}[\ell]\, E\left[\left(\hat{P}_2[\ell']-P_2[\ell']\right)\left(\hat{P}_2[\ell]-P_2[\ell]\right)\right]}{\cdots}\Bigg)$$

All quantities E[...] can be evaluated by referring to the variance and covariance properties of power and cross-power spectrum estimates given earlier. Odd order estimates are identically zero for Gaussian random variables.

$$E[Y_i/T] =$$

$$\bar{n}\,\overline{C_i} \times \left(1 + 2\bar{n}^2 \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4\ |P_{12}[\ell']|^2}{P_1[\ell']^3\, P_2[\ell']^3} + \bar{n}^2 \frac{1}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4}{P_1[\ell']^2\, P_2[\ell']^2} + \bar{n}^2 \frac{1}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4}{P_1[\ell']^2\, P_2[\ell']^2} + \right.$$

$$\left. 0 - \bar{n} \frac{1}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2}{P_1[\ell']\, P_2[\ell']} - \bar{n} \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2\, P_{12}[\ell']^2}{P_1[\ell']^2\, P_2[\ell']^2} + 0 - \bar{n} \frac{1}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2}{P_1[\ell']\, P_2[\ell']}\right) +$$

$$\bar{n}\left(0 - 0 - 0 + \frac{1}{2}\,\frac{2}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]\ P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\, P_2[\ell]^2} + \frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]\, P_{12}[\ell]}{P_1[\ell]\, P_2[\ell]} + \frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]\ P_{12}[\ell]}{P_1[\ell]\, P_2[\ell]} - \right.$$

$$\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]\ P_{12}[\ell]}{P_1[\ell]\, P_2[\ell]} - \frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]\ P_{12}[\ell]}{P_1[\ell]\, P_2[\ell]} + \bar{n}\,\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]}{P_1[\ell]^2\, P_2[\ell]^2} + \bar{n}\,\frac{1}{2}\,\frac{1}{N}$$

$$\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]}{P_1[\ell]^2\, P_2[\ell]^2} - \bar{n}\,\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]}{P_1[\ell]^2\, P_2[\ell]^2} - \bar{n}\,\frac{1}{2}\,\frac{2}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]\ |P_{12}[\ell]\ |^2}{P_1[\ell]^3\, P_2[\ell]^3} -$$

$$\left. \bar{n}\,\frac{1}{2}\,\frac{2}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]\ |P_{12}[\ell]\ |^2}{P_1[\ell]^3\, P_2[\ell]^3} - \bar{n}\,\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]}{P_1[\ell]^2\, P_2[\ell]^2}\right)$$

$$E[Y_i/T] = \bar{n}\,\overline{C_i} \times \left(1 - \frac{2}{N} + 2\bar{n}^2 \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4\ P_{12}[\ell']^2}{P_1[\ell']^3\, P_2[\ell']^3} + \right.$$

$$\left. \bar{n}^2 \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4}{P_1[\ell']^2\, P_2[\ell']^2} - \bar{n} \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^2\, P_{12}[\ell']^2}{P_1[\ell']^2\, P_2[\ell']^2}\right) +$$

$$\bar{n}\left(\frac{1}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]\ P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\, P_2[\ell]^2} - \bar{n}\,\frac{2}{N}\sum_{\ell=1}^{M} \frac{\mathcal{D}[\ell]^3\ P_{12}[\ell]\ |P_{12}[\ell]\ |^2}{P_1[\ell]^3\, P_2[\ell]^3}\right)$$

Finally, we get the following for the bias in the estimate $\Omega_{GW} = Y_i/T$,

$$\text{Bias}[Y_i/T] =$$

$$E[Y_i/T] - \bar{n}\,\overline{C_i} = \bar{n}\,\overline{C_i} \times \left(-\frac{2}{N} + 2\bar{n}^2 \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4\ P_{12}[\ell']^2}{P_1[\ell']^3\, P_2[\ell']^3} + \bar{n}^2 \frac{2}{N}\sum_{\ell'=1}^{M} \frac{\mathcal{D}[\ell']^4}{P_1[\ell']^2\, P_2[\ell']^2} - \right.$$

$$\overline{\eta}\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\,P_{12}[\ell']^2}{P_1[\ell']^2\,P_2[\ell']^2}\Bigg)+$$

$$\overline{\eta}\,\frac{1}{N}\left(\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\,P_2[\ell]^2}-2\,\overline{\eta}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]^3\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^3\,P_2[\ell]^3}\right)$$

This result corresponds to the case where the estimates $\hat{P}_1[\ell]$ and $\hat{P}_2[\ell]$ were made including the epoch *i* for which Y_i is being calculated. We see that the leading term in the bias is $2/N$, plus a number of other terms involving the details of the spectrum. Thus even when $P_{12}[\ell]\to 0$, the bias remains.

Referring to the earlier expression for $\mathcal{C}_i$, $E[\mathcal{C}_i]\approx\overline{\mathcal{C}_i}+\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\,P_2[\ell]^2}$. This expression has a bias that depends on $<\text{signal}>^3$, and therefore is much smaller for weak signals. This corresponds to our earlier observation that $\mathcal{C}_i\propto\frac{Y/T}{\sigma^2}$ had much less bias.

The limiting case of noiseless injection for the H1 – H2 pair gives : $P_{12}[\ell]=P_1[\ell]=P_2[\ell]=\mathcal{D}[\ell]$. Further, all sums $\longrightarrow$ M. Also $\overline{\eta}\,\overline{\mathcal{C}_i}=1$ (for unity injected strength), $\overline{\eta}=\frac{1}{M}$; $\overline{\mathcal{C}_i}=M$. In this case,

$\text{Bias}[Y_i/T]=$

$$E[Y_i/T]-\overline{\eta}\,\overline{\mathcal{C}_i}=1\times\left(-\frac{2}{N}+\frac{6}{MN}-\frac{2}{N}\right)+\frac{1}{N}\left(1-\frac{2}{M}\right)=-\frac{3}{N}+\frac{4}{NM}\longrightarrow-\frac{3}{N}+O\left[\frac{1}{M}\right],$$

which is what has been observed via Monte Carlo injections.

CASE B:

Consider next another simplified analysis in which PSDs are estimated from N discrete segments, but now these *exclude* the segment for which the cross-correlation statistic, (CC), is calculated. Once again, for simplicity in tracking the statistics in the analysis, assume further that no coarse graining of overlapping Hann PSD estimation is applied. The inclusion of such details will change the coefficients of the expansion results, but not the conclusions.

In the case where the epoch *i* is *not* included, a number of terms from Case A are now zero. The answer becomes,

$E[Y_i/T]=$

$$\overline{\eta}\,\overline{\mathcal{C}_i}\times\Bigg(1+2\,\overline{\eta}^2\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^4\,|P_{12}[\ell']|^2}{P_1[\ell']^3\,P_2[\ell']^3}+\overline{\eta}^2\,\frac{1}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^4}{P_1[\ell']^2\,P_2[\ell']^2}+\overline{\eta}^2\,\frac{1}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^4}{P_1[\ell']^2\,P_2[\ell']^2}+$$

$$0-\overline{\eta}\,\frac{1}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2}{P_1[\ell']\,P_2[\ell']}-\overline{\eta}\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2\,P_{12}[\ell']^2}{P_1[\ell']^2\,P_2[\ell']^2}+0-\overline{\eta}\,\frac{1}{N}\sum_{\ell'=1}^{M}\frac{\mathcal{D}[\ell']^2}{P_1[\ell']\,P_2[\ell']}\Bigg)+$$

$$\overline{\eta}\Bigg(0-0-0+\frac{1}{2}\,\frac{2}{N}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\,P_2[\ell]^2}+\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]}{P_1[\ell]\,P_2[\ell]}+\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathcal{D}[\ell]\,P_{12}[\ell]}{P_1[\ell]\,P_2[\ell]}-$$

$$\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]\,P_{12}[\ell]}{P_1[\ell]\,P_2[\ell]} - \frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]\,P_{12}[\ell]}{P_1[\ell]\,P_2[\ell]} + \overline{\mathfrak{n}}\,\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]}{P_1[\ell]^2\,P_2[\ell]^2} + \overline{\mathfrak{n}}\,\frac{1}{2}\,\frac{1}{N}$$

$$\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]}{P_1[\ell]^2\,P_2[\ell]^2} - \overline{\mathfrak{n}}\,\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]}{P_1[\ell]^2\,P_2[\ell]^2} - \overline{\mathfrak{n}}\,\frac{1}{2}\,\frac{2}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^3\,P_2[\ell]^3} -$$

$$\overline{\mathfrak{n}}\,\frac{1}{2}\,\frac{2}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^3\,P_2[\ell]^3} - \overline{\mathfrak{n}}\,\frac{1}{2}\,\frac{1}{N}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]}{P_1[\ell]^2\,P_2[\ell]^2}\Bigg)$$

$$E[Y_i/T] = \overline{\mathfrak{n}}\,\overline{C_i}\times\Bigg(1 - \frac{2}{N} + 2\,\overline{\mathfrak{n}}^2\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^4\,P_{12}[\ell']^2}{P_1[\ell']^3\,P_2[\ell']^3} +$$

$$\overline{\mathfrak{n}}^2\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^4}{P_1[\ell']^2\,P_2[\ell']^2} + -\overline{\mathfrak{n}}\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^2\,P_{12}[\ell']^2}{P_1[\ell']^2\,P_2[\ell']^2}\Bigg) +$$

$$\overline{\mathfrak{n}}\,\frac{1}{N}\Bigg(2\,\overline{C_i} + \sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\,P_2[\ell]^2} - \overline{2\,\mathfrak{n}}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^3\,P_2[\ell]^3} +$$

$$\overline{\mathfrak{n}}^2\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^4}{P_1[\ell']^2\,P_2[\ell']^2} - \overline{\mathfrak{n}}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]}{P_1[\ell]^2\,P_2[\ell]^2}\Bigg)$$

$$\text{Bias}[Y_i/T] = E[Y_i/T] - \overline{\mathfrak{n}}\,\overline{C_i} =$$

$$\overline{\mathfrak{n}}\,\overline{C_i}\times\Bigg(2\,\overline{\mathfrak{n}}^2\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^4\,P_{12}[\ell']^2}{P_1[\ell']^3\,P_2[\ell']^3} + \overline{\mathfrak{n}}^2\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^4}{P_1[\ell']^2\,P_2[\ell']^2} - \overline{\mathfrak{n}}\,\frac{2}{N}\sum_{\ell'=1}^{M}\frac{\mathfrak{D}[\ell']^2\,P_{12}[\ell']^2}{P_1[\ell']^2\,P_2[\ell']^2}\Bigg) +$$

$$\overline{\mathfrak{n}}\,\frac{1}{N}\Bigg(\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^2\,P_2[\ell]^2} - 2\,\overline{\mathfrak{n}}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]\,|P_{12}[\ell]|^2}{P_1[\ell]^3\,P_2[\ell]^3} - \overline{\mathfrak{n}}\sum_{\ell=1}^{M}\frac{\mathfrak{D}[\ell]^3\,P_{12}[\ell]}{P_1[\ell]^2\,P_2[\ell]^2}\Bigg),$$

It is seen that by excluding epoch *i* from the estimation of power spectra, the leading $2/N$ term from the earlier expression is cancelled by an equal and opposite term that was not present before. Now all remaining bias correction terms are proportional to either $P_{12}[\ell]$ and thus become negligible if $P_{12}[\ell] \to 0$, or they are proportional to $\overline{\mathfrak{n}}^2$, which also reduces them to the point where they can be ignored. Note however, that there remains a bias correction when a signal is present and if that signal is large compared to the detector power spectra.

In the limiting case of noiseless injection for the H1 − H2 pair : $P_{12}[\ell] = P_1[\ell] = P_2[\ell] = \mathfrak{D}[\ell]$, and all sums $\longrightarrow$ M. Also $\overline{\mathfrak{n}}\,\overline{\mathfrak{C}_i} = 1$ (for unity injected strength), $\overline{\mathfrak{n}} = \frac{1}{M}$; $\overline{\mathfrak{C}_i} = M$:

$$\text{Bias}[Y_i/T] =$$

$$E[Y_i/T] - \overline{\mathfrak{n}}\,\overline{\mathfrak{C}_i} = 1 \times \left(\frac{6}{MN} - \frac{2}{N}\right) + \frac{1}{N}\left(1 - \frac{2}{M}\right) = -\frac{1}{N} + \frac{4}{NM} \longrightarrow -\frac{1}{N} + O\left[\frac{1}{M}\right],$$

This test has now been performed via Monte Carlo injections and the bias for this case is indeed observed at a strength that is 1 / 3 the bias for Case A above.

Conclusions

It has been shown that the source of bias seen in the extraction of injected signal strength using the estimator for Ω_{GW} can be understood as coming from two contributions: estimation errors in the power and cross-power spectra and, in Case A, correlations that persist between power and cross-power spectra measured with the same data set. By suitably excluding the epoch for which cross-correlation is measured from the data with which power spectra are estimated, the residual bias can then be made proportional to signal strength. Thus, unless and until we are in a regime where the detector power spectra are signal-dominated, the residual bias is not an issue.

The effects discussed here are intrinsic to the use of measured estimates in the place of theoretical quantities in expression for optimal signal filters. Moreover, the effects may be exacerbated by the non-linear manner (e.g., 1/P) in which the measured quantities are used in constructing filters and normalization factors. In any such application, care must be taken to understand and analyze the sources of bias to determine how to minimize them to an acceptable level that does not affect the answer. Places where similar bias effects may be present and non-negligible are transfer function estimates and ratios of spectra or derived quantities, such as those used to determine calibrations. Of course, optimal filters used in other GW searches may also be affected by bias such as that discussed in this note.

APPENDIX A.
Evaluation of bias in other quantities of interest in the stochastic search

The technique used above can be applied to all other quantities of interest to the stochasticsearch. Below, expressions for σ, σ^2, $\frac{\Omega}{\sigma}$, and $\left(\frac{\Omega}{\sigma}\right)^2$ are given.The derivations are omitted since they follow straight forwardly, but tediously, by applying the same method to the starting expressions.In the notation used below, $\epsilon\tilde{Z} \equiv Z - \overline{\overline{Z}} << \overline{Z}$, is the fluctuating part of the measured quantity Z and $< \epsilon\tilde{Z} > = 0$. Z is one of the quantities $P_{1,2}$ or P_{12} .The case where estimates of $P_{1,2}$ do not include the epoch during which P_{12} is estimated will be denoted as ***non − overlapping*** throughout the discussion below; the other case, when measurements of $P_{1,2}$ do include the same epoch when P_{12} is estimated will be denoted as ***overlapping***.

Standard deviations σ and variance σ^2

$$\sigma_{\text{measured}} = \sqrt{\left\langle \frac{1}{\sum_{\mu=1}^{M} \frac{\mathfrak{D}[\mu]^2}{P_1[\mu]\left(1 + \frac{(\epsilon\hat{P}_1[\mu])}{P_1[\mu]}\right) P_2[\mu]\left(1 + \frac{(\epsilon\hat{P}_2[\mu])}{P_2[\mu]}\right)}} \right\rangle}$$

Which, to first order in 1 / N, evaluates to $\sigma_{\text{measured}} \approx \sigma_{\text{true}}\left(1 - \frac{1}{N}\right)$
for both the case of overlapping and non − overlapping estimations of the power and cross − power spectra.

Averaged point estimates, $\overline{Y} = \frac{\sum_{i=1}^{K} \frac{1}{\sigma_i^2} Y_i}{\sum_{i=1}^{K} \frac{1}{\sigma_i^2}}$

$$\overline{Y}_{\text{measured}} = \frac{1}{2}\left\langle \frac{\sum_{i=1}^{K}\left(\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]\, P_{i12}[\ell]\left(1 + \delta\,\frac{(\epsilon\tilde{P}_{i12}[\ell])}{P_{i12}[\ell]}\right)}{P_{i1}[\ell]\left(1 + \frac{(\epsilon\hat{P}_{i1}[\ell])}{P_{i1}[\ell]}\right) P_{i2}[\ell]\left(1 + \frac{(\epsilon\hat{P}_{i2}[\ell])}{P_{i2}[\ell]}\right)}\right)}{\sum_{i=1}^{K}\left(\sum_{\mu=1}^{M} \frac{\mathfrak{D}[\mu]^2}{P_{i1}[\mu]\left(1 + \frac{(\epsilon\hat{P}_{i1}[\mu])}{P_{i1}[\mu]}\right) P_{i2}[\mu]\left(1 + \frac{(\epsilon\hat{P}_{i2}[\mu])}{P_{i2}[\mu]}\right)}\right)} \right\rangle$$

Which, to first order in 1 / N, evaluates to $\overline{Y}_{\text{measured}} \approx \overline{Y}_{\text{true}}\left(1 - \frac{2}{N}\right)$
for the case of overlapping and $\overline{Y}_{\text{measured}} \approx \overline{Y}_{\text{true}}$ for non − overlapping estimations of the power and cross − power spectra.

Variances σ^2 and normalization of Y, $\mathcal{N} \propto \sigma^2$

$$\sigma^2{}_{\text{measured}} = \left\langle \frac{1}{\sum_{\mu=1}^{M} \frac{\mathfrak{D}[\mu]^2}{P_1[\mu]\left(1 + \frac{(\epsilon\hat{P}_1[\mu])}{P_1[\mu]}\right) P_2[\mu]\left(1 + \frac{(\epsilon\hat{P}_2[\mu])}{P_2[\mu]}\right)}} \right\rangle$$

Which, to first order in 1 / N, evaluates to $\sigma^2{}_{\text{measured}} \approx \sigma^2{}_{\text{true}} \left(1 - \frac{2}{N}\right)$
for both the case of overlapping and non − overlapping estimations of the power and cross − power spectra.

Normalized deviates $\frac{\Omega}{\sigma}$

$$\left(\frac{\Omega}{\sigma}\right)_{\text{measured}} = \left\langle \frac{1}{2} \frac{\sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]\ P_{12}[\ell]\left(1 + \frac{(\epsilon\tilde{P}_{i12}[\ell])}{P_{i12}[\ell]}\right)}{P_1[\ell]\left(1 + \frac{(\epsilon\hat{P}_1[\ell])}{P_1[\ell]}\right)P_2[\ell]\left(1 + \frac{(\epsilon\hat{P}_2[\ell])}{P_2[\ell]}\right)}}{\sqrt{\sum_{\mu=1}^{M} \frac{\mathfrak{D}[\mu]^2}{P_1[\mu]\left(1 + \frac{(\epsilon\hat{P}_1[\mu])}{P_1[\mu]}\right)P_2[\mu]\left(1 + \frac{(\epsilon\hat{P}_2[\mu])}{P_2[\mu]}\right)}}} \right\rangle$$

Which, to first order in1 / N, evaluates to $\left(\frac{\Omega}{\sigma}\right)_{\text{measured}} \approx \left(\frac{\Omega}{\sigma}\right)_{\text{true}} \left(1 - \frac{1}{N}\right)$
for the case of overlapping and $\left(\frac{\Omega}{\sigma}\right)_{\text{measured}} \approx$
$\left(\frac{\Omega}{\sigma}\right)_{\text{true}} \left(1 + \frac{1}{N}\right)$ for non − overlapping estimations of the power and cross − power spectra.

Variance of deviates $\left(\frac{\Omega}{\sigma}\right)^2$

$$\left(\frac{\Omega}{\sigma}\right)^2_{\text{measured}} =$$

$$\left\langle \frac{1}{4} \frac{\sum_{\lambda=1}^{M} \frac{\mathfrak{D}[\lambda]\ P_{12}[\lambda]\left(1 + \frac{(\epsilon\tilde{P}_{i12}[\lambda])}{P_{i12}[\lambda]}\right)}{P_1[\lambda]\left(1 + \frac{(\epsilon\hat{P}_1[\lambda])}{P_1[\lambda]}\right)P_2[\lambda]\left(1 + \frac{(\epsilon\hat{P}_2[\lambda])}{P_2[\lambda]}\right)} \sum_{\ell=1}^{M} \frac{\mathfrak{D}[\ell]\ P_{12}[\ell]\left(1 + \frac{(\epsilon\tilde{P}_{i12}[\ell])}{P_{i12}[\ell]}\right)}{P_1[\ell]\left(1 + \frac{(\epsilon\hat{P}_1[\ell])}{P_1[\ell]}\right)P_2[\ell]\left(1 + \frac{(\epsilon\hat{P}_2[\ell])}{P_2[\ell]}\right)}}{\sum_{\mu=1}^{M} \frac{\mathfrak{D}[\mu]^2}{P_1[\mu]\left(1 + \frac{(\epsilon\hat{P}_1[\mu])}{P_1[\mu]}\right)P_2[\mu]\left(1 + \frac{(\epsilon\hat{P}_2[\mu])}{P_2[\mu]}\right)}} \right\rangle$$

Which, to first order in1 / N, evaluates to $\left(\frac{\Omega}{\sigma}\right)^2_{\text{measured}} \approx \left(\frac{\Omega}{\sigma}\right)^2_{\text{true}} \left(1 - \frac{2}{N}\right) + \frac{1}{N}$
for the case of overlapping and $\left(\frac{\Omega}{\sigma}\right)^2_{\text{measured}} \approx \left(\frac{\Omega}{\sigma}\right)^2_{\text{true}} \left(1 + \frac{2}{N}\right) + \frac{2}{N}$
for non − overlapping estimations of the power and cross − power spectra.

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