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# Response of LIGO to Gravitational Waves at High Frequencies and in the Vicinity of the FSR (37.5 kHz)

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# 1 Gravitational waves in the TT gauge

## 1.1 Coordinates of the wave frame

In the linearized theory, gravitational waves are described by small perturbations to the spacetime metric. Assume that in the absence of gravitational waves the spacetime is flat. Then the metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is the tensor of linear metric perturbations caused by the gravitational wave. In the transverse traceless (TT) coordinates the perturbations take the form:

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

where the two amplitudes $h_+$ and $h_\times$ correspond to two different polarizations of the gravitational wave:

$$h_+(t, \mathbf{r}) = h_+(t + z/c), \tag{3}$$

$$h_\times(t, \mathbf{r}) = h_\times(t + z/c). \tag{4}$$

For most of this analysis, time can be regarded as a fixed dimension and no time-dependent coordinate transformations will be necessary. We therefore can safely neglect the time components of the 4-dimensional tensors and consider only their spatial (3-dimensional) parts:

$$\mathbf{h}_0 = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

## 1.2 Coordinates of the detector frame

A laser interferometer defines its own coordinate system so that the $x$ and $y$ axes run along the two interferometer arms and the origin is at the beam-splitter. In general, these coordinates are oriented differently from the coordinates of the gravitational wave introduced above. The two coordinate systems can be related by rotational transformation

$$\mathbf{x} = \mathbf{R}^T \mathbf{x}_0, \tag{6}$$

where $\mathbf{x}_0$ are the coordinates associated with the gravitational wave and $\mathbf{x}$ are the coordinates associated with the detector. The transformation of the coordinates induces the transformation of the metric:

$$\mathbf{h} = \mathbf{R}^T \mathbf{h}_0 \, \mathbf{R}. \tag{7}$$

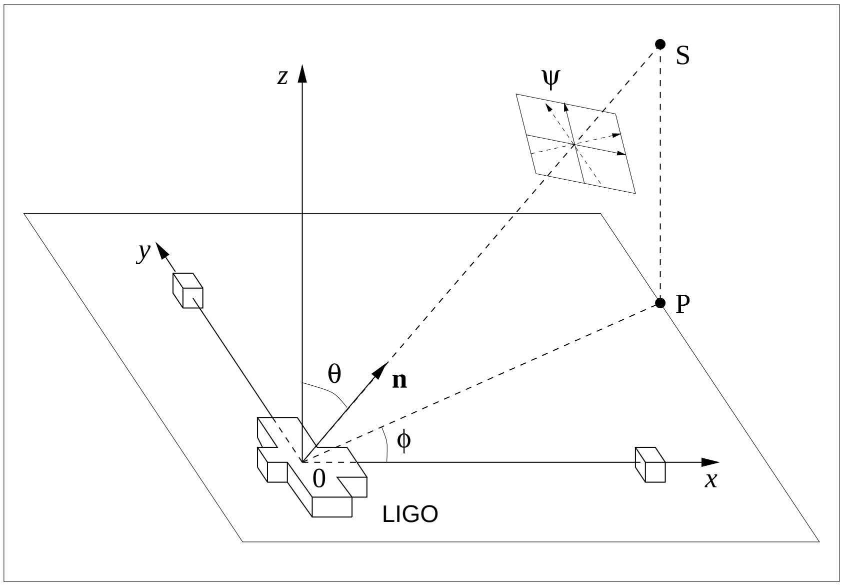


Figure 1: Orientation of the detector with respect to the source of gravitational waves.

Explicit formulas for the rotational matrix $\mathbf{R}$ can be constructed as follows. Denote the direction to the source of gravitational waves by a unit vector $\mathbf{n}$. The detector coordinate system $xyz$ and the source of gravitational waves $S$ are shown schematically in Fig. 1. The components of the vector $\mathbf{n}$ in the detector coordinate frame are:

$$n_x = \sin\theta\cos\phi, \tag{8}$$

$$n_y = \sin\theta\sin\phi, \tag{9}$$

$$n_z = \cos\theta, \tag{10}$$

where $\theta$ and $\phi$ are the spherical angles of the source. The third angle $\psi$ describes the orientation of the principal axes of polarization of the gravitational waves with respect to the plane of incidence ($SOP$ in Fig. 1).

The rotation matrix $R_{ij}$ is defined by the Euler angles: $\phi$, $\theta$, and $\psi$, and can be constructed as a product of the following three rotations.

First, we rotate the coordinate frame $xyz$ around the $z$-axis by angle $\phi$. The axes $x$ and $y$ remain in the horizontal plane. As a result, the $x$-axis comes to point along $\vec{OP}$, which coincides with the projection of $\vec{n}$ onto the plane of the detector. This rotation is given by

$$\mathbf{R}_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

Second, we rotate the coordinate frame around the $y$-axis by angle $\theta$. As a result, the $z$-axis comes to point to the source ($\vec{OS}$). Also, the axes $x$ and $y$ become perpendicular to $\vec{OS}$. This rotation is given by

$$\mathbf{R}_y(\theta) \;=\; \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{12}$$

Finally, we rotate the coordinate frame around the $z$-axis by angle $\psi$. As a result, the axes $x$ and $y$ become aligned with the polarization axes of the gravitational wave. This rotation is given by

$$\mathbf{R}_z(\psi) \;=\; \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

The combined effect of all three rotations can be found as a product

$$\mathbf{R} \;=\; \mathbf{R}_z(\psi)\, \mathbf{R}_y(\theta)\, \mathbf{R}_z(\phi). \tag{14}$$

As a result, the gravitational wave tensor will have all components:

$$\mathbf{h}_0 = \begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}. \tag{15}$$

## 2 Propagation of light in the detector frame

In what follows we will frequently use the perturbation expansion keeping only terms which are first order in $h$. Whenever we encounter a quantity $A(p)$ which is of order $h$, we will neglect all terms in $p$ which are of order $h$ and higher.

### 2.1 Null trajectory

Consider the interval corresponding to the metric, Eq.(1), in the detector coordinate frame:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 + h_{ij}(t + \mathbf{n} \cdot \mathbf{r}/c) dx^i dx^j. \qquad (16)$$

Propagation of light takes place along the null geodesic:

$$ds^2 = 0. \qquad (17)$$

For light traveling in the $x$-direction, the condition for null interval becomes

$$c^2 dt^2 = [1 + h_{xx}(t + n_x x/c)]\, dx^2. \qquad (18)$$

Introduce the coordinate velocity of the photon:

$$v_x = \frac{dx}{dt} = \pm \frac{c}{\sqrt{1 + h_{xx}(t + n_x x/c)}}. \qquad (19)$$

To first order in $h$ the velocity can be approximated as

$$v_x \approx \pm c \left[1 - \frac{1}{2} h_{xx}(t + n_x x/c)\right]. \qquad (20)$$

Then the trajectory of the photon is given by

$$x(t) = x_0 + \int\limits_{t_0}^{t} v_x(t', x')\, dt', \qquad (21)$$

where $x'$ belongs to the unperturbed photon trajectory: $x' = x_0 + c(t' - t_0)$.

Equivalently, we can describe the trajectory in the inverse form:

$$t(x) = t_0 + \int\limits_{x_0}^{x} \frac{dx'}{v(t', x')}, \qquad (22)$$

in which case $t'$ would belong to the unperturbed photon trajectory: $t' = t_0 + (x' - x_0)/c$. This inverse form is more convenient for calculations of the photon propagation times. To first order in $h$ it can be written explicitly as

$$t(x) = t_0 + \frac{x - x_0}{c} + \frac{1}{2c} \int\limits_{x_0}^{x} h_{xx}\left(t_0 + \frac{x' - x_0}{c} + \frac{n_x x'}{c}\right) dx'. \qquad (23)$$

### 2.2 Photon round-trip time

Consider two test masses with coordinates $x_a$ and $x_b$ and assume that a photon is launched from one test mass and is bounced by the other. Let the coordinates of the test masses be

$$x_a = l, \qquad \text{and} \qquad x_b = l + L, \tag{24}$$

and the unperturbed propagation time between the masses be

$$T = \frac{L}{c}. \tag{25}$$

As we know, in the TT gauge the coordinates of the test masses, do not change under the influence of the gravitational wave. Therefore, the duration of the forward trip can be found from Eq.(23) by specifying the boundary conditions in Eq.(24). Namely,

$$\begin{aligned} T_1 &= t(l+L) - t(l) \\ &= \frac{L}{c} + \frac{1}{2c} \int_{l}^{l+L} h_{xx}\left(t_0 + \frac{x-l}{c} + \frac{n_x x}{c}\right) dx. \end{aligned} \tag{26}$$

Assume that the end of the trip corresponds to time $t$. In this case we can express $t_0$ as a function of $t$:

$$t_0 + \frac{L}{c} = t. \tag{27}$$

Then $t$ can be viewed as a running parameter and the propagation time can be written as a function of $t$:

$$T_1(t) = T + \frac{1}{2c} \int_{l}^{l+L} h_{xx}\left(t - \frac{L+l}{c} + \frac{x}{c} + \frac{n_x x}{c}\right) dx. \tag{28}$$

Similarly, we find that the duration of the return trip is

$$T_2(t) = T + \frac{1}{2c} \int_{l}^{l+L} h_{xx}\left(t + \frac{l}{c} - \frac{x}{c} + \frac{n_x x}{c}\right) dx. \tag{29}$$

In this case, too, $t$ corresponds to the end of the trip and can be viewed as a running parameter.

The round-trip time of the photon propagation is

$$T_{\text{r.t.}}(t) = T_1[t - T_2(t)] + T_2(t). \tag{30}$$

To first order in $h$ this expression can be approximated as

$$T_{\text{r.t.}}(t) = T_1(t - T) + T_2(t). \tag{31}$$

Deviations of this round-trip time from its unperturbed value ($2T$) are given by

$$\delta T(t) = \frac{1}{2c} \int\limits_{l}^{l+L} \left[ h_{xx} \left( t - 2T + \frac{x-l}{c} + \frac{n_x x}{c} \right) + h_{xx} \left( t - \frac{x-l}{c} + \frac{n_x x}{c} \right) \right] dx. \quad (32)$$

By making the substitution: $x \to x - l$ we obtain the final result:

$$\delta T(t) = \frac{1}{2c} \int\limits_{0}^{L} \left[ h_{xx} \left( t - 2T + \frac{x}{c} + \frac{n_x (x+l)}{c} \right) + h_{xx} \left( t - \frac{x}{c} + \frac{n_x (x+l)}{c} \right) \right] dx. \quad (33)$$

# 3 Laplace-domain transfer function

The deviation in the round-trip time, Eq.(33), can also be written in the Fourier or Laplace domain. Define the Laplace transform of an arbitrary function of time $A(t)$ by

$$\tilde{A}(s) = \int\limits_{0}^{\infty} e^{-st} \, A(t) \, dt. \quad (34)$$

For convenience we introduced the integral:

$$I(t) = \int\limits_{0}^{L} h_{xx} \left( t + \alpha + \beta x \right) \, dx. \quad (35)$$

Its Laplace transform is given by

$$\tilde{I}(s) = e^{s\alpha} \left( \frac{e^{s\beta L} - 1}{s\beta} \right) \tilde{h}_{xx}(s), \quad (36)$$

Then the variation of the round-trip time Eq.(33) can be written in terms of $I(t)$ as follows:

$$\delta T(t) = \frac{1}{2c} \left[ I_1(t) + I_2(t) \right]. \quad (37)$$

where the parameters $\alpha$ and $\beta$ takes the following values

$$\alpha_1 = n_x \tau, \qquad \beta_1 = \frac{n_x - 1}{c}, \quad (38)$$

$$\alpha_2 = n_x \tau - 2T, \qquad \beta_2 = \frac{n_x + 1}{c}, \quad (39)$$

and also $\tau = l/c$.

Thus we obtain the formula for $\delta T$ in Laplace domain:

$$\delta\tilde{T}(s) = \frac{1}{2c}\left[e^{s\alpha_1}\,\left(\frac{e^{s\beta_1 L}-1}{s\beta_1}\right) + e^{s\alpha_2}\,\left(\frac{e^{s\beta_2 L}-1}{s\beta_2}\right)\right]\tilde{h}_{xx}(s). \tag{40}$$

It is convenient to rewrite the result as

$$\frac{\delta\tilde{T}(s)}{T} = e^{sn_x\tau}\,D(s,n_x)\,\tilde{h}_{xx}(s), \tag{41}$$

where $D(s, n_x)$ is the corresponding transfer function:

$$D(s,n_x) = \frac{1}{2sT}\left[\frac{1-e^{-(1-n_x)sT}}{1-n_x} - e^{-2sT}\cdot\frac{1-e^{(1+n_x)sT}}{1+n_x}\right], \tag{42}$$

There are a couple of limiting cases that we can check now. First, consider the low-frequency limit: $sT \to 0$. In this case $D(s, \mathbf{n}) \to 1$, and therefore

$$\frac{\delta\tilde{T}(s)}{T} = \tilde{h}_{xx}(s), \tag{43}$$

which agrees with the standard result, usually obtained in the long-wavelength regime.

The other simple limiting case occurs when the gravitational wave is incident upon the detector from the direction which is perpendicular to the $x$-arm, i.e. $n_x = 0$. In this case

$$\frac{\delta\tilde{T}(s)}{T} = \left(\frac{1-e^{-2sT}}{2sT}\right)\tilde{h}_{xx}(s), \tag{44}$$

which agrees with the standard calculations of the detector response for optimal orientation (see Refs. [6, 11, 10]).

# 4 Response of a Fabry-Perot cavity

Thus far, we considered the bouncing photon as a particle, assuming that there is a beginning and an end for the photon round trips. In practice, measurements of photon propagation times are done with optical interferometry in which photons are represented by continuous electromagnetic waves. We therefore extend the previous calculation to include continuous waves. This will allow us to include the storage effect of Fabry-Perot cavity.

## 4.1 Round-trip phase of light

Assume that the light is given by a plane monochromatic wave with frequency $\omega$ and wavenumber $k$. Define the phase of the light acquired during propagation between two points $x_a$ and $x_b$ as

$$\Psi(t) = -\int_{x_a}^{x_b} k\, dl = -k \int_{x_a}^{x_b} \sqrt{g_{xx}(t, \vec{r})}\, dx. \tag{45}$$

For forward propagation, the phase is

$$\Psi_1(t) = -kL - \frac{k}{2} \int_{l}^{l+L} h_{xx}(t', x)\, dx. \tag{46}$$

Similarly, for the return trip:

$$\Psi_2(t) = -kL - \frac{k}{2} \int_{l}^{l+L} h_{xx}(t', x)\, dx. \tag{47}$$

Then the round-trip phase can be found as

$$\Psi_{\mathrm{r.t.}}(t) = \Psi_1(t - T) + \Psi_2(t) = -2kL + \delta\Psi(t), \tag{48}$$

where $\delta\Psi$ is the linear perturbation due to the gravitational wave:

$$\delta\Psi(t) = -\omega\, \delta T(t). \tag{49}$$

Laplace-domain version of this equation is

$$\delta\tilde{\Psi}(s) = -kL\, e^{s n_x \tau}\, D(s, n_x)\, \tilde{h}_{xx}(s). \tag{50}$$

### 4.2 Fabry-Perot effect

Consider a Fabry-Perot cavity held on resonance by control systems. Assume that there are no disturbances, and the cavity is in the steady state. The field in the cavity will be essentially static: $E(t) = \bar{E}$.

The gravitational wave will cause variations in the light propagation phase which will produce temporal variations in the cavity field. The general form for the field equation is

$$E(t) = t_a\, A + q\, e^{-i\delta\Psi(t)}\, E(t - 2T), \tag{51}$$

where $A$ is the complex amplitude of the input laser field and $q$ is the round-trip reflectivity: $q = r_a r_b e^{-2ikL}$.

The solution can be found in terms of first order perturbations:

$$E(t) = \bar{E} + \delta E(t), \tag{52}$$

where $\bar{E}$ is the (static) equilibrium field and $\delta E$ is a small perturbation produced by the gravitational wave. The equilibrium field correspond to the situation $\Psi = 0$ and is given by

$$\bar{E} = \frac{t_a\, A}{1 - q}. \tag{53}$$

Then the field perturbation satisfies the equation:

$$\delta E(t) - q\, \delta E(t - 2T) = -iq\bar{E}\, \delta\Psi(t). \tag{54}$$

Laplace transformation of both sides of this equation yields

$$\delta\tilde{E}(s) = -iq\bar{E}\, H(s,\alpha)\, \delta\tilde{\Psi}(s), \tag{55}$$

where $H(s,\alpha)$ is the cavity transfer function:

$$H(s;\alpha) = \frac{1}{1 - q\, e^{-2sT}}, \tag{56}$$

where $\alpha = kL$ modulo $\pi$. This transfer function describes the storage property of Fabry-Perot cavities. Note that the LIGO arm cavities are set to the center of the resonance, which means that $\alpha = 0$. In this case,

$$H(s;0) = \frac{1}{1 - r_a r_b\, e^{-2sT}}. \tag{57}$$

Combining the response of the phase to gravitational waves with the filtering property of the Fabry-Perot cavity we obtain the total field response:

$$\delta\tilde{E}(s) = iqkL\bar{E}\, e^{sn_x\tau}\, H(s,\alpha)\, D(s,n_x)\, \tilde{h}_{xx}(s). \tag{58}$$

### 4.3 Michelson interferometer with Fabry-Perot arms

In LIGO detectors, two Fabry-Perot cavities are combined to form a Michelson interferometer, and the signal is extracted from the antisymmetric port (Fig.2).

Assume that the beam splitter is at the dark fringe, then the signal is proportional to

$$\delta\tilde{V}(s) = e^{-s(n_x\tau_x+2T)}\delta\tilde{E}_x(s) - e^{-s(n_y\tau_y+2T)}\delta\tilde{E}_y(s), \tag{59}$$

where $\delta E_x$ and $\delta E_y$ are the corresponding field perturbations in $X$ and $Y$ arms.

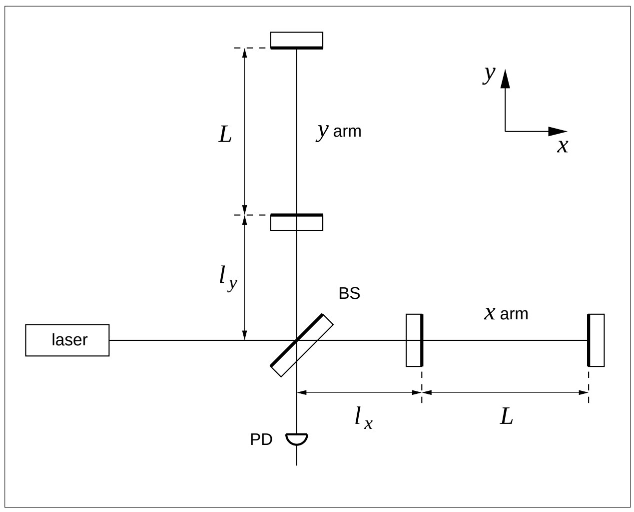


Figure 2: Michelson interferometer with Fabry-Perot arm cavities.

For simplicity, we assume that the arms are equal, then the formula for the signal is

$$\delta\tilde{V}(s) = e^{-2sT}H(s,\alpha)\,\left[D(s,n_x)\,\tilde{h}_{xx}(s) - D(s,n_y)\,\tilde{h}_{yy}(s)\right], \tag{60}$$

where we omitted the irrelevant multiplicative constant.

This allows us to introduce two transfer functions:

$$G_+(s) \equiv \frac{\delta\tilde{V}(s)}{\delta\tilde{h}_+(s)}, \tag{61}$$

$$G_\times(s) \equiv \frac{\delta\tilde{V}(s)}{\delta\tilde{h}_\times(s)}. \tag{62}$$

These transfer functions also depend on the coordinates of the source: $\phi, \theta, \psi$. Note that

$$G_\times(s)|_\psi = G_+(s)|_{\psi+\pi/4}\,. \tag{63}$$

Therefore, it is sufficient to consider one transfer function:

$$G(s) \equiv G_+(s). \tag{64}$$

# 5 Angular dependence of the detector response

Consider the dependence of the response function on the angles:

$$G = G(s|\phi, \theta, \psi). \tag{65}$$

For simplicity, we fix two angles and vary the third.

## 5.1 $\Phi$-dependence

Take $\theta = 0$ and also $\psi = 0$, and vary $\phi$. The changes in the response function are shown in Fig.3.

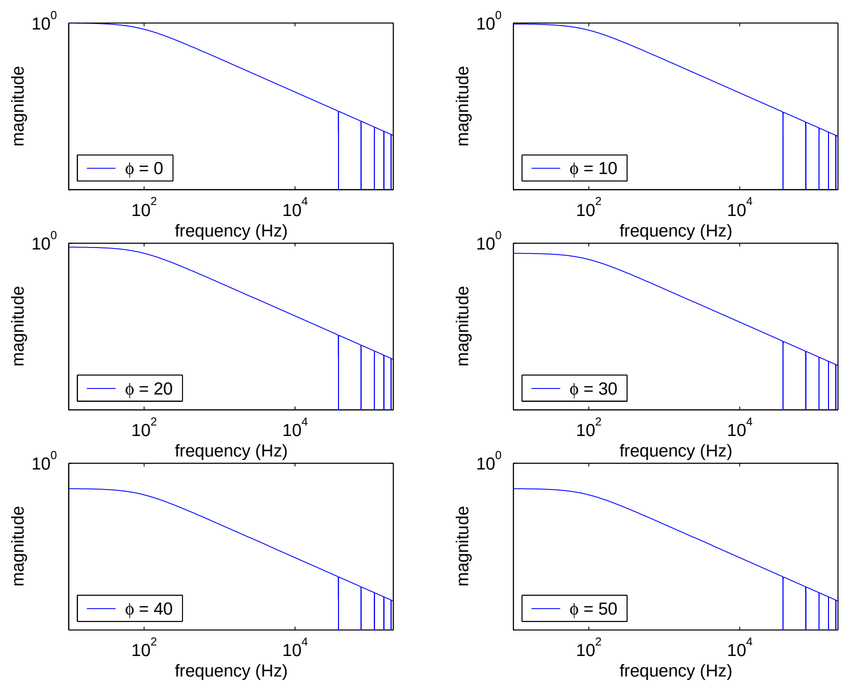


Figure 3: Dependence of the response function on the angle $\phi$ (the values of the angle are given in degrees).

### 5.2 Θ-dependence

Take now $\phi = 0$ and also $\psi = 0$, and vary $\theta$. The changes in the response function are shown in Fig.4.

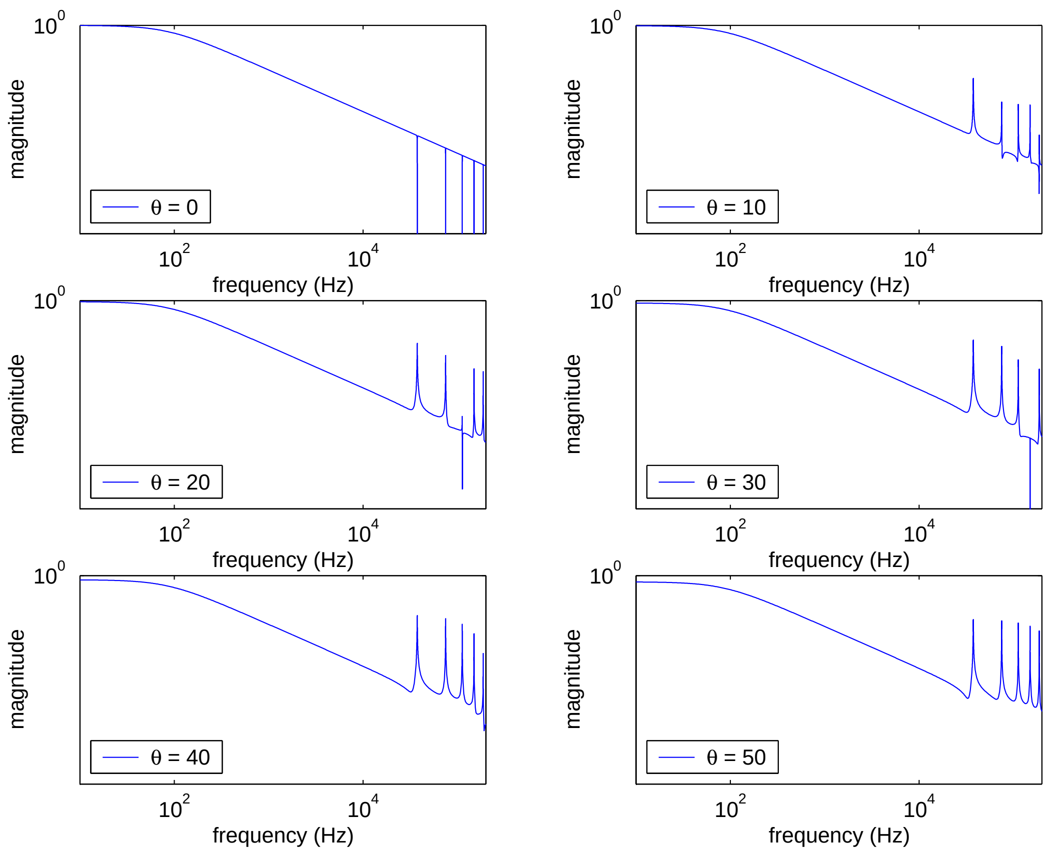


Figure 4: Dependence of the response function on the angle $\theta$ (the values of the angle are given in degrees).

## 6 Antenna patterns at different frequencies

Conversely we can fix the frequency and study the angular dependence of the response function.

### 6.1 DC pattern

Consider the angular dependence of the detector response function $|G|$. Assume for simplicity that $\psi = 0$. At low frequencies, the angular dependence (antenna pattern) is shown in Fig.5 (left).

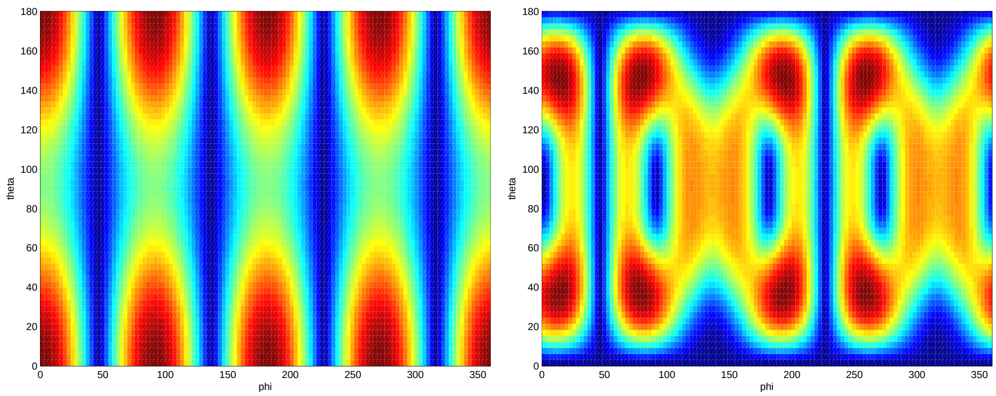


Figure 5: The dependence of the $|G|$ on the location of the source in the sky. Left: the frequency of G.W. is 0. Right:the frequency of G.W. is equal to FSR.

### 6.2 FSR pattern

No significant change occurs as the frequency of the gravitational wave changes through the kilohertz region. The drastic change occurs when the frequency approaches the free spectral range (FSR) of the arm cavities:

$$f_{\mathrm{FSR}} = \frac{1}{2T}. \tag{66}$$

The angular dependence of the detector response in this case is shown in Fig.5 (right). There are several maxima of the response function. Numerical optimization shows that the first maximum occurs approximately at

$$\theta = 37.5^\circ, \qquad \phi = 14^\circ. \tag{67}$$

### 6.3 Frequency response of LIGO for maximum at FSR

Consider the source location which yields maximum signal at FSR. Namely, its position is given by Eq.(67). For this location, the response of the detector (Bode plot) is shown in Fig.6.

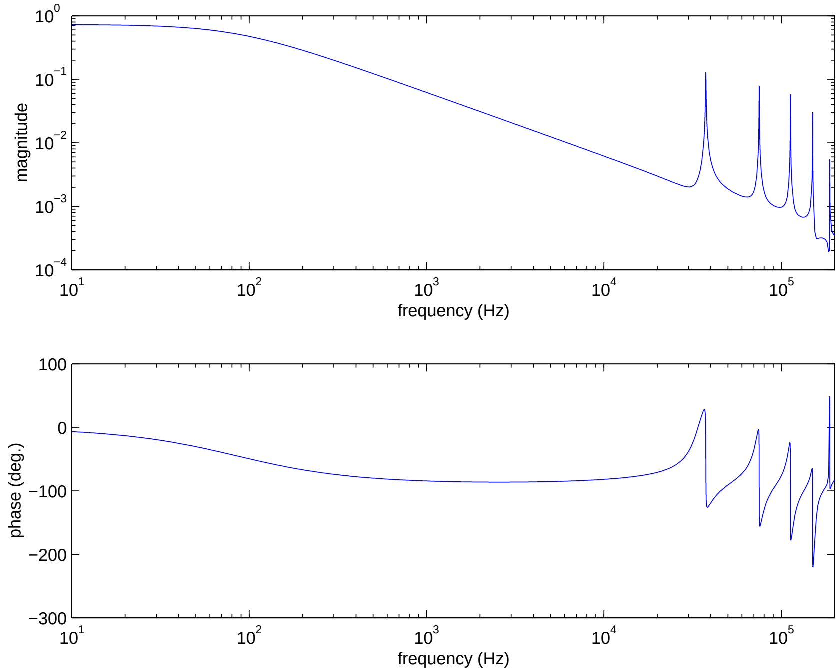


Figure 6: Response of LIGO to gravitational waves at $\theta = 37.5°$ and $\phi = 14°$.

## References


[1] D. Sigg, "Strain calibration in LIGO," LIGO Technical Note T970101-B, California Institute of Technology, Pasadena, California, 1997.

[2] R. Schilling, "Angular and frequency response of LISA," *Classical and Quantum Gravity*, vol. 14, pp. 1513–1519, 1997.

[3] F. B. Estabrook and H. D. Wahlquist, "Response of Doppler spacecraft tracking to gravitational radiation," *General Relativity and Gravitation*, vol. 6, pp. 439–447, 1975.

[4] R. Forward, "Wideband laser-interferometer gravitational-radiation experiment," *Physical Review D*, vol. 17, pp. 379–390, 1978.

[5] F. Estabrook, "Response functions of free mass gravitational wave antennas," *General Relativity and Gravitation*, vol. 17, pp. 719–724, 1985.

[6] J.-Y. Vinet, "Recycling interferometric antennas for periodic gravitational waves," *Journal De Physique*, vol. 47, pp. 639–643, 1986.

[7] B. F. Schutz and M. Tinto, "Antenna patterns of interferometric detectors of gravitational waves – i. linearly polarized waves," *Monthly Notices of the Royal Astronomical Society*, vol. 224, pp. 131–154, 1987.

[8] S. V. Dhurandhar and M. Tinto, "Astronomical observations with a network of detectors of gravitational waves – i. mathematical framework and solution of the five detector problem," *Monthly Notices of the Royal Astronomical Society*, vol. 234, pp. 663–676, 1988.

[9] J. Mizuno, "Comparison of optical configurations for laser-interferometric gravitational-wave detectors," MPQ technical report 203, Max-Planck-Institute for Quantum Optics, Garching, Germany, 1995.

[10] J. Mizuno, A. Rüdiger, R. Schilling, W. Winkler, and K. Danzmann, "Frequency response of Michelson- and Sagnac-based interferometers," *Optics Communications*, vol. 138, pp. 383–393, 1997.

[11] P. Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors.* Singapore: World Scientific, 1994.

[12] D. Sigg and R. Savage, "Analysis proposal to search for gravitational waves at multiples of the ligo arm cavity free-spectral-range frequency," LIGO Tech. Note T030296, California Institute of Technology, Pasadena, 2003.

[13] J. Markowicz, R. Savage, and P. Schwinberg, "Development of a readout scheme for high-frequency gravitational waves," LIGO Technical Document T030186, California Institute of Technology, Pasadena, California, 2003.