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Note: Coating thermal noise for arbitrary-shaped beams

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This is a brief note (intended for distribution to the coating thermal noise community) to summarize how coating brownian noise (in particular) will change with arbitrary mirror shapes.

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I. INTRODUCTION

I present several different calculations to demonstrate that coating thermal noise should be proportional to

$$S \propto d \int d^2 r |P(r)|^2 \propto d \int d^2 K |\tilde{P}(K)|^2 \qquad (1)$$

where d is the thickness of the coating, P is a normalized function proportional to the beam intensity profile, and $\tilde{P}$ is the two-dimensional fourier transform of P.

In these calculations, I use the fluctuation-dissipation theorem. I assume the elastic response is at all times quasi-stationary as described in **??**. I assume the mirror is half-infinite.

Finally, in the second calculation, I assume the **coating has the same elastic properties as the bulk**, save for the losses, which are far larger in the coating.

II. CALCULATION 1: SCALING ARGUMENTS

My result should not be surprising – it *must* have this form, based simply on scaling arguments, because no other length scale (besides the small thickness of the coating) exists in the elasticity problem we solve!

In this section, I first sketch the argument for thermoelastic noise (simpler, because *no* other lenth scale exists), then describe a similar argument for thermal noise

A. Preliminaries - Scaling for thermoelastic noise

In this section, based solely on the known result that the thermoelastic noise for a gaussian beam has $S \propto r_o^3$, I deduce that the thermoelastic noise integral must satisfy

$$S \propto \int d^2 K |K| |\tilde{P}(K)|^2 \qquad (2)$$

- *Step 1*: The thermoelastic noise power spectrum must be proportional to

$$S \propto \int d^2 K G(K) |\tilde{P}(K)|^2$$

with a proportionality constant independent of beam size or shape.

Reason: According to the fluctuation-dissipation theorem, the noise is proportional to the power dissipation rate $W_{\rm diss}$ associated with a fluctuating pressure of shape $P(r)$ on the mirror surface. Manifestly (for half-infinite mirrors), $W_{\rm diss}$ must be proportional to a translation-invariant inner product on P, of form

$$W_{\rm diss} \propto \int d^2 R \int d^2 R' V(R-R') P(R) P(R') \qquad (3)$$

$$\propto \int d^2 K G(K) |P(K)|^2 \qquad (4)$$

- *Step 2*: The kernel G must be scale invariant, and therefore satisfy $G(\lambda K) = \lambda^p G(K)$, and thereforem be of form

$$G(K) = K^p c_1$$

for some constant c_1. After all, no other scale exists in the (static) elasticity problem we are solving.

- *Step 3*: Finally, to recover the usual result for gaussian beams (i.e. $S \propto 1/r_o^3$), we must have $p = 1$. Therefore, we find Eq. (2).

B. Scaling for thermal noise

We proceed as above.

- *Step 1a*: The brownian-noise power spectrum must be proportional to

$$S \propto \int d^2 K G(K, d) |\tilde{P}(K)|^2$$

with a proportionality constant independent of beam size or shape, or of coating thickness

- *Step 1b*: In the limit of small coating thickness d, the first-order contribution to this integral is (practically by definition) the coating contribution to the brownian noise:

$$G(K, d) \approx G_o(K) + d G_1(K) + \dots$$

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- *Step 2*: The kernel G_1 must be scale invariant, and satisfy

$$G_1(K) = K^p c_1$$

for some constant c_1.

- *Step 3*: Finally, to recover the usual result for gaussian beams (i.e. $S \propto d/r_o^2$), we must have $p = 0$.

III. CALCULATION 2: USING NAKAGAWA

To check this simple scaling argument, we use an explicit approach, that relies on the elastic green's functions conveniently tabulated in an appendix of Nakagawa et al[1].

[This discussion applies to Nakagawa et al's approach to coating thermal noise – they consider a homogeneous slab of material with losses confined to a thin region of depth d near the surface. In particular, we implicitly assume the coating and substrate have the same elastic properties.]

Setup: From Nakagawa et al's Eq. (1) , we know

$$S \propto \int d^2R \int d^2R' P(R)P(R') \mathrm{Im}\chi_{zz}(R - R') \qquad (5)$$

where $\mathrm{Im}\chi_{zz}(R, R')$ is given by their Eqs. (4-5):

$$\mathrm{Im}\chi_{zz}(r) = \phi \frac{1-\sigma^2}{\pi E} \left[F(r, 0) - F(r, d)\right] \qquad (6)$$

$$F(r, z) = \frac{1}{\sqrt{r^2 + 4z^2}} \times \left(1 + \frac{z^2/(1-\sigma)}{r^2 + 4z^2} + 12 \frac{z^4/(1-\sigma)}{(r^2 + 4z^2)^2}\right) \qquad (7)$$

Fourier representation: We can equivalently represent this integral in the fourier domain, as

$$S \propto \int d^2K |\tilde{P}(K)|^2 \left[\tilde{F}(K, 0) - \tilde{F}(K, d)\right] \qquad (8)$$

where [Nakagawa et al Eq. A1]

$$\tilde{F}(K, d) = 2\pi \frac{e^{-2Kd}}{K} \left[1 + \frac{Kd}{1-\sigma} + \frac{(Kd)^2}{1-\sigma}\right] \qquad (9)$$

In other words

$$S \propto \int_0^\infty 2KdK |\tilde{P}(K)|^2 \times \left[\frac{-1 + \exp[-2Kd]}{K} + \frac{d \exp[-2Kd](1 + Kd)}{1-\sigma}\right] \qquad (10)$$

Small-d limit: Naturally, $\tilde{P}(K)$ drops to zero well before $K \approx 1/d$; therefore, we may take a small-d limit. We therefore conclude

$$S \propto d \int KdK |\tilde{P}(K)|^2$$

APPENDIX A: THERMOELASTIC NOISE OF HALF-INFINITE MIRRORS

To evaluate the thermoelastic noise associated with a given beam shape $P(r)$, we must evaluate the integral I_A [Eq. (**??**)] given the solution y^a to a model elasticity problem [Eq. (**??**)]. As discussed in Sec. **??**, if the mirror is sufficiently large compared to the beam shape $P(r)$, we can effectively treat the mirror as half-infinite (i.e. filling the whole volume $z < 0$) in the elasticity problem. In this case, the bulk acceleration term in the elasticity problem drops out [i.e. $V_A \to \infty$ in Eq. (**??**)] and we seek only the elastic response of a half-infinite medium to an imposed surface stress. This last problem has been discussed extensively in the literature — cf., e.g., [24, 25] — and there exist simple fourier-based computational techniques to generate and manage solutions. We apply these known solutions from the literature to evaluate the thermoelastic integral I_A.

1. Elastic solutions for the expansion (Θ)

In the case of half-infinite mirrors, the response y^a to the imposed pressure profile $P(r)$ can be found in the literature [cf. Eqs. (8.18) and (8.19) of Landau and Lifshitz's book on elasticity [24], where, however, the half-infinite volume is chosen *above* the $z = 0$ plane rather than below; see also Nakagawa et al. [25], especially their Appendix A]. These expressions allow us to explicitly relate the expansion Θ to the imposed pressure profile $P(r)$:

$$\Theta(\vec{r}, z) = \int G^{(\Theta)}(\vec{r}, z; r') P(r') d^2\vec{r'} \qquad \text{(A1a)}$$

$$G^{(\Theta)}(\vec{r}, z; \vec{r}_o) = -\frac{(1+\sigma)(1-2\sigma) z H(-z)}{2\pi E \left|(\vec{r} - \vec{r}_o)^2 + z^2\right|^{3/2}} \qquad \text{(A1b)}$$

where $H(x)$ is a step function which is 1 when $x > 0$ and 0 otherwise.

Because we have complete transverse translation symmetry, we can make our results more tractable by fourier-transforming in the transverse dimensions:

$$\tilde{\Theta}(K, z) \equiv \int e^{-i\vec{K}\cdot\vec{R}} \Theta(R, z) d^2R\,, \qquad \text{(A2)}$$

$$\tilde{P}(\vec{K}) \equiv \int e^{-i\vec{K}\cdot\vec{R}} P(R) d^2R\,. \qquad \text{(A3)}$$

For example, the convolution relating light intensity profile to the associated elastic response, Eq. (A1a), can be re-expressed as

$$\tilde{\Theta}(K, z) = G^{\Theta}(z, \vec{K}) \tilde{P}(\vec{K}) \qquad \text{(A4)}$$

$$\tilde{G}^{(\Theta)}\left(z, \vec{K}\right) = -\frac{(1+\sigma)(1-2\sigma)}{2\pi E} e^{-|Kz|}\,. \qquad \text{(A5)}$$

2. Thermoelastic integral I_A

Inserting the solution discussed above into Eq. (**??**) and using fourier-transform techniques to simplify the resulting integral, we find

$$I_A = \left(\frac{(1+\sigma)(1-2\sigma)}{2\pi E} \right)^2 \int d^2\vec{K} \; |K| \left| \tilde{P}(K) \right|^2 . \quad \text{(A6)}$$

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[2] LIGO II Conceptual Project Book, available at http://www.ligo.caltech.edu/docs/M/M990288-A1.pdf

[3] The LIGO Scientific Collaboration (LSC) maintains a website describing plans for the first-generation upgrade: http://www.ligo.caltech.edu/advLIGO/. This site was created for (and contains) the advanced LIGO NSF review. An older advanced LIGO website is at http://www.ligo.caltech.edu/~ligo2/.

[4] Peter Fritschel maintained a website which provided a succinct summary of an older advanced LIGO design, including material parameters: http://www.ligo.caltech.edu/~ligo2/scripts/l2refdes.htm (quoted 28-June-2003)

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