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Transverse Vibrations of Tapered Cantilever Beams with End Loads

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Free vibrations of nonuniform cantilever beams with an end mass have been investigated, using the equations of Bernoulli–Euler. Two configurations of interest are treated: (a) constant width and linearly variable thickness; and (b) constant thickness and linearly variable width. Charts have been plotted for each case from which the fundamental frequency, the second harmonic, and the third harmonic can be easily determined for various taper ratios and ratios of end mass to beam mass. The Tables from which these charts were plotted are also included.

INTRODUCTION

TAPERED cantilever beams with and without end loads are often used for electrical contacts and for springs in electromechanical devices. For either application, it is important to be able to determine easily and accurately the natural frequencies of vibration. The purpose of this paper is to make design information readily available for natural frequencies for these types of beams.

Several papers have dealt with the free vibrations of tapered cantilever beams without end loads. The ones most applicable to design have been those by Cranch and Adler,[1] MacDuff and Felgar,[2] and Housner and Keightley,[3] Siddall and Isakson,[4] and Pinney[5] have also investigated the problem.

As far as the authors have been able to discover, engineering data for natural frequencies for the general case of a tapered cantilever beam with end load are not available in the literature. Houbolt and Anderson,[6] in a paper on the use of the iteration method for determining frequencies of nonuniform beams, present, as an example, a particular case of a nonuniform cantilever with intermediate load.

In this paper, the free vibrations of a cantilever beam with end load and (a) constant width and linearly variable thickness and (b) constant thickness and linearly variable width have been investigated, using Bernoulli-Euler equations. These equations neglect the effects of rotary inertia and shear, and give accurate results for cases in which the wavelength under consideration is large as compared with the lateral dimension of the beam. See, for example, Kolsky.[7] The equations were solved on a computer and curves were plotted from which fundamental frequency, second harmonic, and third harmonic can be determined for various taper ratios and ratios of end mass to beam mass. The methods presented yield accuracies to four significant figures in the Tables.

BEAM OF CONSTANT WIDTH AND LINEARLY VARIABLE THICKNESS

For the beam shown in Fig. 1 with end mass M and beam mass m, it is assumed that the end mass is concentrated at the free end of the beam. The thickness h at a distance x from the free end is given by

$$h=h_1+(h_0-h_1)(x/l). \tag{1}$$

[1] E. T. Cranch and A. A. Adler, "Bending Vibrations of Variable Section Beams," J. Appl. Mech. 23, 103–108 (1956).

[2] J. N. MacDuff and R. P. Felgar, "Vibration Design Charts," ASME Trans. 79, 1459–1475 (1957).

[3] G. W. Housner and W. O. Keightley, "Vibrations of Linearly Tapered Cantilever Beams," J. Eng. Mech. Div. Am. Soc. Civil Engrs. 88, 95–123 (Apr. 1962).

[4] J. N. Siddall and G. Isakson, "Approximate Analytical Methods for Determining Natural Modes and Frequencies of Vibration," Mass. Inst. Technol. Rept. ONR Proj. NR-035-259, pp. 141–146 (Jan. 1951).

[5] E. Pinney, "Vibration Modes of Tapered Beams," Am. Math. Monthly 54, 391–394 (1947).

[6] J. C. Houbolt and R. A. Anderson, "Calculations of Uncoupled Modes and Frequencies in Bending or Torsion of Nonuniform Beams," NACA TN-1522, pp. 19, 40, 45, 71 (Feb. 1948).

[7] H. Kolsky, *Stress Waves in Solids* (Oxford University Press, New York, 1953), pp. 52, 71.

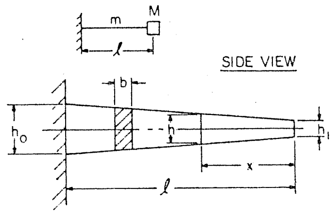

FIG. 1. Cantilever beam of constant width and linearly variable thickness.

The Bernoulli-Euler equation of a vibrating beam is

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)=-\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2}, \tag{2}$$

where $\rho A/g$ is the mass per unit length.

This equation applies to a beam with any type of support with the appropriate boundary conditions.

For a cantilever beam with end load, the following boundary conditions hold:

At $x=0$:

$$\partial^2 y/\partial x^2=0;$$
$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial^2 x}\right)=-V=-M\frac{\partial^2 y}{\partial t^2},\ \text{where } M \text{ is the end mass.} \tag{3a}$$

At $x=l$:

$$\partial y/\partial x=0 \quad\text{and}\quad y=0. \tag{3b}$$

Assuming a sustained free vibration at a frequency ω of

$$y(x,t)=z(x)\sin\omega t,$$

Eq. (2) becomes

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 z}{dx^2}\right)=\left(\frac{\rho A}{g}\right)\omega^2 z. \tag{4}$$

For the beam tapered as shown,

$$I=\tfrac{1}{12}bh^3=(b/12)[h_1+(h_0-h_1)x/l]^3,$$
$$A=bh=b[h_1+(h_0-h_1)x/l],$$

where I is the moment of inertia and A is the cross-sectional area.

Substituting in Eq. (4),

$$E\frac{d^2}{dx^2}\left\{\frac{b}{12}\left[h_1+(h_0-h_1)\frac{x}{l}\right]^3\frac{d^2 z}{dx^2}\right\}$$
$$=\rho\frac{b\omega^2}{g}\left[h_1+(h_0-h_1)\frac{x}{l}\right]z. \tag{5}$$

With X defined by

$$X=h_1+(h_0-h_1)x/l,$$

Eq. (5) becomes

$$d^2/dX^2[X^3(d^2z/dX^2)]=k^4Xz, \tag{6}$$

where

$$k^4=12\rho\omega^2 l^4/Eg(h_0-h_1)^4.$$

Introducing the linear operator D such that

$$D=(1/X)(d/dX)[X^2(d/dX)],$$

Eq. (6) can be put in the form

$$(D+k^2)(D-k^2)z=0,$$

which, from the theory of linear operators, has a solution obtained from

$$(D+k^2)z=0 \quad\text{and}\quad (D-k^2)\overset{z}{\cancel{Z}}=0,$$

or

$$(d/dX)[X^2(dz/dX)]\pm k^2Xz=0. \tag{7}$$

Using Eq. (7), Eq. (6) has the general solution

$$z(X)=\frac{1}{\sqrt{X}}[AJ_1(2k\sqrt{X})+BY_1(2k\sqrt{X})$$
$$+CI_1(2k\sqrt{X})+DK_1(2k\sqrt{X})], \tag{8}$$

where J_1 and Y_1 are first-order Bessel functions of the first and second kind, and I_1 and K_1 are first-order modified Bessel functions of the first and second kind.

The boundary conditions, Eqs. (3), transform into:

At $x=0$ or $X=h_1$:

$$d^2z/dX^2=0$$
$$\left.\frac{d^3z}{dX^3}\right)_{X=h_1}=\left.\frac{k^4Rz}{2h_1}[\alpha^2-1]\right)_{h_1}, \tag{9a}$$

where

$$R=M/m \quad\text{and}\quad \alpha=h_0/h_1.$$

At $x=l$ or $X=h_0$:

$$\frac{dz}{dX}=0 \quad\text{and}\quad z=0. \tag{9b}$$

Imposing the boundary conditions, Eqs. (9), on the general solution, Eq. (8), gives the following determinental equation for obtaining the natural frequencies.

$$\begin{vmatrix} J_1(2S\sqrt{\alpha}) & Y_1(2S\sqrt{\alpha}) & I_1(2S\sqrt{\alpha}) & K_1(2S\sqrt{\alpha}) \\ -J_0(2S\sqrt{\alpha}) & -Y_0(2S\sqrt{\alpha}) & -I_0(2S\sqrt{\alpha}) & K_0(2S\sqrt{\alpha}) \\ \left[\left(\frac{2-S^2}{S}\right)J_1(2S)-2J_0(2S)\right] & \left[\left(\frac{2-S^2}{S}\right)Y_1(2S)-2Y_0(2S)\right] & \left[\left(\frac{2-S^2}{S}\right)I_1(2S)-2I_0(2S)\right] & \left[\left(\frac{2+S^2}{S}\right)K_1(2S)+2K_0(2S)\right] \\ \left\{\left[S^4(\alpha^2-1)\frac{R}{2}+4S^2-6\right]J_1(2S) + S(6-S^2)J_0(2S)\right\} & \left\{\left[S^4(\alpha^2-1)\frac{R}{2}+4S^2-6\right]Y_1(2S) + S(6-S^2)Y_0(2S)\right\} & \left\{\left[S^4(\alpha^2-1)\frac{R}{2}-6-4S^2\right]I_1(2S) + S(6+S^2)I_0(2S)\right\} & \left\{\left[S^4(\alpha^2-1)\frac{R}{2}-6-4S^2\right]K_1(2S) - S(6+S^2)K_0(2S)\right\} \end{vmatrix} = 0, \quad (10)$$

where $S=k\sqrt{H_1}$ and $S\sqrt{\alpha}=k\sqrt{h_0}$.

The determinant was solved for $2k\sqrt{h_1}$ for various values of R and α for the fundamental, second-harmonic, and third-harmonic frequencies.

Solving

$$k^4=12\rho\omega^2 l^4/Eg(h_0-h_1)^4$$

for ω and making the substitution $\alpha=h_0/h_1$ gives

$$\omega=[k^2h_1^2(\alpha-1)^2/l^2](Eg/12\rho)^{\frac{1}{2}}$$

or

$$\omega(12\rho/Eg)^{\frac{1}{2}}=[2k\sqrt{h_1}(\alpha-1)/2]^2\cdot[h_1/l^2], \quad (11)$$

where ρ = weight density, lb/in^3.

Knowing the values of $2k\sqrt{h_1}$ for various values of α and R from the solution of the determinant, it is possible to build up a Table to give the factor $[2k\sqrt{h_1}\times(\alpha-1)/2]^2$. These data are given in Table I. From this Table, $\omega(12\rho/Eg)^{\frac{1}{2}}$ vs R was plotted for given values of α for $h_1/l^2=1$. Plots for the fundamental frequency, second harmonic, and third harmonic are given in Figs. 3, 4, and 5, respectively. For values of h_1/l^2 other

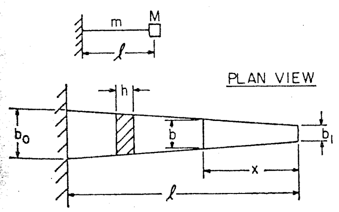

FIG. 2. Cantilever beam of constant thickness and linearly variable width.

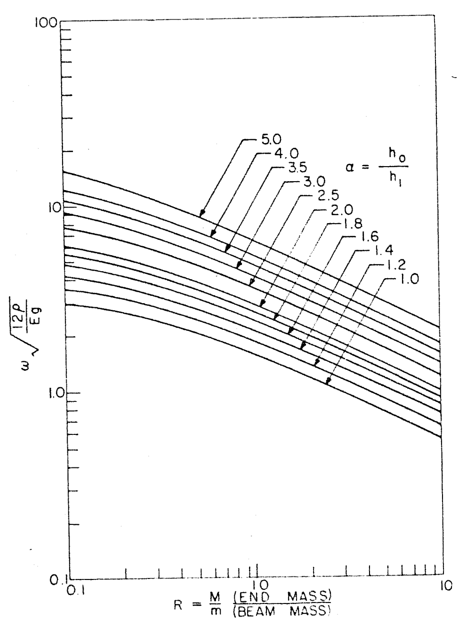

FIG. 3. Fundamental frequency for cantilever beam of constant width and variable thickness with $h_1/l^2=1$.

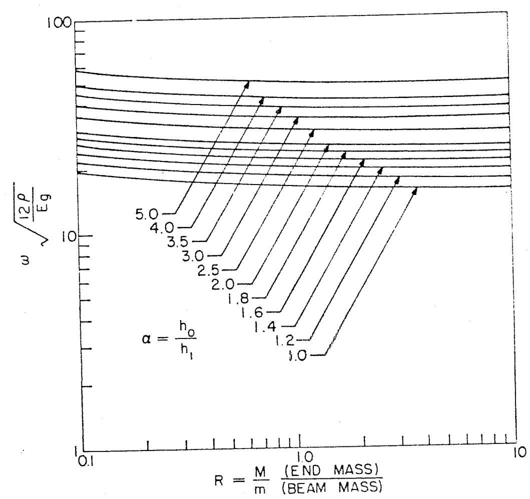

FIG. 4. Second-harmonic frequency for cantilever beam of constant width and variable thickness with $h_1/l^2=1$.

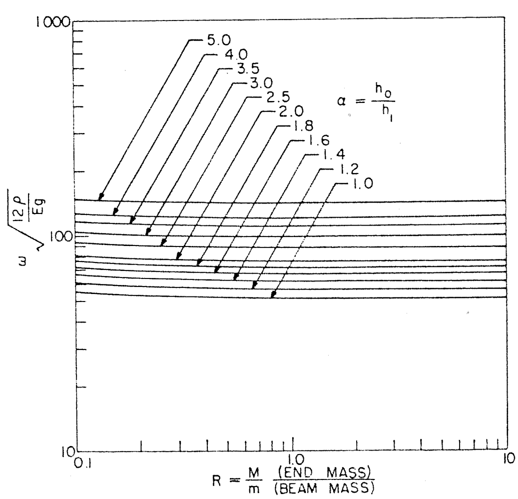

FIG. 5. Third-harmonic frequency for cantilever beam of constant width and variable thickness with $h_1/l^2=1$.

α	R	Fundamental frequency	Second harmonic	Third harmonic
1.0	0	3.51600	22.034	61.697
	0.1	2.96783	19.356	55.518
	0.2	2.61275	18.208	53.559
	0.4	2.16799	17.176	52.063
	0.6	1.89247	16.701	51.445
	0.8	1.70062	16.427	51.108
	1.0	1.55730	16.250	50.896
	1.5	1.31430	15.996	50.601
	2.0	1.15818	15.861	50.448
	3.0	0.96281	15.720	50.291
	4.0	0.84153	15.647	50.211
	5.0	0.75695	15.602	50.162
	10.0	0.54137	15.512	50.064
1.2	0	4.30907	25.035	68.557
	0.1	3.59114	21.724	61.146
	0.2	3.13864	20.388	58.990
	0.4	2.58502	19.231	57.403
	0.6	2.24640	18.711	56.763
	0.8	2.01359	18.416	56.417
	1.0	1.84072	18.225	56.200
	1.5	1.54958	17.956	55.900
	2.0	1.36359	17.813	55.746
	3.0	1.13186	17.664	55.588
	4.0	0.98849	17.587	55.507
	5.0	0.88870	17.541	55.458
	10.0	0.63497	17.446	55.360
1.4	0	5.1209	27.984	75.231
	0.1	4.2164	24.000	66.553
	0.2	3.6605	22.483	64.221
	0.4	2.9927	21.212	62.562
	0.6	2.5924	20.652	61.902
	0.8	2.3187	20.336	61.550
	1.0	2.1165	20.137	61.331
	1.5	1.7778	19.852	61.027
	2.0	1.5626	19.703	60.871
	3.0	1.2954	19.547	60.712
	4.0	1.1306	19.469	60.631
	5.0	1.0161	19.420	60.584
	10.0	0.7253	19.323	60.485
1.6	0	5.9492	30.892	81.769
	0.1	4.8422	26.200	71.785
	0.2	4.1783	24.508	69.301
	0.4	3.3944	23.132	67.578
	0.6	2.9311	22.536	66.904
	0.8	2.6165	22.207	66.546
	1.0	2.3854	21.995	66.326
	1.5	2.0000	21.701	66.019
	2.0	1.7561	21.544	65.863
	3.0	1.4542	21.386	65.707
	4.0	1.2686	21.303	65.625
	5.0	1.1396	21.253	65.576
	10.0	0.8129	21.154	65.479
1.8	0	6.7919	33.775	88.195
	0.1	5.4679	28.341	76.871
	0.2	4.6914	26.481	74.249
	0.4	3.7897	25.004	72.481
	0.6	3.2631	24.376	71.795
	0.8	2.9081	24.034	71.436
	1.0	2.6482	23.814	71.213
	1.5	2.2169	23.507	70.903
	2.0	1.9447	23.348	70.748
	3.0	1.6089	23.182	70.587
	4.0	1.4028	23.102	70.506
	5.0	1.2599	23.048	70.459
	10.0	0.8981	22.944	70.358
2.0	0	7.6474	36.633	94.527
	0.1	6.0932	30.426	81.848
	0.2	5.1998	28.404	79.103
	0.4	4.1786	26.838	77.282
	0.6	3.5888	26.184	76.589
	0.8	3.1937	25.822	76.230
	1.0	2.9053	25.598	76.004
	1.5	2.4288	25.281	75.699
	2.0	2.1290	25.120	75.542
	3.0	1.7600	24.950	75.377
	4.0	1.5339	24.864	75.299
	5.0	1.3772	24.812	75.256
	10.0	0.9813	24.707	75.152
2.5	0	9.8358	43.720	110.061
	0.1	7.6491	35.453	93.867
	0.2	6.4514	33.049	90.869
	0.4	5.1265	31.280	88.963
	0.6	4.3798	30.565	88.257
	0.8	3.8857	30.178	87.891
	1.0	3.5280	29.937	87.666
	1.5	2.9403	29.602	87.357
	2.0	2.5736	29.430	87.203
	3.0	2.1240	29.255	87.049
	4.0	1.8496	29.165	87.058
	5.0	1.6600	29.111	86.923
	10.0	1.1815	29.002	86.825
3.0	0	12.082	50.749	125.28
	0.1	9.1936	40.267	105.45
	0.2	7.6757	37.514	102.25
	0.4	6.0442	35.574	100.30
	0.6	5.1416	34.810	99.580
	0.8	4.5510	34.402	99.212
	1.0	4.1254	34.148	98.985
	1.5	3.4310	33.798	98.676
	2.0	2.9991	33.618	98.522
	3.0	2.4724	33.436	98.365
	4.0	2.1518	33.345	98.283
	5.0	1.9299	33.288	98.236
	10.0	1.3729	33.175	98.141
3.5	0	14.375	57.749	140.27
	0.1	10.724	44.918	116.72
	0.2	8.8752	41.848	113.36
	0.4	6.9353	39.756	111.36
	0.6	5.8800	38.950	110.64
	0.8	5.1938	38.524	110.27
	1.0	4.7024	38.260	110.04
	1.5	3.9041	37.899	109.73
	2.0	3.4100	37.714	109.58
	3.0	2.8086	37.526	109.42
	4.0	2.4426	37.430	109.34
	5.0	2.1904	37.373	109.29
	10.0	1.5572	37.259	109.20
4.0	0	16.706	64.736	155.10
	0.1	12.241	49.438	127.74
	0.2	10.052	46.076	124.24
	0.4	7.8042	43.853	122.21
	0.6	6.5977	43.011	121.49
	0.8	5.8185	42.570	121.12
	1.0	5.2629	42.296	120.89
	1.5	4.3629	41.924	120.58
	2.0	3.8084	41.734	120.43
	3.0	3.1334	41.541	120.27
	4.0	2.7245	41.444	120.20
	5.0	2.4430	41.388	120.15
	10.0	1.7357	41.270	120.06
5.0	0	21.463	78.716	184.43
	0.1	15.230	58.180	149.20
	0.2	12.347	54.299	145.52
	0.4	9.4864	51.852	143.44
	0.6	7.9840	50.948	142.72
	0.8	7.0246	50.478	142.35
	1.0	6.3423	50.192	142.12
	1.5	5.2468	49.801	141.82
	2.0	4.5745	49.604	141.66
	3.0	3.7605	49.401	141.51
	4.0	3.2674	49.303	141.43
	5.0	2.9289	49.241	141.39
	10.0	2.0794	49.120	141.30

TABLE I. Factor $[2k\sqrt{h_1}(\alpha-1)/2]^2$.

Third harmonic
110.061
93.867
90.869
88.963
88.257
87.891
87.666
87.357
87.203
87.049
87.058
86.923
86.825
125.28
105.45
102.25
100.30
99.580
99.212
98.985
98.676
98.522
98.365
98.283
98.236
98.141
140.27
116.72
113.36
111.36
110.64
110.27
110.04
109.73
109.58
109.42
109.34
109.29
109.20
155.10
127.74
124.24
122.21
121.49
121.12
120.89
120.58
120.43
120.27
120.20
120.15
120.06
184.43
149.20
145.52
143.44
142.72
142.35
142.12
141.82
141.66
141.51
141.43
141.39
141.30

than one, multiply the values of $\omega(12\rho/Eg)^{\frac{1}{2}}$ from the figures (or Table) by the required value of h_1/l^2.

The special case of the beam with no end load can easily be analyzed by taking $R=0$ in Table I.

BEAM OF CONSTANT THICKNESS AND LINEARLY VARIABLE WIDTH

For a beam tapered in the horizontal plane, as shown in Fig. 2, the width b at a distance x from the free end is given by

$$b=b_1+(b_0-b_1)(x/l). \qquad (12)$$

Also,

$$I=\tfrac{1}{12}bh^3=\tfrac{1}{12}[b_1+(b_0-b_1)x/l]h^3,$$
$$A=bh=[b_1+(b_0-b_1)x/l]h.$$

Substituting these values into Eq. (4) gives

$$\frac{d^2}{dx^2}\left\{\left[b_1+(b_0-b_1)\frac{x}{l}\right]\frac{d^2z}{dx^2}\right\}=\frac{12\rho\omega^2}{Egh^2}\left[b_1+(b_0-b_1)\frac{x}{l}\right]z. \qquad (13)$$

Let

$$u=(1/l)x$$

so that

$$\frac{dz}{dx}=\left(\frac{1}{l}\right)\frac{dz}{du} \quad\text{and}\quad \frac{d^2z}{dx^2}=\left(\frac{1}{l^2}\right)\frac{d^2z}{du^2}.$$

Making this substitution and differentiating, Eq. (13) becomes

$$\left[1+\left(\frac{b_0}{b_1}-1\right)u\right]\frac{d^4z}{du^4}+2\left(\frac{b_0}{b_1}-1\right)\frac{d^3z}{du^3}$$
$$=\frac{12l^4\rho\omega^2}{Egh^2}\left[1+\left(\frac{b_0}{b_1}-1\right)u\right]z. \qquad (14)$$

Making the substitution

$$\beta=b_0/b_1 \quad\text{and}\quad K^4=(12\rho\omega^2)/(Egh^2),$$

Eq. (14) becomes

$$\frac{d^4z}{du^4}+\frac{2(\beta-1)}{[1+(\beta-1)u]}\frac{d^3z}{du^3}=l^4K^4z, \qquad (15)$$

with the following boundary conditions:

At $x=0$ or $u=0$:

$$d^2z/du^2=0$$
$$\left(\frac{d^3z}{du^3}\right)_{u=0}=\left(\frac{l^4K^4Rz(\beta+1)}{2}\right)_0,$$

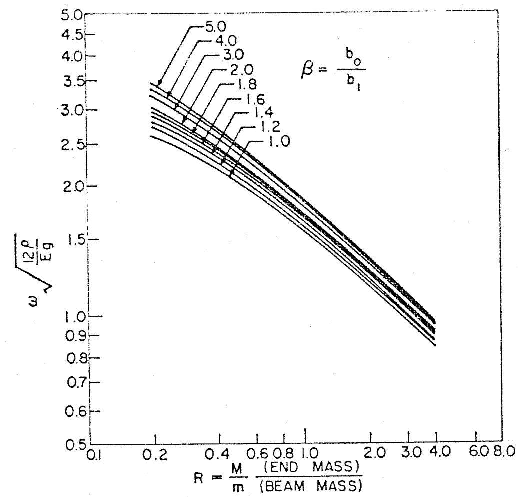

FIG. 6. Fundamental frequency for cantilever beam of constant thickness and variable width with $h/l^2=1$.

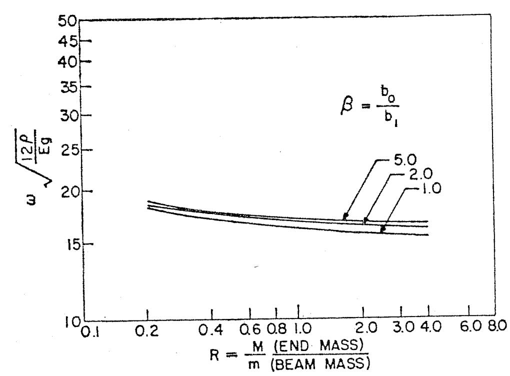

FIG. 7. Second-harmonic frequency for cantilever beam of constant thickness and variable width with $h/l^2=1$.

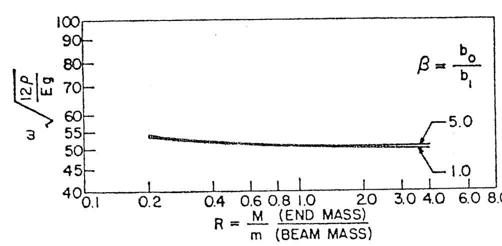

FIG. 8. Third-harmonic frequency for cantilever beam of constant thickness and variable width with $h/l^2=1$.

where

$$R=M/m.$$

At $x=l$ or $u=1$:

$$dz/du=0 \quad\text{and}\quad z=0.$$

Equation (15) was solved by numerical integration. For various values of β and R, the values of (lK) corresponding to the fundamental frequency, second

TABLE II. Factor $[lK]^2$.

β	R	Fundamental frequency	Second harmonic	Third harmonic
1.0	0	3.5160	22.035	61.70
	0.2	2.6127	18.208	53.55
	0.4	2.1680	17.176	52.06
	0.6	1.8926	16.701	51.44
	0.8	1.7007	16.428	51.11
	1.0	1.5573	16.250	50.89
	2.0	1.1582	15.861	50.45
	3.0	0.9628	15.720	50.30
	4.0	0.8416	15.647	50.21
1.2	0	3.7168	22.415	62.06
	0.2	2.7202	18.348	53.58
	0.4	2.2440	17.312	52.13
	0.6	1.9527	16.844	51.54
	0.8	1.7517	16.576	51.21
	1.0	1.6017	16.403	51.02
	2.0	1.1879	16.026	50.59
	3.0	0.9862	15.891	50.45
	4.0	0.8616	15.822	50.37
1.4	0	3.8923	22.743	62.39
	0.2	2.8100	18.451	53.58
	0.4	2.3061	17.414	52.17
	0.6	2.0017	16.951	51.60
	0.8	1.7924	16.690	51.29
	1.0	1.6374	16.522	51.11
	2.0	1.2113	16.157	50.71
	3.0	1.0048	16.026	50.57
	4.0	0.8772	15.961	50.50
1.6	0	4.0485	23.030	62.68
	0.2	2.8863	18.530	53.57
	0.4	2.3581	17.492	52.20
	0.6	2.0418	17.035	51.64
	0.8	1.8260	16.779	51.35
	1.0	1.6664	16.614	51.17
	2.0	1.2301	16.259	50.79
	3.0	1.0197	16.134	50.67
	4.0	0.8900	16.070	50.59
1.8	0	4.1873	23.286	62.95
	0.2	2.9519	18.590	53.55
	0.4	2.4019	17.552	52.22
	0.6	2.0756	17.102	51.68
	0.8	1.8540	16.849	51.41
	1.0	1.6908	16.693	51.22
	2.0	1.2457	16.344	50.87
	3.0	1.0318	16.222	50.74
	4.0	0.9002	16.159	50.68
2.0	0	4.3152	23.520	63.20
	0.2	3.0088	18.636	53.54
	0.4	2.4395	17.601	52.23
	0.6	2.1045	17.155	51.71
	0.8	1.8777	16.908	51.44
	1.0	1.7111	16.750	51.28
	2.0	1.2589	16.414	50.92
	3.0	1.0422	16.294	50.81
	4.0	0.9090	16.233	50.74
2.5	0	4.5852	24.021	63.74
	0.2	3.1244	18.715	53.48
	0.4	2.5141	17.685	52.24
	0.6	2.1609	17.252	51.77
	0.8	1.9238	17.012	51.52
	1.0	1.7506	16.862	51.35
	2.0	1.2837	16.542	51.04
	3.0	1.0615	16.429	50.92
	4.0	0.9252	16.372	50.87
3.0	0	4.8057	24.441	64.24
	0.2	3.2120	18.760	53.44
	0.4	2.5690	17.735	52.24
	0.6	2.2020	17.312	51.80
	0.8	1.9569	17.082	51.57
	1.0	1.7790	16.937	51.42
	2.0	1.3014	16.630	51.12
	3.0	1.0752	16.522	51.02
	4.0	0.9367	16.467	50.97
3.5	0	4.9894	24.802	64.66
	0.2	3.2808	18.786	53.38
	0.4	2.6115	17.768	52.24
	0.6	2.2329	17.354	51.81
	0.8	1.9822	17.128	51.60
	1.0	1.8002	16.988	51.45
	2.0	1.3145	16.691	51.18
	3.0	1.0787	16.589	51.08
	4.0	0.9452	16.536	51.04
4.0	0	5.1456	25.119	65.06
	0.2	3.3365	18.800	53.33
	0.4	2.6452	17.788	52.23
	0.6	2.2575	17.381	51.83
	0.8	2.0017	17.162	51.62
	1.0	1.8168	17.025	51.48
	2.0	1.3243	16.737	51.22
	3.0	1.0929	16.637	51.12
	4.0	0.9516	16.586	51.08
5.0	0	5.3977	25.655	65.74
	0.2	3.4210	18.810	53.25
	0.4	2.6948	17.810	52.22
	0.6	2.2934	17.415	51.84
	0.8	2.0301	17.203	51.64
	1.0	1.8406	17.072	51.52
	2.0	1.3386	16.798	51.28
	3.0	1.1038	16.703	51.19
	4.0	0.9606	16.655	51.15

harmonic, and third harmonic were found. Solving

$$K^4 = (12\rho\omega^2)/(Egh^2)$$

for ω gives

h

$$\omega = \cancel{l}K^2(Eg/12\rho)^{\frac{1}{2}},$$

or

$$\omega(12\rho/Eg)^{\frac{1}{2}} = (lK)^2\cdot[h/l^2], \tag{16}$$

where ρ = weight density, lb/in.[3]

Knowing the values of (lK) for various values of β and R from the solution of Eq. (15), Table II was developed to give the factor $(lK)^2$. From this Table, $\omega(12\rho/Eg)^{\frac{1}{2}}$ vs R was plotted for given values of β for $h/l^2=1$. Figures 6, 7, and 8 give the plots for the fundamental frequency, second harmonic, and third harmonic, respectively. For values of h/l^2 other than one, multiply the values of $\omega(12\rho/Eg)^{\frac{1}{2}}$ from the Figures (or Table) by the required value of h/l^2.

The case of the beam with no end load can be analyzed by taking $R=0$ in Table II. The results for the fundamental frequency with no end load agree with those obtained by Ono.[8]

Determination of frequencies involved numerical integration, since the solution to Eq. (15) could not be

[8] Akimasa Ono, "Vibration of a Cantilever of Variable Section," J. Soc. Mech. Engrs. (Tokyo) 28, 429 (1925).

written in terms of known functions (see Ref. 1). Let $Z(u)=AZ_1(u)+BZ_2(u)+CZ_3(u)+DZ_4(u)$, where A, B, C, and D are arbitrary constants and Z_1, Z_2, Z_3, and Z_4 are independent solutions of Eq. (~~18~~ 15), with the following initial conditions:

$$\begin{array}{llll} Z_1(0)=1, & Z_2(0)=0, & Z_3(0)=0, & Z_4(0)=1, \\ Z_1'(0)=0, & Z_2'(0)=1, & Z_3'(0)=0, & Z_4'(0)=0, \\ Z_1''(0)=0, & Z_2''(0)=0, & Z_3''(0)=1, & Z_4''(0)=0, \\ Z_1'''(0)=F, & Z_2'''(0)=0, & Z_3'''(0)=0, & Z_4'''(0)=1. \end{array}$$

The boundary conditions

$$\begin{array}{ll} Z(1)=0, & Z''(0)=0, \\ Z'(1)=0, & Z'''(0)=FZ(0)=[l^4K^4R(\beta+1)/2][Z(0)] \end{array}$$

demand that

$$\begin{vmatrix} Z_1(1) & Z_2(1) \\ Z_1'(1) & Z_2'(1) \end{vmatrix} =0 \quad \text{for} \quad F\neq 1 \quad .$$

Thus, Eq. (15) was solved twice, corresponding to initial condition given for $Z_1(u)$ and $Z_2(u)$. The values of $l^4K^4R(\beta+1)/2$ that satisfy the above determinant where then used to determine the factor $(lK)^2$ for the entries of Table II.

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