12.7.1

Received 10 May 1971

Transverse Vibrations of Double-Tapered Cantilever Beams

H. H. MABIE

Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

C. B. ROGERS

Sandia Corporation, Albuquerque, New Mexico 87115

The differential equation is developed from the Bernoulli–Euler equation for the free vibrations of a double-tapered cantilever beam. The beam tapers linearly in the horizontal and in the vertical planes simultaneously. From a computer solution of this equation, a table has been developed from which the fundamental frequency, second, third, fourth, and fifth harmonic can easily be obtained for various taper ratios. Charts are plotted for selected taper ratios in the vertical plane to show the effect of taper ratios on frequency.

INTRODUCTION

This analysis is a continuation of the work[1,2] started by the authors on the vibration of tapered cantilever beams. This type of beam linearly tapered in either the horizontal or the vertical plane finds wide application for electrical contacts and for springs in electromechanical devices. Occasionally, however, it is advantageous to use a beam which tapers linearly in the horizontal and in the vertical plane simultaneously, and it is necessary to be able to determine easily and accurately the natural frequencies of vibration.

The vibration of tapered cantilever beams has been considered by many investigators; however, only a few have studied the case of the beam tapered linearly in the horizontal and the vertical plane simultaneously. Martin[3] in 1956, working on the vibration of turbine blades, developed relations expressing the ratios between frequencies of tapered and untapered beams. Housner and Keightley[4] in 1962 analyzed a rectangular double-tapered cantilever beam and a conical-shell beam. Lindberg[5] in 1963 developed a method of deriving a dynamic stiffness matrix for determining the frequency of any nonuniform beam and applied this method in general terms to a linearly double-tapered cantilever beam. Rao[6] in 1964 developed a method for determining the ratio of the frequency of a tapered beam to the frequency of a uniform beam.

The methods mentioned above are interesting but are in general too complicated for average design use. Because of this, the present investigation was undertaken to provide a quick and accurate method of determining the first five frequencies of double-tapered cantilever beams over a wide range of taper ratios.

The differential equation of motion for a vibrating double-tapered cantilever beam has been derived and, because of its complexity, has been solved on a computer. From this solution, tables have been developed from which the fundamental frequency, second, third, fourth, and fifth harmonic can be obtained for various taper ratios. Charts have also been plotted to show the effect of taper ratios. In this analysis of transverse vibrations, shear and rotary inertia were neglected.

I. BEAM OF LINEARLY VARIABLE THICKNESS AND OF LINEARLY VARIABLE WIDTH

For the beam shown in a plan and elevated view of Fig. 1, the thickness h at a distance x from the free end is given by

$$h = h_1 + (h_0 - h_1)(x/l) \tag{1}$$

and the width b is given by

$$b = b_1 + (b_0 - b_1)(x/l). \tag{2}$$

The Bernoulli–Euler equation of a vibrating beam is

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2}, \tag{3}$$

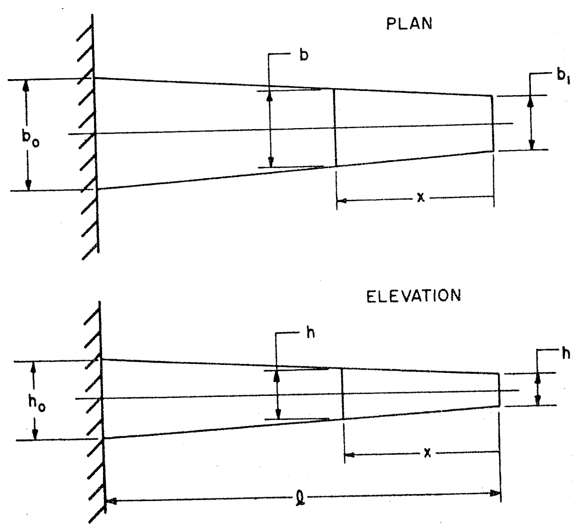

FIG. 1. Cantilever beam tapered linearly in horizontal and in vertical planes simultaneously.

TABLE I. Factor $(lk)^2$ $\alpha=h_0/h_1$, $\beta=b_0/b_1$.

α	β	Fundamental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
$\alpha=1.0$	1.0	3.5160	22.035	61.70	120.9	199.8
	1.2	3.7168	22.415	62.06	121.2	200.2
	1.4	3.8927	22.742	62.39	121.6	200.5
	1.6	4.0485	23.029	62.68	121.9	200.8
	1.8	4.1886	23.286	62.95	122.1	201.1
	2.0	4.3152	23.519	63.20	122.4	201.4
	2.5	4.5852	24.021	63.74	123.0	202.0
	3.0	4.8057	24.440	64.24	123.5	202.5
	3.5	4.9894	24.802	64.67	124.0	203.0
	4.0	5.1456	25.123	65.06	124.4	203.5
	5.0	5.3977	25.656	67.54	125.2	204.4
$\alpha=1.2$	1.0	4.3089	25.032	68.56	133.5	220.2
	1.2	4.5501	25.466	68.99	133.9	220.6
	1.4	4.7607	25.837	69.36	134.3	221.0
	1.6	4.9484	26.163	69.71	134.7	221.4
	1.8	5.1162	26.455	70.01	135.0	221.7
	2.0	5.2684	26.719	70.29	135.3	222.0
	2.5	5.5928	27.289	70.91	136.0	222.7
	3.0	5.8574	27.767	71.47	136.5	223.3
	3.5	6.0787	28.179	71.96	137.1	223.9
	4.0	6.2665	28.540	72.40	137.6	224.4
	5.0	6.5695	29.155	73.19	138.5	225.4
$\alpha=1.4$	1.0	5.1207	27.982	75.22	145.7	239.8
	1.2	5.4019	28.463	75.72	146.2	240.3
	1.4	5.6482	28.878	76.14	146.7	240.8
	1.6	5.8666	29.241	76.53	147.1	241.2
	1.8	6.0629	29.566	76.88	147.4	241.6
	2.0	6.2400	29.862	77.19	147.8	241.9
	2.5	6.6193	30.500	77.90	148.5	242.7
	3.0	6.9285	31.036	78.52	149.2	243.4
	3.5	7.1867	31.498	79.07	149.8	244.0
	4.0	7.4066	31.904	79.57	150.3	244.6
	5.0	7.7612	32.594	80.42	151.3	245.7
$\alpha=1.6$	1.0	5.9492	30.891	81.76	157.7	259.0
	1.2	6.2705	31.422	82.32	159.2	259.5
	1.4	6.5521	31.880	82.79	158.7	260.0
	1.6	6.8022	32.281	83.21	159.2	260.5
	1.8	7.0262	32.641	83.59	159.6	260.9
	2.0	7.2291	32.966	83.96	159.9	261.3
	2.5	7.6624	33.672	84.73	160.8	262.1
	3.0	8.0163	34.265	85.41	161.5	262.9
	3.5	8.3123	34.776	86.03	162.2	263.6
	4.0	8.5632	35.227	86.56	162.8	264.2
	5.0	8.9694	35.992	87.52	163.9	265.4
$\alpha=1.8$	1.0	6.7912	33.772	88.19	169.4	277.6
	1.2	7.1535	34.352	88.79	170.0	278.3
	1.4	7.4704	34.851	89.32	170.5	278.8
	1.6	7.7518	35.291	89.79	171.0	279.3
	1.8	8.0044	35.684	90.21	171.5	279.8
	2.0	8.2329	36.041	90.59	171.9	280.2
	2.5	8.7208	36.813	91.45	172.8	281.1
	3.0	9.1192	37.462	92.20	173.6	282.0
	3.5	9.4526	38.023	92.85	174.3	282.7
	4.0	9.7356	38.517	93.45	174.9	283.4
	5.0	10.1927	39.358	94.50	176.1	284.7
$\alpha=2.0$	1.0	7.6469	36.632	94.52	180.8	295.9
	1.2	8.0497	37.260	95.18	181.5	296.6
	1.4	8.4019	37.803	95.75	182.1	297.2
	1.6	8.7149	38.278	96.26	182.6	297.8
	1.8	8.9958	38.705	96.73	183.1	298.3
	2.0	9.2495	39.092	97.14	183.6	298.7
$\alpha=2.0$	2.5	9.7925	39.931	98.09	184.6	299.7
	3.0	10.2355	40.637	98.88	185.4	300.6
	3.5	10.6061	41.247	99.60	186.2	301.5
	4.0	10.9210	41.785	100.24	186.9	302.2
	5.0	11.4291	42.700	101.36	188.2	303.6
$\alpha=2.5$	1.0	9.8345	43.713	110.04	208.8	340.4
	1.2	10.3388	44.462	110.84	209.7	341.3
	1.4	10.7807	45.109	111.53	210.4	342.0
	1.6	11.1723	45.679	112.15	211.0	342.7
	1.8	11.5240	46.188	112.70	211.6	343.3
	2.0	11.8419	46.652	113.21	212.1	343.8
	2.5	12.5217	47.656	114.34	213.3	345.0
	3.0	13.0762	48.501	115.28	214.3	346.1
	3.5	13.5402	49.234	116.14	215.3	347.1
	4.0	13.9345	49.880	116.90	216.1	348.0
	5.0	14.5710	50.982	118.24	217.6	349.5
$\alpha=3.0$	1.0	12.0798	50.740	125.26	236.1	383.6
	1.2	12.6871	51.608	126.18	237.0	384.6
	1.4	13.2183	52.358	126.99	237.9	385.4
	1.6	13.6893	53.019	127.71	238.6	386.2
	1.8	14.1120	53.612	128.37	239.3	386.9
	2.0	14.4940	54.150	128.96	239.9	387.5
	2.5	15.3100	55.319	130.26	241.3	388.9
	3.0	15.9768	56.306	131.35	242.5	390.2
	3.5	16.5340	57.160	132.34	243.5	391.3
	4.0	17.0074	57.914	133.24	244.5	392.3
	5.0	17.7721	59.201	134.79	246.2	394.1
$\alpha=3.5$	1.0	14.3724	57.739	140.23	262.7	425.7
	1.2	15.0823	58.725	141.30	263.8	426.8
	1.4	15.7030	59.577	142.23	264.8	427.8
	1.6	16.2538	60.329	143.04	265.6	428.7
	1.8	16.7477	61.004	143.78	266.4	429.5
	2.0	17.1943	61.618	144.46	267.1	430.2
	2.5	18.1476	62.952	145.93	268.7	431.8
	3.0	18.9251	64.075	147.19	270.0	433.2
	3.5	19.5753	65.052	148.30	271.2	434.5
	4.0	20.1287	65.913	149.33	272.3	435.6
	5.0	21.0204	67.388	151.07	274.3	437.7
$\alpha=4.0$	1.0	16.7028	64.717	155.05	289.0	467.0
	1.2	17.5167	65.826	156.25	290.2	468.3
	1.4	18.2278	66.782	157.28	291.3	469.4
	1.6	18.8582	67.626	158.21	292.3	470.4
	1.8	19.4234	68.383	159.04	293.1	471.3
	2.0	19.9344	69.071	159.79	293.9	472.1
	2.5	21.0250	70.568	161.44	295.7	473.9
	3.0	21.9146	71.832	162.87	297.2	475.4
	3.5	22.6576	72.074	164.12	298.5	476.8
	4.0	23.2893	73.900	165.23	299.7	478.1
	5.0	24.3079	75.560	167.21	301.9	480.4
$\alpha=5.0$	1.0	21.4573	78.682	184.33	340.5	547.9
	1.2	22.4790	80.026	185.80	342.1	549.5
	1.4	23.3714	81.186	187.06	343.4	550.8
	1.6	24.1621	82.212	188.18	344.5	552.0
	1.8	24.8702	83.132	189.17	345.6	553.1
	2.0	25.5096	83.973	190.11	346.5	554.1
	2.5	26.8749	85.794	192.10	348.7	556.2
	3.0	27.9873	87.336	193.82	350.5	558.1
	3.5	28.9164	88.678	195.33	352.1	559.8
	4.0	29.7047	89.865	196.70	353.6	561.3
	5.0	30.9770	91.895	199.09	356.1	564.1

Fourth harmonic	Fifth harmon
184.6	299.7
185.4	300.6
186.2	301.5
186.9	302.2
188.2	303.6
208.8	340.4
209.7	341.3
210.4	342.0
211.0	342.7
211.6	343.3
212.1	343.8
213.3	345.0
214.3	346.1
215.3	347.1
216.1	348.0
217.6	349.5
236.1	383.6
237.0	384.6
237.9	385.4
238.6	386.2
239.3	386.9
239.9	387.5
241.3	388.9
242.5	390.2
243.5	391.3
244.5	392.3
246.2	394.1
262.7	425.7
263.8	426.8
264.8	427.8
265.6	428.7
266.4	429.5
267.1	430.2
268.7	431.8
270.0	433.2
271.2	434.5
272.3	435.6
274.3	437.7
289.0	467.0
290.2	468.3
291.3	469.4
292.3	470.4
293.1	471.3
293.9	472.1
295.7	473.9
297.2	475.4
298.5	476.8
299.7	478.1
301.9	480.4
340.5	547.9
342.1	549.5
343.4	550.8
344.5	552.0
345.6	553.1
346.5	554.1
348.7	556.2
350.5	558.1
352.1	559.8
353.6	561.3
356.1	564.1

where $\rho A/g$ is the mass per unit length (ρ weight density, A cross-sectional area, g gravitational constant), E the modulus of elasticity, and I the moment of inertia. If one assumes a sustained free vibration at a frequency ω of $y(x,t)=z(x)\sin\omega t$, Eq. 3 becomes

$$\frac{d^2}{dx^2}\left(EI\frac{d^2z}{dx^2}\right)=\left(\frac{\rho A}{g}\right)\omega^2 z. \tag{4}$$

For the beam tapered in two directions as shown,

$$I=\frac{1}{12}bh^3=\frac{1}{12}\left[b_1+(b_0-b_1)\frac{x}{l}\right]\left[h_1+(h_0-h_1)\frac{x}{l}\right]^3 \tag{5}$$

and

$$A=bh=\left[b_1+(b_0-b_1)\frac{x}{l}\right]\left[h_1+(h_0-h_1)\frac{x}{l}\right]. \tag{6}$$

Substituting Eqs. 5 and 6 into Eq. 4 and letting $u=x/l$ (so that u will always be in the interval $[0,1]$ for numerical integration) gives the following equation:

$$\frac{d^4z}{du^4}+\frac{2d^3z}{du^3}\left\{\frac{3(h_0-h_1)}{[h_1+(h_0-h_1)u]}+\frac{(b_0-b_1)}{[b_1+(b_0-b_1)u]}\right\}$$
$$+\frac{6d^2z}{du^2}\left\{\frac{(b_0-b_1)(h_0-h_1)}{[b_1+(b_0-b_1)u][h_1+(h_0-h_1)u]}+\frac{(h_0-h_1)^2}{[h_1+(h_0-h_1)u]^2}\right\}$$
$$=\frac{12l^4\rho\omega^2}{Eg}\left[\frac{1}{h_1+(h_0-h_1)u}\right]^2 z. \tag{7}$$

To simplify Eq. 7, let $\alpha=h_0/h_1$, $\beta=b_0/b_1$, and

$$k^4=12\rho\omega^2/Egh_1^2. \tag{8}$$

Equation 7 then becomes

$$\frac{d^4z}{du^4}+\frac{2d^3z}{du^3}\left[\frac{3(\alpha-1)}{1+(\alpha-1)u}+\frac{\beta-1}{1+(\beta-1)u}\right]$$
$$+\frac{6d^2z}{du^2}\left\{\frac{(\beta-1)(\alpha-1)}{[1+(\beta-1)u][1+(\alpha-1)u]}+\frac{(\alpha-1)^2}{[1+(\alpha-1)u]^2}\right\}=\frac{(lk)^4z}{[1+(\alpha-1)u]^2}. \tag{9}$$

Equation 9 is the equation of motion for a double-tapered beam. It was solved by numerical integration to give values of (lk) for various taper ratios for the following boundary conditions:

at $x=0$ or $u=0$, $d^2z/du^2=0$ and $z=0$,

at $x=l$ or $u=1$, $dz/du=0$ and $z=0$.

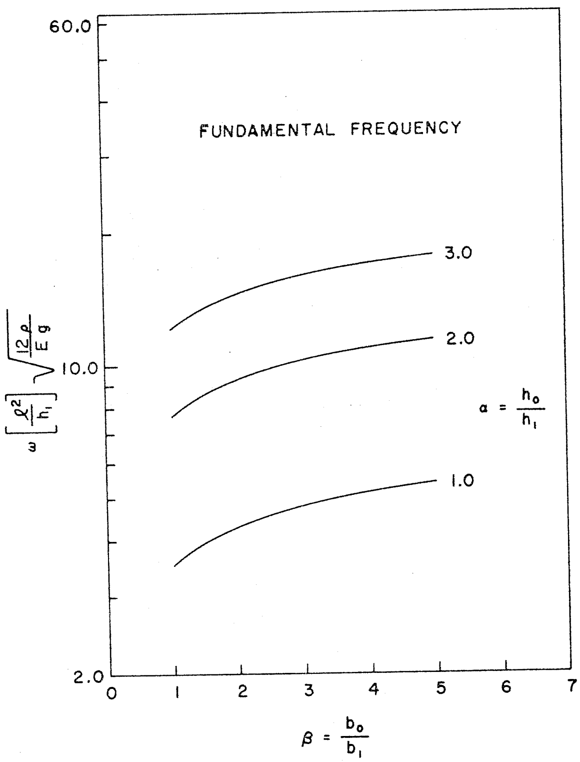

FIG. 2. Effect of taper ratios on fundamental frequency.

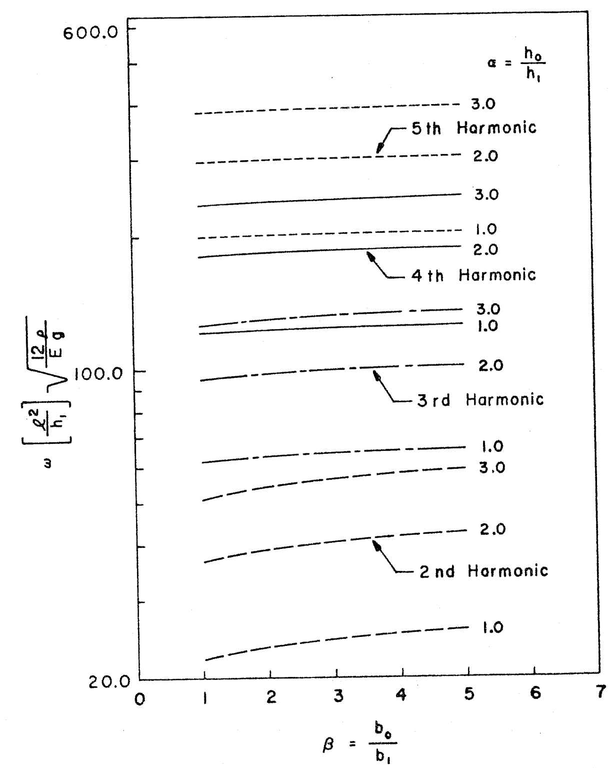

FIG. 3. Effect of taper ratios on second, third, fourth, and fifth harmonic frequencies.

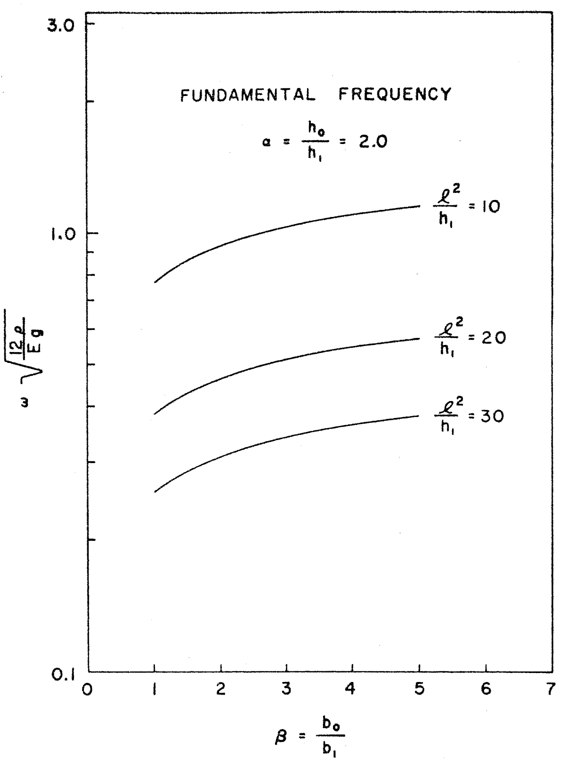

FIG. 4. Effect of factor l^2/h_1 on fundamental frequency.

With these boundary conditions, the solution becomes that of a double-tapered cantilever beam which is truncated and tapers from the fixed end only.

An equation for ω can be written in terms of (lk) from Eq. 8 as

$$\omega = \frac{(lk)^2}{l^2/h_1}\left(\frac{Eg}{12\rho}\right)^{\frac{1}{2}}. \tag{10}$$

Table I was developed from the solution of Eq. 9 to give values of $(lk)^2$ for various values of α and β for the fundamental, second, third, fourth, and fifth harmonics. With the values from this table, frequencies can easily be calculated using Eq. 10. It should be mentioned that this table can also be used to calculate the frequencies of a uniform beam where $\alpha=\beta=1.0$ or for a beam tapered in only one plane.

From Table I, curves were plotted of $\omega(l^2/h_1)\times(12\rho/Eg)^{\frac{1}{2}}$ vs β for $\alpha=1.0$, 2.0, and 3.0 for the five harmonics as shown in Figs. 2 and 3. Figure 4 shows curves for the fundamental frequency of $\omega(12\rho/Eg)^{\frac{1}{2}}$ vs β for $l^2/h_1=10$, 20, and 30 with $\alpha=2.0$.

From Figs. 2 and 3 it can be seen that the taper ratio β in the horizontal plane has much less effect upon frequency than does the ratio α in the vertical plane. Also, the effect of β decreases as the harmonic increases so that at the fourth and fifth harmonics β has practically no effect on frequency. It is interesting to note from Fig. 3 that there is an overlapping of the curves for the third and fourth harmonics depending upon the value of α. This is also true for the fourth and fifth harmonics.

Figure 4 was plotted to show the effect of the factor l^2/h_1 upon frequency. Several values of l^2/h_1 are given which might be encountered in an actual design. As can be seen, frequencies decrease as this factor increases.

A. Example

Determine the fundamental frequency of vibration of a beryllium copper cantilever beam with the following dimensions:

$$h_1=0.010\text{ in.},\quad b_1=0.020\text{ in.},$$
$$h_0=0.020\text{ in.},\quad b_0=0.060\text{ in.},$$
$$l=0.500\text{ in.},$$
$$\alpha=h_0/h_1=2,\quad \beta=b_0/b_1=3.$$

From Table I, for $\alpha=2$ and $\beta=3$, $(lk)^2=10.236$. Also, $l^2/h_1=0.500^2/0.010=25$. For beryllium copper,

$$\rho=0.297\text{ lb/in.}^3,$$
$$E=20\times10^6\text{ psi (heat treated).}$$

From Eq. 10,

$$\omega=\frac{(lk)^2}{l^2/h_1}\left(\frac{Eg}{12\rho}\right)^{\frac{1}{2}}$$
$$=\frac{10.236}{25}\left(\frac{20\times10^6\times386}{12\times0.297}\right)^{\frac{1}{2}}$$
$$=19\,100\text{ rad/sec},$$
$$f=19\,100/2\pi=3040\text{ cps}.$$

ACKNOWLEDGMENT

The authors express their appreciation to Calvin G. Gray, Jr., for his work in setting up the computer program.

[1] H. H. Mabie and C. B. Rogers, "Transverse Vibrations of Tapered Cantilever Beams with End Loads," J. Acoust. Soc. Amer. **36**, 463–469 (1964).
[2] H. H. Mabie and C. B. Rogers, "Transverse Vibrations of Tapered Cantilever Beams with End Support," J. Acoust. Soc. Amer. **44**, 1739–1741 (1968).
[3] A. I. Martin, "Some Integrals Relating to the Vibration of a Cantilever Beam and Approximation for the Effect of Taper on Overtone Frequencies," Aeron. Quart. **VII**, 109–123 (May 1956).
[4] G. W. Housner and W. O. Keightley, "Vibrations of Linearly Tapered Cantilever Beams," J. Eng. Mech. Div. Amer. Soc. Civil Engrs. **88**, 95–123 (Apr. 1962).
[5] G. M. Lindberg, "Vibration of Non-Uniform Beams," Aeron. Quart. **XIV**, 387–395 (Nov. 1963).
[6] J. S. Rao, "The Fundamental Flexural Vibration of a Cantilever Beam of Rectangular Cross Section with Uniform Taper," Aeron. Quart. **XVI**, 139–144 (May 1965).