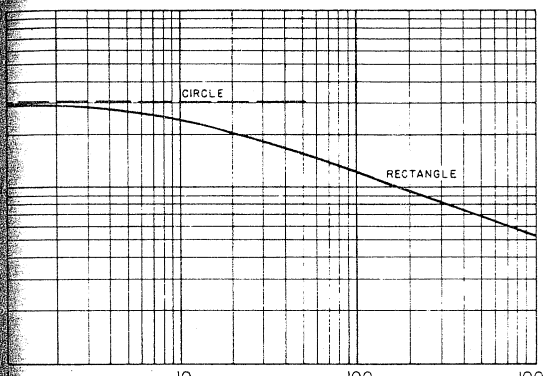

1. Variation of the accession to inertia of a rectangle at low frequencies, as a function of its aspect ratio a/b; with the value for a circular piston inserted for comparison.

x' being the distances from one end of the rectangle to respective source and field points, and varying from zero to a. If both these points are on the same side of the rectangle, $r=|x-x'|$; if they are on opposite sides, then $r^2=|x-x'|^2+b^2$. It is sufficient to consider source points on one of the pair of parallel sides, and double the result to take into account those on the opposite side. Also, it is clear that if both x and x' are allowed to range from zero to a, every pair of distances will be counted twice, with equal contributions due to the symmetry of Eq. 2 in source and field. The computation is simplified by only considering pairs of points such that $x'\leqslant x$, so that the range of integration over x' is now only from zero to x, and the absolute-value symbols can be dropped from the expressions for r; then this result is once again doubled.

At this point Z_a, the contribution to Z of the first pair of sides of the rectangle, is given by

$$\frac{2\omega\rho}{\pi}\left\{\int_0^a\int_0^x[(x-x')^2+b^2]^{\frac12}dx'dx-\int_0^a\int_0^x(x-x')dx'dx\right\}. \quad (3)$$

The integration is made easy by the change $X=x-x'$, $Y=x+x'-a$ and interpretation in rectangular coordinates. The integrand is not a function of Y, and the resulting single integrals are found in standard tables.

To the result of the final integration must be added the contribution of the other pair of sides, which is obtained by simply interchanging a and b. Thus we get finally:

$$Z\sim(j\omega\rho/2\pi)A^{\frac32}f(a/b), \quad (4)$$

where

$$f(z)=2z^{\frac12}\sinh^{-1}z^{-1}+2z^{-\frac12}\sinh^{-1}z+\tfrac23 z^{\frac32}+\tfrac23 z^{-\frac32}-\tfrac23(z+z^{-1})^{\frac32} \quad (5)$$

$A=ab$, the area of the piston.

This result coincides with that obtained by H. Stenzel in Ref. 1. The function here called $f(z)$ is, except for a factor, the ... of his Eq. (54) for $n=0$. The value of Stenzel's expansion in general n can be derived through computation of the more general integral $\oint\oint r^{n-1}\mathbf{dl}'\cdot\mathbf{dl}$ by the method explained in this ...

A graph of the nondimensional equivalent mass, $f(a/b)$, is shown in Fig. 1. It is clearly seen how the accession to inertia diminishes as the rectangle becomes more elongated. The asymptotic value for z tending to infinity is

$$f(z)\sim z^{-\frac12}[1+2\ln(2z)] \quad (6)$$

The value of $f(a/b)$, except for a numerical factor, is the slope at the origin of Stenzel's graph for the reactance ratio P_m, Fig. 20 of Ref. 1. He also exhibits curves for a circular piston. The circle, as the most compact shape, is expected to have the maximum possible accession to inertia; its value (Ref. 3) $16/(3\pi^{\frac12})$, is shown for comparison in Fig. 1. The square, however, has a value that comes very close, 2.9732096··· as against 3.0090111···.

Acknowledgment: This investigation was supported by the Office of Naval Research Acoustics Program (Code 468), under contract.

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Transverse Vibrations of Tapered Cantilever Beams with End Support

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Free vibrations of nonuniform cantilever beams with an end support have been investigated, using the equations of Bernoulli–Euler. Two configurations of interest are treated: (a) constant width and linearly variable thickness; and (b) constant thickness and linearly variable width. Charts have been plotted for each case from which the fundamental frequency, the second harmonic, and the third harmonic can be easily determined for various taper ratios. The Tables from which these charts were plotted are also included.

Outlined below is a brief summary of an analysis of transverse vibrations of tapered cantilever beams with end support. This analysis is a continuation of the work[1] started by the authors on the vibration of cantilever beams as used for electrical contacts and springs in electromechanical devices. This work is similar to that published by Conway and Dubil,[2] but it is more applicable to design because of the way the results are presented (tables and charts) and because of the large number of taper ratios considered. Moreover, two cases are considered: (1) taper in vertical plane and (2) taper in horizontal plane. Thus, the free vibrations of a cantilever beam with end support and (a) constant width and linearly variable thickness and (b) constant thickness and linearly variable width have been investigated using the Bernoulli–Euler equations. These equations neglect the effects of rotary inertia and shear and give accurate results for cases in which the wavelength under consideration is large as compared with the lateral dimension of the beam. The equations were solved on a computer, and curves were plotted from which fundamental frequency, second harmonic, and third harmonic can be determined for various taper ratios. The method presented yields accuracies to five significant figures in the Tables.

Beam of Constant Width and Linearly Variable Thickness. For the beam shown in Fig. 1 the thickness h at a distance x from the supported end is given by

$$h=h_1+(h_0-h_1)x/l. \quad (1)$$

Fig. 1. Cantilever beam of constant width and linearly variable thickness with end support.

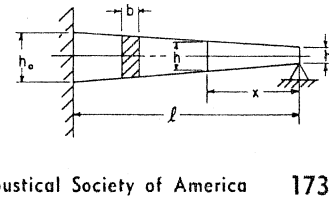

TABLE I. Factor $[2k\sqrt{(h_1)}^{\frac{1}{2}}(\alpha-1)/2]^2$.

α	Fundamental frequency	Second harmonic	Third harmonic
1.05	15.907	51.307	106.94
1.1	16.392	52.636	109.60
1.2	17.349	55.260	114.85
1.4	19.222	60.386	125.09
1.6	21.049	65.378	135.03
1.8	22.841	70.262	144.74
2.0	24.600	75.055	154.24
2.5	28.891	86.722	177.32
3.0	33.062	98.045	199.66
3.5	37.143	109.10	221.42
4.0	41.150	119.96	242.74
4.5	45.099	130.65	263.70
5.0	48.999	141.20	284.35
6.0	56.675	161.96	324.94

The Bernoulli–Euler equation of a vibrating beam is

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)=-\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2}, \qquad (2)$$

where $\rho A/g$ is the mass per unit length. For a cantilever beam with end support, the following boundary conditions hold:

$$\text{at } x=0,\quad y=0 \text{ and } \partial^2y/\partial x^2=M=0$$
$$\text{at } x=l\quad y=0 \text{ and } \partial y/\partial x=\theta=0.$$

Assuming a sustained free vibration at a frequency ω of $y(x,t)=z(x)\sin\omega t$, Eq. 2 becomes

$$\frac{d^2}{dx^2}\left(EI\frac{d^2z}{dx^2}\right)=\left(\frac{\rho A}{g}\right)\omega^2 z. \qquad (3)$$

For the beam tapered as shown

$$I=\tfrac{1}{12}bh^3=\tfrac{1}{12}b[h_1+(h_0-h_1)x/l]^3$$
$$A=bh=b[h_1+(h_0-h_1)x/l],$$

where I is the moment of inertia and A is the cross-sectional area.

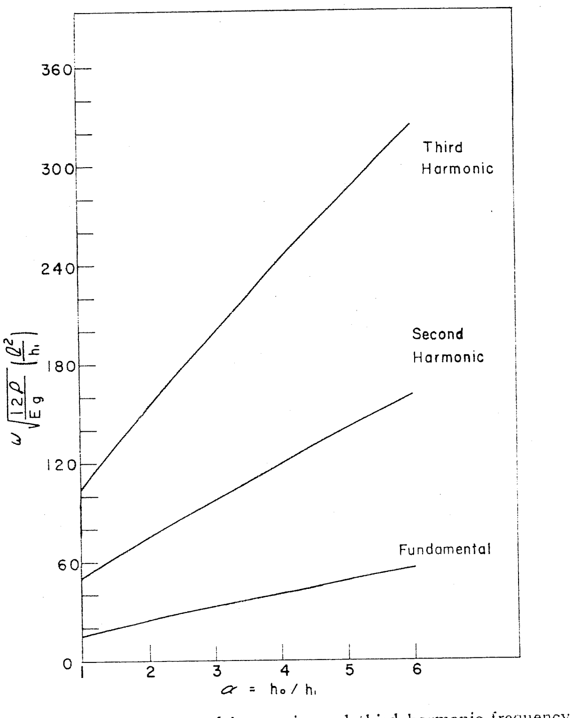

FIG. 2. Fundamental, second harmonic, and third harmonic frequency for cantilever beam of constant width and linearly variable thickness with end support.

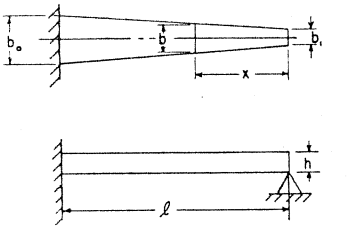

FIG. 3. Cantilever beam of constant thickness and linearly variable width with end support.

Substituting these values for I and A into Eq. 3 and letting

$$X=h_1+(h_0-h_1)x/l$$

and

$$k^4=12\rho\omega^2l^4/Eg(h_0-h_1)^4,$$

Eq. 3 becomes

$$X^4\frac{d^4z}{dX^4}+6X^3\frac{d^3z}{dX^3}+6X^2\frac{d^2z}{dX^2}=k^4X^2z. \qquad (4)$$

Equation 4 has the general solution

$$z(X)=X^{-\frac{1}{2}}[AJ_1(2k\sqrt{X})+BY_1(2k\sqrt{X})$$
$$+CI_1(2k\sqrt{X})+DK_1(2k\sqrt{X})], \qquad (5)$$

where J_1 and Y_1 are first-order Bessel functions of the first and second kind, and I_1 and K_1 are first-order modified Bessel functions of the first and second kind.

Imposing the boundary conditions, the determinantal equation that resulted was solved for the factor $2k(h_1)^{\frac{1}{2}}$ for various values of taper ratios ($\alpha=h_0/h_1$) for the fundamental, second harmonic, and third harmonic frequencies.

Solving

$$k^4=12\rho\omega^2l^4/Eg(h_0-h_1)^4$$

for ω and making the substitution $\alpha=h_0/h_1$, gives

$$\omega=\left[\frac{k^2h_1^2(\alpha-1)^2}{l^2}\right]\left(\frac{Eg}{12\rho}\right)^{\frac{1}{2}}$$

or

$$\omega\left(\frac{12\rho}{Eg}\right)^{\frac{1}{2}}\left(\frac{l^2}{h_1}\right)=\left[\frac{2k(h_1)^{\frac{1}{2}}(\alpha-1)}{2}\right]^2. \qquad (6)$$

Knowing the values of $2k(h_1)^{\frac{1}{2}}$ for various values of α from the solution of the determinant, Table I was developed to give the factor $[2k(h_1)^{\frac{1}{2}}(\alpha-1)/2]^2$. From this Table,

$$\omega[12\rho/Eg]^{\frac{1}{2}}[l^2/h_1]$$

versus α was plotted in Fig. 2 for the fundamental frequency, second harmonic, and third harmonic.

Beam of Constant Thickness and Linearly Variable Width.

For a beam tapered in the horizontal plane as shown in Fig. 3, the width b at a distance x from the supported end is given by

$$b=b_1+(b_0-b_1)x/l. \qquad (7)$$

TABLE II. Factor $(lK)^2$.

β	Fundamental frequency	Second harmonic	Third harmonic
1.0	15.417	49.964	104.24
1.2	15.604	50.139	104.43
1.4	15.751	50.275	104.55
1.6	15.867	50.386	104.67
1.8	15.963	50.477	104.78
2.0	16.044	50.555	104.86
2.5	16.195	50.702	105.00
3.0	16.299	50.807	105.12
3.5	16.374	50.884	105.23
4.0	16.431	50.944	105.29
5.0	16.507	51.024	105.39

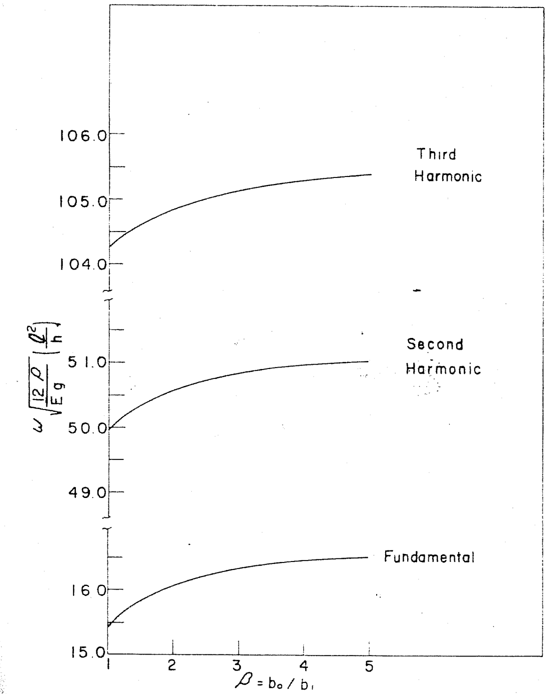

FIG. 4. Fundamental, second harmonic, and third harmonic frequency for cantilever beam of constant thickness and linearly variable width with end support.

Using Eq. 7 to determine the equations for moment of inertia (I) and area (A) and substituting into Eq. 3 and letting $u=(1/l)x$, the following equation results:

$$\left[1+\left(\frac{b_0}{b_1}-1\right)u\right]\frac{d^4z}{du^4}+2\left(\frac{b_0}{b_1}-1\right)\frac{d^3z}{du^3} =\frac{12l^4\rho\omega^2}{Egh^2}\left[1+\left(\frac{b_0}{b_1}-1\right)u\right]z. \quad (8)$$

Making the substitutions

$$\beta=b_0/b_1 \quad \text{and} \quad K^4=12\rho\omega^2/Egh^2,$$

Eq. 8 becomes

$$\frac{d^4z}{du^4}+\frac{2(\beta-1)}{[1+(\beta-1)u]}\frac{d^3z}{du^3}=(lK)^4z \quad (9)$$

with the following boundary conditions:

$$\text{at } x=0 \text{ or } u=0,\ d^2z/du^2=0 \text{ and } z=0$$
$$\text{at } x=l \text{ or } u=1,\ dz/du=0 \text{ and } z=0.$$

Equation 9 was solved by numerical integration. For various values of β, the values of (lK) corresponding to the fundamental frequency, second harmonic, and third harmonic were found. Solving

$$K^4=12\rho\omega^2/Egh^2$$

for ω gives

$$\omega=hK^2(Eg/12\rho)^{\frac{1}{2}}$$

or

$$\omega(12\rho/Eg)^{\frac{1}{2}}(l^2/h)=(lK)^2. \quad (10)$$

Knowing the values of (lK) for various values of β from the solution of Eq. 9, Table II was developed to give the factor $(lK)^2$. From this Table,

$$\omega(12\rho/Eg)^{\frac{1}{2}}(l^2/h)$$

vs β was plotted in Fig. 4 for the fundamental frequency, second harmonic, and third harmonic.

[1] H. H. Mabie and C. B. Rogers, "Transverse Vibrations of Tapered Cantilever Beams with End Loads," J. Acoust. Soc. Amer. 36, 463–469 (1964).

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Measurements of the Particle Velocity and Pressure of the Ambient Noise in a Shallow Bay*

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Sound perception by marine animals may be affected by the particle velocity or the pressure of ambient noise. Levels of ambient noise measured with a velocity hydrophone in very shallow water were found to be considerably above the associated pressure levels, particularly at low frequencies. This difference is probably due to nearfield components of ambient particle motion and may be important in acoustic signal detection by mechanoreceptor organs.

THOSE WORKING IN MARINE BIOACOUSTICS HAVE RECENTLY BEEN made aware of two important phenomena; masking by ambient noise[1,2] and detection of particle motion.[3–5] On the basis of these findings, further investigations of hearing in an aquatic environment should take into account both the pressure and the particle velocity of noise. The author is not aware of any such field-velocity measurements, and this is understandable, considering the state of the art in measuring this parameter.

Velocity and pressure measurements of the ambient noise were made at shallow-water test sites during experiments on the use of hearing by sharks. These sites, in Biscayne Bay, Florida, were in 25 to 45 cm of water over a soft mud bottom. Snapping shrimp were not abundant, and individual snaps could be easily distinguished. Fishes were constantly foraging in the area, producing frequent swimming and jumping sounds. The sensitive axis of the velocity hydrophone was set perpendicular to tidal currents, when present. All velocity measurements were in a horizontal plane. Wind speeds were 6 to 15 mph, creating a rippled surface, but no breaking waves. Three series of recordings were made at approximately high tide between 10 A.M. and 7 P.M., May to July 1968.

The velocity hydrophone consisted of a Geo Space HS-1 refraction geophone, 4.5-Hz resonant frequency, mounted on a PVC plate that was suspended from an outside housing.[5] Although this choice of resonant frequency necessitated careful leveling of the geophone, it permitted working at frequencies down to 20 Hz. The nearfield directional characteristics of this hydrophone were found to be typically dipole. Response at 20, 40, 80, and 160 Hz was down 3 dB at 45° and 22 dB at 90° to the sensitive axis; while

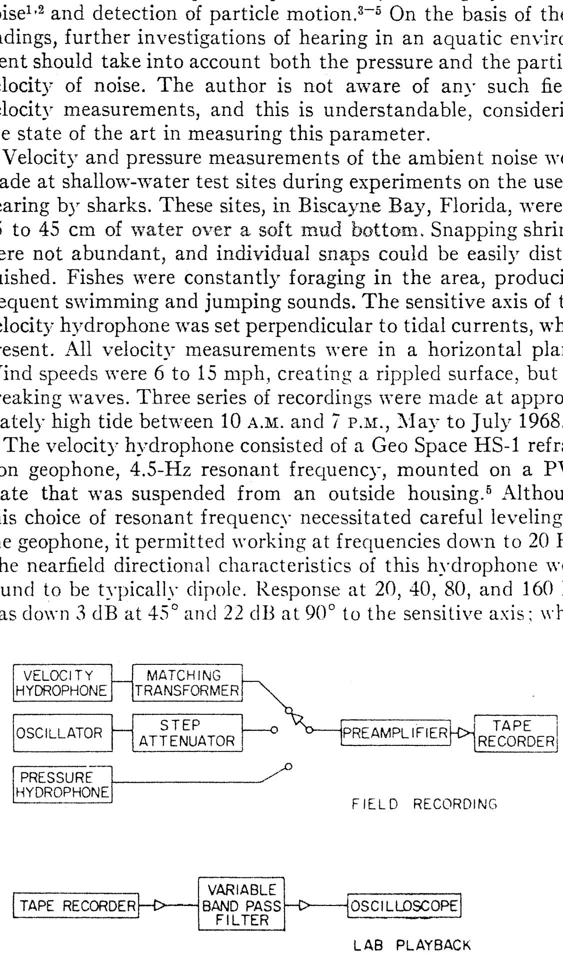

FIG. 1. Equipment for field recording and laboratory analysis of noise samples.