Mode calculation based on Sigg's radiation pressure induced instability model

Eiichi Hirose
Department of Physics, Syracuse University, Syracuse NY 13244
Feb 5, 2007

There are two modes in two mirror resonator [1]. One is potentially unstable while the other is stable.

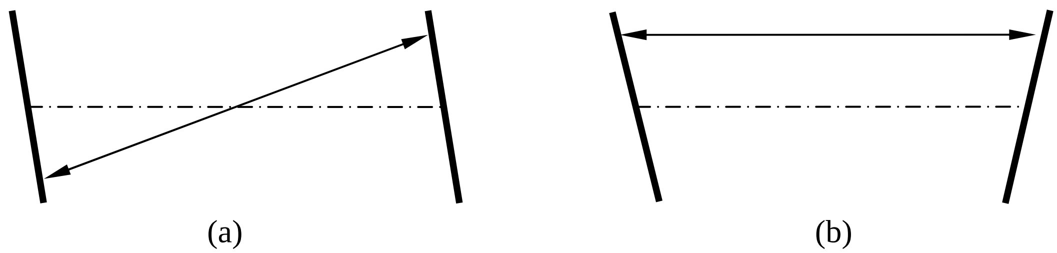

Figure 1: Two different resonator configuration (a) stable (b) unstable

Now, we want to calculate resonance frequency of the system considering small oscillation. In order to do this, let's set up coordinates in the following way.

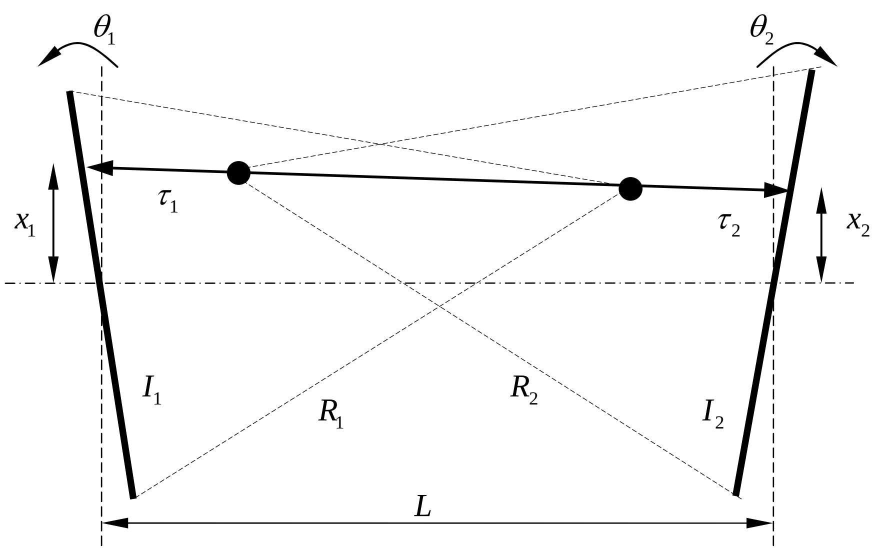

Figure 2: Coordinate setting

Here, I_1 and I_2 are moment of inertia of each mirror. τ_1 , τ_2 are torque due to radiation pressure, which can be expressed using laser power inside the cavity. x_1 and x_2 are distance from the center line to points on which radiation pressure is acting. L, R_1, R_2 are cavity length, radius of curvature of mirror 1, and radius of curvature of mirror 2, respectively. Centers of radius of curvature (ROC) should, in reality, sit outside the mirrors since ROC is much bibber than arm length in LIGO. But, the figure two is drawn in convenient way.

We assume Hook's law for restoring force due to torsion of wires. Namely, restoring torque is assumed to be some constant times small angle. Define the constant as μ_1, μ_2. Then, we can immediately write down kinetic energy and potential.

$$K = \frac{1}{2} I_1 \dot{\theta}_1^{\,2} + \frac{1}{2} I_2 \dot{\theta}_2^{\,2}$$
$$V = \frac{1}{2} \mu_1 \theta_1^{\,2} + \frac{1}{2} \mu_2 \theta_2^{\,2} - \int \tau_1 d\theta_1 - \int \tau_2 d\theta_2$$

where

$$\tau_1 = \frac{2Px_1}{c}$$
$$\tau_2 = \frac{2Px_2}{c}$$

P is the laser power stored inside the cavity, and c is the speed of light. Minus sign of radiation-pressure-induced torque works as if it reduces energy due to torsion spring. Relation between x and θ in resonator is the following [2].

$$x_1 = \frac{g_2}{1 - g_1 g_2} L\theta_1 + \frac{1}{1 - g_1 g_2} L\theta_2$$
$$x_2 = \frac{1}{1 - g_1 g_2} L\theta_1 + \frac{g_1}{1 - g_1 g_2} L\theta_2$$

Here, g_1 and g_2 are g parameters.

$$g_1 = 1 - L / R_1$$
$$g_2 = 1 - L / R_2$$

Therefore, the third and fourth term of potential can be rewritten in the following way.

$$\int \tau_1 d\theta_1 + \int \tau_2 d\theta_2$$
$$= \frac{2P}{c} \int d\theta_1 \left(\frac{g_2}{1 - g_1 g_2} L\theta_1 + \frac{1}{1 - g_1 g_2} L\theta_2 \right) + \frac{2P}{c} \int d\theta_2 \left(\frac{1}{1 - g_1 g_2} L\theta_1 + \frac{g_1}{1 - g_1 g_2} L\theta_2 \right)$$

$$= \frac{P}{c}\left(\frac{g_2 L}{1-g_1 g_2}\theta_1^{\,2} + \frac{g_1 L}{1-g_1 g_2}\theta_2^{\,2} \right) + \frac{2P}{c}\frac{L}{1-g_1 g_2}\int (\theta_2 d\theta_1 + \theta_1 d\theta_2)$$

$$= \frac{P}{c}\left(\frac{g_2 L}{1-g_1 g_2}\theta_1^{\,2} + \frac{g_1 L}{1-g_1 g_2}\theta_2^{\,2} \right) + \frac{2P}{c}\frac{L}{1-g_1 g_2}\theta_1\theta_2$$

$$\because d(\theta_1\theta_2) = \theta_2 d\theta_1 + \theta_1 d\theta_2$$

Thus, potential of this system is

$$V = \frac{1}{2}\mu_1\theta_1^{\,2} + \frac{1}{2}\mu_2\theta_2^{\,2} - \frac{P}{c}\frac{g_2 L}{1-g_1 g_2}\theta_1^{\,2} - \frac{P}{c}\frac{g_1 L}{1-g_1 g_2}\theta_2^{\,2} - \frac{2P}{c}\frac{L}{1-g_1 g_2}\theta_1\theta_2 .$$

When we express kinetic and potential terms as $K = 1/2\, K_{ij}\dot{\theta}_i\dot{\theta}_j$ and $V = 1/2 V_{ij}\theta_i\theta_j$, ($i, j = 1,2$) to see small oscillation, K_{ij} and V_{ij} will be the following.

$$K_{ij} = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$$

$$V_{ij} = \begin{pmatrix} \mu_1 - \frac{2P}{c}\frac{g_2 L}{1-g_1 g_2} & -\frac{2P}{c}\frac{L}{1-g_1 g_2} \\ -\frac{2P}{c}\frac{L}{1-g_1 g_2} & \mu_2 - \frac{2P}{c}\frac{g_1 L}{1-g_1 g_2} \end{pmatrix}$$

Calculate eigenvalues by taking $\det(V - \lambda K) = 0$. Namely,

$$\begin{vmatrix} \mu_1 - \frac{2P}{c}\frac{g_2 L}{1-g_1 g_2} - \lambda I_1 & -\frac{2P}{c}\frac{L}{1-g_1 g_2} \\ -\frac{2P}{c}\frac{L}{1-g_1 g_2} & \mu_2 - \frac{2P}{c}\frac{g_1 L}{1-g_1 g_2} - \lambda I_2 \end{vmatrix} = 0 .$$

After some calculation, we get a characteristic equation.

$$I_1 I_2 \lambda^2 - I_1\left(\mu_2 - \frac{2P}{c}\frac{g_1 L}{1-g_1 g_2} \right)\lambda - I_2\left(\mu_1 - \frac{2P}{c}\frac{g_2 L}{1-g_1 g_2} \right)\lambda$$

$$+\left(\mu_2 - \frac{2P}{c}\frac{g_1 L}{1-g_1 g_2} \right)\left(\mu_1 - \frac{2P}{c}\frac{g_2 L}{1-g_1 g_2} \right) - \frac{4P^2}{c^2}\left(\frac{L}{1-g_1 g_2} \right)^2 = 0$$

If we have $I_1 = I_2 = I$ and $\mu_1 = \mu_2 = \mu$, the equation above will become much simpler. And fortunately, that is the case in LIGO
.

$$I^2\lambda^2 - 2I\left(\mu - \frac{PL}{c}\frac{g_1+g_2}{1-g_1g_2}\right)\lambda + \left(\mu^2 - \frac{2PL}{c}\frac{g_1+g_2}{1-g_1g_2}\mu - \frac{4P^2L^2}{c^2}\frac{1}{1-g_1g_2}\right) = 0$$

This gives eigenvalues.

$$\lambda = \frac{\mu}{I} + \frac{PL}{Ic}\left(\frac{-(g_1+g_2) \pm \sqrt{4+(g_1-g_2)^2}}{1-g_1g_2}\right)$$

We see that laser power stored in the cavity makes system's mode split into two since $\mu/I \equiv \omega^2$ is free torsion pendulum's natural frequency. One is higher than original torsion pendulum's frequency while the other is lower than the original. In other words, original torsion pendulum is degenerate case of the resonator configuration.

Eigenvectors are

$$\begin{pmatrix} 2 \\ g_1 - g_2 - \sqrt{4+(g_1-g_2)^2} \end{pmatrix}\begin{matrix} + \\ - \end{matrix} \text{ , or } \begin{pmatrix} 2 \\ g_1 - g_2 + \sqrt{4+(g_1-g_2)^2} \end{pmatrix}\begin{matrix} + \\ + \end{matrix}.$$

There are two different modes, which correspond to each sketch in figure 5.

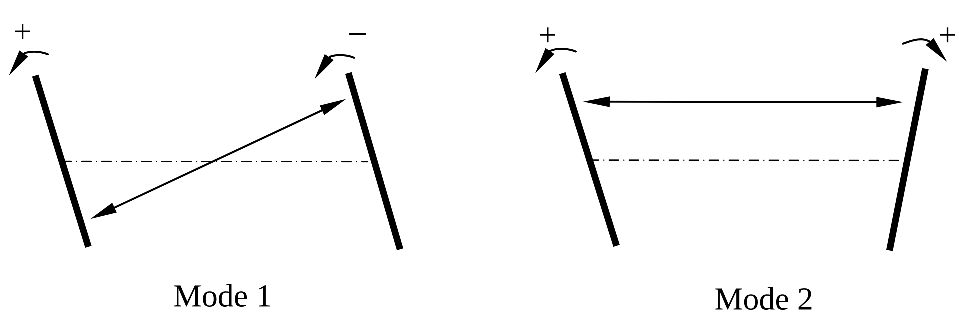

Figure 3: Four modes

Now is the time for us to input some realistic values into the equations. The main parameters are listed in table 1.

Parameter	4K		2K	
	value	unit	value	unit
frequency	$2\pi 0.5$	rad/s	$2\pi 0.5$	rad/s
inertia I	0.0474	kg m^2	0.0474	kg m^2
L	4000	m	2000	m
ROC (ITM)	13760	m	14200	m
ROC (ETM)	7290	m	7400	m
g_1	0.709	-	0.859	-
g_2	0.451	-	0.730	-
speed of light c	300000000	m/s	300000000	m/s

Table 1: Main parameters

It is convenient to express eigenfrequencies with PSL laser power instead of laser power stored in the cavity since we really do not know the actual power there. In order to do this, we have to estimate energy enhancement on the way to the arm cavity. Here, we use energy loss at mode cleaner: 33%, energy enhancement factor at power recycling mirror: 50, energy split at beam splitter: 50%, and finally energy enhancement factor at Fabry-Perot cavity: 140. Therefore, we estimate laser power stored in the cavity from PSL laser power in the following way.

$$P_{caviy} = 140 \times 0.5 \times 40 \times 0.67 \times P_{PSL} \quad \text{[W]}$$

Therefore, with above PSL laser power, eigenfrequecies of the system can be plotted by

$$\eta = \frac{1}{2\pi}\sqrt{\frac{\mu}{I} + \frac{PL}{Ic}\left(\frac{-(g_1+g_2) \pm \sqrt{4+(g_1-g_2)^2}}{1-g_1 g_2}\right)} \quad \text{[Hz]}$$

Plotted are H1 and H2 respectively.

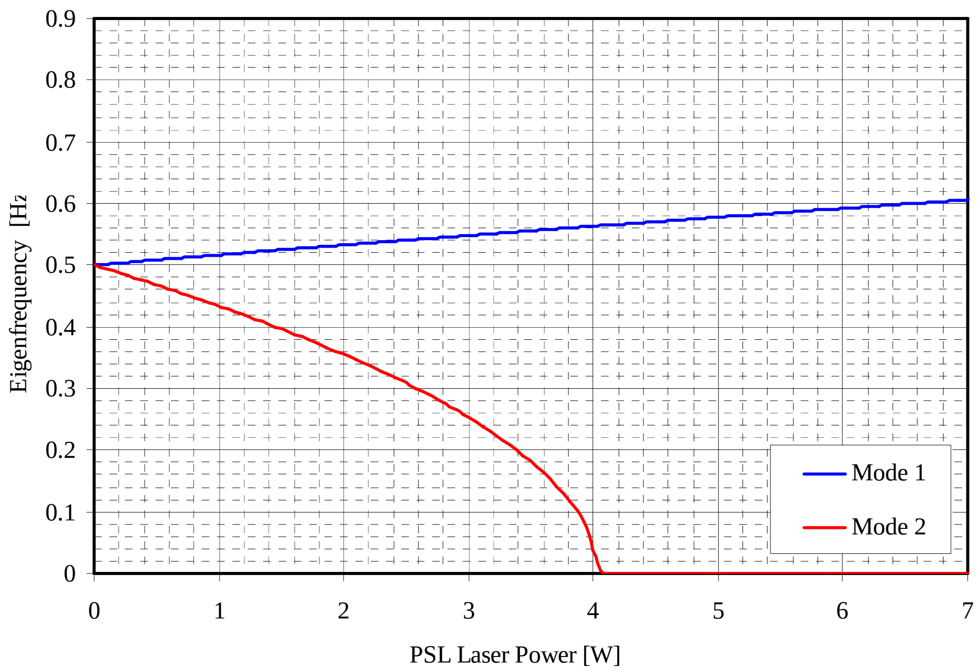

Figure 4: 4K eigenfrequency in different modes

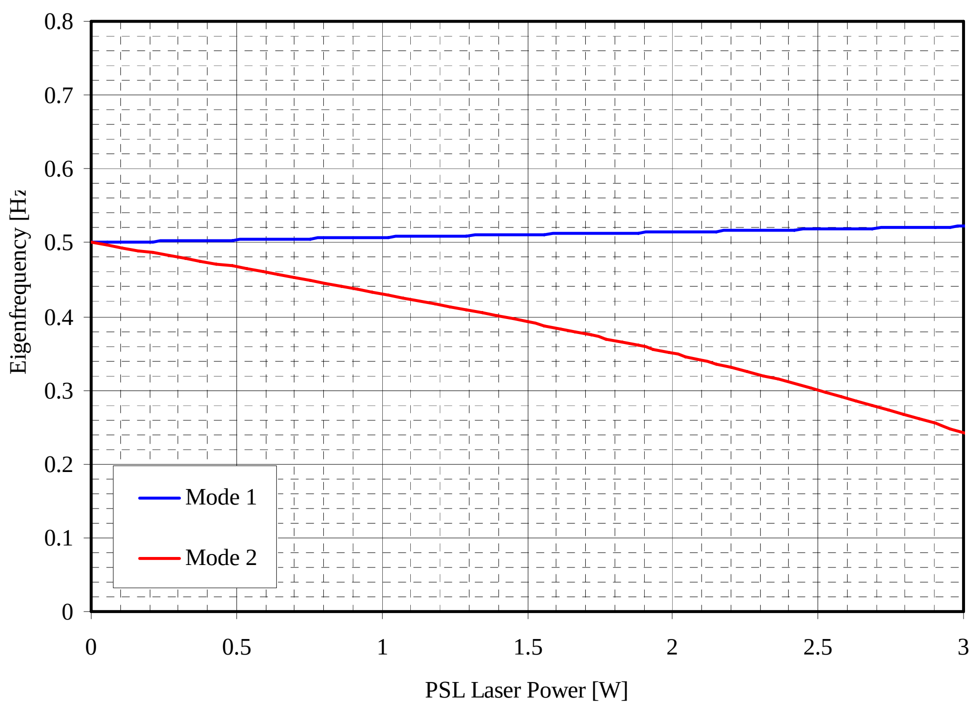

Figure 5: 2K eigenfrequencies in different modes

As seen in figure 3 and 4, Mode 2 is unstable in the sense that resonance frequency becomes zero. The reason is that radiation pressure decrease restoring of pendulum wire and eventually makes restoring force zero as laser power goes up.

References

[1] Daniel Sigg, LIGO-T030120-00

[2] A. E. Siegman, Lasers, University Science Books