Parameter Estimation Using Short Fourier Transforms

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Abstract

We investigate using short Fourier transforms (SFTs) to estimate the parameters of the gravitational wave signal of a spinning neutron star [1]. We motivate estimating the parameters directly from the complex amplitudes of the SFTs, and discuss certain technical problems. By using the square of the complex amplitudes, i.e. the power spectra, we demonstrate the extension of the PowerFlux method [2] to estimating A_+, $A_\times$ and ψ simultaneously. It remains whether these extensions improve sensitivity enough to out-weigh the additional computational cost.

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1 Introduction

We follow the standard model [1] of the gravitational wave signal, as seen by an interferometric gravitational wave detector, of a freely precessing axisymmetric neutron star, focusing only on the $f = 2f_{\mathrm{rotation}}$ component expected from a tri-axial ellipsoid. The template is:

$$h(t) = A_1 h_1(t) + A_2 h_2(t) + A_3 h_3(t) + A_4 h_4(t) \tag{1}$$

where

$$h_1(t) = a(t)\cos 2\pi f t \qquad h_3(t) = a(t)\sin 2\pi f t \tag{2}$$
$$h_2(t) = b(t)\cos 2\pi f t \qquad h_4(t) = b(t)\sin 2\pi f t \tag{3}$$

and $a(t)$ and $b(t)$ are related to the beam patterns F_+, $F_\times$. The A parameters are:

$$A_1 = A_+ \cos 2\psi \cos\phi_0 - A_\times \sin 2\psi \sin\phi_0 \tag{4a}$$
$$A_2 = A_+ \sin 2\psi \cos\phi_0 + A_\times \cos 2\psi \sin\phi_0 \tag{4b}$$
$$A_3 = -A_+ \cos 2\psi \sin\phi_0 - A_\times \sin 2\psi \cos\phi_0 \tag{4c}$$
$$A_4 = -A_+ \sin 2\psi \sin\phi_0 + A_\times \cos 2\psi \cos\phi_0 \tag{4d}$$

where A_+, $A_\times$ are the amplitudes of the plus, cross polarisations, ϕ_0 is the initial phase, and ψ is the polarisation.

We note in passing that ψ is a rotation of the spatial metric peturbation matrix H describing the gravitational wave: the three angles (α, δ, ψ), where α and δ are the right ascension and declination of the neutron star respectively, constitute an Euler rotation from the wave reference frame to the celestial reference frame (See [1]).

1.1 Inverting $A_{1,2,3,4}$ to find A_+, $A_\times$, ϕ_0, ψ

A method to find A_+, $A_\times$, ϕ_0, ψ in terms of $A_{1,2,3,4}$ is given in [3]. Another method, derived here, first substitutes $\theta_1 = \phi_0 - 2\psi$, $\theta_2 = \phi_0 + 2\psi$ into equations (4), and takes the sum and difference of appropriate expressions to give

$$A_2 + A_3 = (A_\times - A_+)\sin\theta_1 \tag{5a}$$
$$A_4 - A_1 = (A_\times - A_+)\cos\theta_1 \tag{5b}$$
$$A_2 - A_3 = (A_\times + A_+)\sin\theta_2 \tag{5c}$$
$$A_4 + A_1 = (A_\times + A_+)\cos\theta_2\,; \tag{5d}$$

from which we find

$$\theta_1 = \tan^{-1}\frac{A_2 + A_3}{A_4 - A_1} \tag{6}$$
$$\theta_2 = \tan^{-1}\frac{A_2 - A_3}{A_4 + A_1}\,. \tag{7}$$

The appropriate ranges of θ_1 and θ_2, and thus of ϕ_0 and ψ, depend on the ranges chosen for A_+ and $A_\times$. This becomes clearer when we consider the expressions

$$A^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2 = A_+^2 + A_\times^2 \tag{8}$$
$$D = A_1 A_4 - A_2 A_3 = A_+ A_\times \tag{9}$$

and solve for A_+ and $A_\times$, giving 4 possible solutions:

$$A_+ = \pm\sqrt{\frac{A^2 \pm \sqrt{A^4 - 4D^2}}{2}} \tag{10}$$

$$A_\times = \frac{D}{A_+} \,. \tag{11}$$

These solutions are in fact equivalent to interchanging and/or negating A_+ and $A_\times$. If we impose the conditions $A_+ \geq A_\times \geq 0$, this selects only one solution, and $-\pi \leq \theta_1, \theta_2 < \pi$.

Inverting θ_1, θ_2 to find ϕ_0, ψ are a little complicated by the cyclical $\mod 2\pi$ nature of angles, which leads to the expression

$$\theta_{1,2}(\phi_0, \psi) = \phi_0 \pm 2\psi \equiv \theta_{1,2}(\phi_0 + \pi, \psi + \frac{\pi}{2}) \mod 2\pi \tag{12}$$

This implies that the ranges of ϕ_0 and ψ are reduced to $0 \leq \phi_0, 2\psi < \pi$. Commonly though, ψ is chosen to have a range of $-\frac{\pi}{4} \leq \psi < \frac{\pi}{4}$, resulting in a full range for ϕ_0 of $0 \leq \phi_0 < 2\pi$.

2 Estimation from complex amplitudes

We first investigated estimating A_+, $A_\times$, ϕ_0, ψ from the complex ampitudes of 30 minute SFTs. If we could obtain a well-defined estimate of A_+, $A_\times$ from a single SFT, then a series of SFTs produced over the course of a data run would yield a population $\{(A_+, A_\times)_i\}$ of estimates; from which, a mean and standard deviation could be used respectively as a detection statistic, and a measure of the significance of the detection statistic, e.g. whether the detection statistic deviates significantly from zero.

The A parameters are computed from the data as described in [1, 3]:

$$A_1 = 2\frac{B\Re(F_a) - C\Re(F_b)}{D} \tag{13a}$$

$$A_2 = 2\frac{A\Re(F_b) - C\Re(F_a)}{D} \tag{13b}$$

$$A_3 = 2\frac{B\Im(F_a) - C\Im(F_b)}{D} \tag{13c}$$

$$A_4 = 2\frac{A\Im(F_b) - C\Im(F_a)}{D} \tag{13d}$$

where

$$A = (a|\, a) \qquad B = (b|\, b) \qquad C = (a|\, b) \tag{14}$$

$$D = AB - C^2 \tag{15}$$

and

$$F_a = \left(x|\, a(t) e^{2\pi i f t} \right) \tag{16a}$$

$$F_b = \left(x|\, b(t) e^{2\pi i f t} \right) \,, \tag{16b}$$

where we define

$$(x|\, y) = \sum_{j=0}^{N-1} x_j y_j = \left(\sum_{k=0}^{N-1} \tilde{x}_k \tilde{y}_k^* \right)^* \tag{17}$$

The $\tilde{x}_k$ are the input SFT data.

2.1 Problems and discussion

We initially assumed that $a(t)$ and $b(t)$, being slowly varying, could be treated as constant over the duration of each SFT. However this creates a few problems:

Firstly, that A, B, and C are no longer independent: $C = AB/2$, which implies $D = 0$ and thus equations (13) are singular. This can be solved in a sense by approximating the (discrete) inner products $(x|y)$ by a continuous integral $\int_{t=t_0}^{t_0+T} x(t)y(t)dt$, for which it is possible to find fairly compact analytic expressions. This results in an accurate calculation of A, B, C and ensures that $D \neq 0$; however, D can still be of order 10^{-6}, which would result in a 10^6 "magnification" of any noise in the SFT, which of course is always true for a real search. This can be solved by averaging the $A_{1,2,3,4}$s from several SFTs together; however this is no longer the estimation from a single SFT we initially sought.

Secondly, F_a and F_b are no longer independent: $F_a = aF$ and $F_b = bF$ where $F = \left(x\middle| e^{2\pi ift}\right)$. F has only 2 independent components ($\Re(F)$, $\Im(F)$) but we are trying to use it to find 4 parameters $A_{1,2,3,4}$. This can be solved by using analytic expressions for the Fourier transforms of $a(t)e^{2\pi ift}$ and $b(t)e^{2\pi ift}$. The effect of the $a(t)$ and $b(t)$ are to split the f peak (from the Fourier transform of $e^{2\pi ift}$) into lines at $f \pm f_{\text{Earth}}$ and $f \pm 2f_{\text{Earth}}$. The summation for the inner products should only need to sum over a few bins around these peaks for reasonable accuracy.

A more serious problem is implicit in (16): that F_a and F_b are implicitly dependent upon f, the signal frequency. As this is generally unknown, a search over values of f would be required, which would scale as $T^{-1}_{\text{observation}}$. Thus we see that parameter estimation from complex amplitudes naturally leads to a coherent search, of which the $\mathcal{F}$ statistic [1] is the canonical example. We instead prefer to find a more optimal incoherent search, i.e. one that scales with T^{-1}_{SFT}.

3 Estimation from power

We now investigate estimating A_+, $A_\times$, and ψ from the squared complex amplitudes, i.e. the power, of the SFTs. We consider the signal in the form

$$h(t) = A_+F_+(\psi, t)\cos\Phi(t) + A_\times F_\times(\psi, t)\sin\Phi(t), \tag{18}$$

where F_+ and $F_\times$ are the beam pattern response functions, and Φ is the phase, which includes the intial phase. The phase contains modulations from doppler shifts due to the relative motion between the source and the detector and the frequency evolution of the source. Over the duration of a single 30 minute SFT, the beam pattern functions and frequency of the signal are approximately constant. Thus the strain at discrete time t_j measured from the start of the SFT, where j is the discrete time index, can be approximated as

$$\begin{aligned} h_j \approx F_+(\psi, t_{1/2})A_+\cos(\phi_0 + 2\pi f(t_{1/2})t_j) \\ + F_\times(\psi, t_{1/2})A_\times\sin(\phi_0 + 2\pi f(t_{1/2})t_j), \end{aligned} \tag{19}$$

where $t_{1/2}$ is the time at the midpoint of the SFT, and here ϕ_0 is the approximate phase at the start of the SFT, *not* the initial phase at the start of the observation as before; i.e.

$$\phi_0 = \Phi(t_{1/2}) - 2\pi f(t_{1/2})(T_{\text{SFT}}/2). \tag{20}$$

Using these approximations, the signal can be treated as the sum of pure sinusoids during the time of one SFT, $T_{\rm SFT}$.

The SFT of this data is given by

$$\tilde{h} = \sum_{j=0}^{N-1} h_j e^{-2\pi ijk/N} \Delta t, \tag{21}$$

where Δt is one divided by the sample rate of the data. Ignoring the mismatch in frequency (which is unknown in a search over frequency), the normalized signal power is:

$$\frac{2|\tilde{h}|^2}{T_{\rm SFT}} = 0.5(A_+^2 F_+^2 + A_\times^2 F_\times^2) T_{\rm SFT}, \tag{22}$$

where it is taken to be understood that this is the power from the SFT bin with the signal, and that F_+ and $F_\times$ are constants evaluated at the midpoint of each SFT.

3.1 Derivation of the PowerFlux method

Equation (22) represents the expected signal power for an elliptically polarized signal from one SFT. If we label the SFTs using index α, and consider a linearly polarized signal with $A_\times = 0$, we can define the noise weighed sum of the square deviations in power as

$$g = \sum_\alpha \frac{[P_\alpha - 0.5 A_+^2 F_{+\alpha}^2 T_{\rm SFT}]^2}{S_\alpha^2}, \tag{23}$$

where S_α is the one-sided power spectral density for the appropriate frequency bin, and

$$P_\alpha = \frac{2|\tilde{x}_\alpha|^2}{T_{\rm SFT}}, \tag{24}$$

with $\tilde{x}_\alpha$ the SFT data from the appropriate frequency bin.

A natural way one way to estimate A_+^2, analogous to χ^2 minimization, is to find the value that minimizes g. Thus, we need to solve

$$\frac{\partial g}{\partial A_+^2} = -\sum_\alpha \frac{(P_\alpha - 0.5 A_+^2 F_{+\alpha}^2 T_{\rm SFT}) F_{+\alpha}^2 T_{\rm SFT}}{S_\alpha^2} = 0. \tag{25}$$

Solving for A_+^2 gives,

$$A_+^2 = 4 \sum_\alpha \frac{F_{+\alpha}^2}{S_\alpha^2} \frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2} \Big/ \sum_\alpha \frac{F_{+\alpha}^4}{S_\alpha^2}. \tag{26}$$

Note that equation (26) is the PowerFlux method as defined in [2], though the derivation given here is different. The noise and beam pattern weighting following naturally from the minimization of g in equation (23), but this appears to be equivalent to the maximization of signal to noise ratio given in [2]. Finally, a value for ψ has to be chosen to evaluate $F_{+\alpha}$, and thus a search using this method has to include a search over values of ψ.

3.2 Generalization to estimate A_+^2 and $A_\times^2$

Having obtained the PowerFlux formula by minimizing g given in equation (23), the obvious generalization to an elliptically polarized signal is to redefine g as

$$g = \sum_\alpha \frac{[P_\alpha - 0.5(A_+^2 F_{+\alpha}^2 + A_\times^2 F_{\times\alpha}^2)T_{\rm SFT}]^2}{S_\alpha^2}. \tag{27}$$

Note that for this to work that $F_{+\alpha}^2$ and $F_{\times\alpha}^2$ have to be linearly independent functions of α, which should be true. Thus minimizing g with repect to A_+^2 and $A_\times^2$ gives

$$\frac{\partial g}{\partial A_+^2} = -\sum_\alpha \frac{[P_\alpha - 0.5(A_+^2 F_{+\alpha}^2 + A_\times^2 F_{\times\alpha}^2)T_{\rm SFT}]F_{+\alpha}^2 T_{\rm SFT}}{S_\alpha^2} = 0, \tag{28a}$$

$$\frac{\partial g}{\partial A_\times^2} = -\sum_\alpha \frac{[P_\alpha - 0.5(A_+^2 F_{+\alpha}^2 + A_\times^2 F_{\times\alpha}^2)T_{\rm SFT}]F_{\times\alpha}^2 T_{\rm SFT}}{S_\alpha^2} = 0. \tag{28b}$$

Solving for A_+^2 and $A_\times^2$ gives

$$A_+^2 = \frac{4}{\mathcal{D}}\left[\sum_\alpha \frac{F_{\times\alpha}^4}{S_\alpha^2}\sum_\alpha \frac{F_{+\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2} - \sum_\alpha \frac{F_{+\alpha}^2 F_{\times\alpha}^2}{S_\alpha^2}\sum_\alpha \frac{F_{\times\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2}\right], \tag{29a}$$

$$A_\times^2 = \frac{4}{\mathcal{D}}\left[\sum_\alpha \frac{F_{+\alpha}^4}{S_\alpha^2}\sum_\alpha \frac{F_{\times\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2} - \sum_\alpha \frac{F_{+\alpha}^2 F_{\times\alpha}^2}{S_\alpha^2}\sum_\alpha \frac{F_{+\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2}\right], \tag{29b}$$

where

$$\mathcal{D} = \sum_\alpha \frac{F_{+\alpha}^4}{S_\alpha^2}\sum_\alpha \frac{F_{\times\alpha}^4}{S_\alpha^2} - \left(\sum_\alpha \frac{F_{+\alpha}^2 F_{\times\alpha}^2}{S_\alpha^2}\right)^2. \tag{29c}$$

Equations (29) are a natural extension of the PowerFlux method to elliptically polarized signals. One could use the sum of A_+^2 and $A_\times^2$ as the detection statistic. A value for ψ has to be chosen to evaluate $F_{+\alpha}^2$ and $F_{\times\alpha}^2$, and thus this method still has to include a search over values of ψ. Whether this method improves sensitivity has yet to be shown, and it also appears to be of the order of 10 times the numerical complexity of PowerFlux.

3.3 Generalization to estimate A_+^2, $A_\times^2$, and ψ

We can re-write F_+ and $F_\times$ in terms of ψ and two functions independent of ψ, a and b:

$$F_+(\psi, t) = \sin\zeta[\cos 2\psi\, a(t) + \sin 2\psi\, b(t)], \tag{30a}$$

$$F_\times(\psi, t) = \sin\zeta[\cos 2\psi\, b(t) - \sin 2\psi\, a(t)]. \tag{30b}$$

The normalized signal power can be written as,

$$\frac{2|\tilde{h}_\alpha|^2}{T_{\rm SFT}} = 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\rm SFT}, \tag{31}$$

where the amplitudes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are

$$\mathcal{A} = \sin^2\zeta(A_+^2\cos^2 2\psi + A_\times^2\sin^2 2\psi), \tag{32a}$$

$$\mathcal{B} = \sin^2\zeta(A_+^2\sin^2 2\psi + A_\times^2\cos^2 2\psi), \tag{32b}$$

$$\mathcal{C} = \sin^2\zeta(A_+^2 - A_\times^2)2\cos 2\psi\sin 2\psi. \tag{32c}$$

It is easy to invert these equations, using:

$$\tan 4\psi = \frac{\mathcal{C}}{\mathcal{A} - \mathcal{B}}, \tag{33a}$$

$$A_+^2 + A_\times^2 = \frac{\mathcal{A} + \mathcal{B}}{\sin^2 \zeta}, \tag{33b}$$

$$A_+^2 - A_\times^2 = \frac{\mathcal{A} - \mathcal{B}}{\sin^2 \zeta \cos 4\psi}. \tag{33c}$$

Thus, we can redefine g as

$$g = \sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]^2}{S_\alpha^2}. \tag{34}$$

Note that for this to work that $F_{+\alpha}^2$ and $F_{\times\alpha}^2$ have to be linearly independent functions of α, which should be true. Thus minimizing g with repect to A_+^2 and $A_\times^2$ gives

$$\frac{\partial g}{\partial \mathcal{A}} = -\sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]a_\alpha^2 T_{\mathrm{SFT}}}{S_\alpha^2} = 0, \tag{35a}$$

$$\frac{\partial g}{\partial \mathcal{B}} = -\sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]b_\alpha^2 T_{\mathrm{SFT}}}{S_\alpha^2} = 0, \tag{35b}$$

$$\frac{\partial g}{\partial \mathcal{C}} = -\sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]a_\alpha b_\alpha T_{\mathrm{SFT}}}{S_\alpha^2} = 0. \tag{35c}$$

Thus, the amplitudes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ can be found by inverting equations (35).

In this method, the value for ψ no longer has to be searched over, but the computational complexity of the search has increased. Whether this method improves sensitivity has yet to be shown.

4 Conclusion

We found that estimating A_+, $A_\times$, ϕ_0, ψ from the complex amplitudes of a single SFTs presents several difficulties, and that using multiple SFTs naturally leads to a more computationally expensive coherent-like search. On the other hand, by using the power of the SFTs we were able to extend the PowerFlux method to the estimation of A_+, $A_\times$, and ψ. Further work would determine whether these extensions increase sensitivity, and whether this merits the additional computational cost.

References

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