

5 February 1989 (revised version¹)

Pressure Specification for LIGO

by Alex Abramovici

1 The Pressure in the Pipes

1.1 Noise Due to the Presence of Gas in the Pipes

Molecules crossing the beam path inside the Fabry-Perot cavities in a gravity wave detector cause fluctuations in the index of refraction of the medium between the mirrors. This, in turn, leads to fluctuations in the optical path which are perceived by the interferometer as fluctuations in the length of the arms. It can be shown² that the spectral density of the equivalent gravity wave signal (for unit signal to noise ratio) is:

$$\delta h^2(f) = \frac{8}{L^2} \int_0^L \frac{\eta}{v_0 w(z)} \left(\frac{n_0 - 1}{\eta_0} \right)^2 \exp \left[-2\pi f \frac{w(z)}{v_0} \right] dz \quad (1)$$

where:

- L - the length of the interferometer arms.

¹revision follows review by R. E. Spero

²for the theory, see S. E. Whitcomb *Optical Pathlength Fluctuations in an Interferometer due to Residual Gas*, provided as an Appendix

- η - the number of gas molecules per unit volume. η_0 is the number of molecules per unit volume at atmospheric pressure, so that $\eta = \eta_0 p/p_0$, with p_0 the standard atmospheric pressure of 760 torr.
- n_0 - the index of refraction of the gas at atmospheric pressure.
- v_0 - the rms speed of molecules in a plane transverse to the light beam, at the operating temperature³: $v_0 = (2kT/\mu)^{1/2}$, where k is the Boltzmann constant, T is the absolute temperature and μ is the mass of the molecule. $\mu = M/\mathcal{N}$, where M is the molecular weight and \mathcal{N} is the Avogadro number.
- $w(z)$ - the gaussian beam radius at coordinate z .
- f - the frequency of the fluctuations.

Replacing the above into Eq.(1) yields:

$$\delta h^2(f) = \frac{8}{L^2} \int_0^L \frac{p}{p_0} \frac{(n_0 - 1)^2}{\eta_0 w(z)} \left(\frac{M}{2kT\mathcal{N}} \right)^{\frac{1}{2}} \exp \left[-\frac{2\pi f w(z) M^{\frac{1}{2}}}{(2kT\mathcal{N})^{\frac{1}{2}}} \right] dz \quad (2)$$

For a LIGO type cavity, e. g. consisting of a flat mirror and a mirror with $R = 5 \text{ km}$, 4 km apart, $w(z)$ is practically a constant. If the pressure is constant, the integral can be readily evaluated⁴. It is reasonable to replace $w(z)$ in Eq. (2) by 2.28 cm , the geometric mean of the largest and smallest value of $w(z)$ for the 4 km cavity. Eqs. (1,2) refer to a single arm. For two arms, the amount of noise power doubles, but the signal power quadruples, so that a unit signal to noise ratio is obtained for half the $\delta h^2(f)$ given by Eqs. (1,2). One notes that, for molecules with mass numbers in the lower tens, the exponential in Eq. (2) is very close to one for frequencies below 1 kHz . Replacing all the constants by their numerical values yields:

$$\delta h^2(f) = 9.67 \times 10^{-34} (n_0 - 1)^2 p M^{\frac{1}{2}} \quad (\text{Hz}^{-1}) \quad (3)$$

where the pressure p is to be expressed in torr. Values of $n_0 - 1$ and M for several gasses of interest are given in Table 1.

³room temperature, for the LIGO

⁴If p is a function of z , one has to take the average value, over the length of the beam

Table 1: $n_0 - 1$, M , C (Eq. 6) and D (Eq. 13) for several gasses

Gas	$(n_0 - 1) \times 10^4$	M	C	D
H ₂	1.4	2	1.0	1.0
He	0.36	4	10.7	0.71
O ₂	2.8	32	0.062	0.25
N ₂	3.0	28	0.058	0.27
CO ₂	4.5	44	0.021	0.21
CO	3.4	28	0.045	0.27
Ar	2.8	40	0.056	0.22
Water	2.5	18	0.104	0.33

If several molecular species are present, the combined power spectral density of equivalent strain noise is evaluated by using:

$$\delta h^2(f) = \sum_i \delta h_i^2(f) \quad (4)$$

1.2 Pressure Specification for the Pipes

The specification for the tolerable pressure in the pipes is obtained as follows:

1. Determine the desired strain sensitivity h_d .
2. Divide h_d by the square root of the corresponding band width $(\Delta f)^{1/2}$ in order to obtain the spectral density of the signal, $\delta h_d(f)$.
3. Assume that a certain molecular species dominates the composition of the gas in the pipes. Evaluate the corresponding pressure specification by setting $\delta h_d(f) = \delta h(f)$ in Eq. (3):

$$p = \frac{10^{33}}{(n_0 - 1)^2 M^{\frac{1}{2}} \Delta f} h_d^2 \quad (\text{torr}) \quad (5)$$

and by using Table 1. If the pressure specification p_{H_2} for hydrogen has been calculated, than for another gas G :

$$p_G = C p_{H_2} \quad ; \quad C = \frac{(n_{0H_2} - 1)^2 M_{H_2}^{\frac{1}{2}}}{(n_{0G} - 1)^2 M_G^{\frac{1}{2}}} \quad (6)$$

The values of C for several gasses are given in Table 1 (column 4).

For example⁵ for $h_d = 5.5 \times 10^{-29}$, $\Delta f = 1.6 \times 10^{-8} \text{ Hz}$ and a residual gas dominated by hydrogen, the admissible residual pressure is $6.9 \times 10^{-9} \text{ torr}$, for unit signal to noise ratio. If the dominant molecular species were water, a similar signal to noise ratio would require the pressure not to exceed $p_{H_2O} = 7.1 \times 10^{-10} \text{ torr}$.

2 The Pressure in the Chambers

2.1 Damping of the Test Mass by Residual Gas

The main effect of the presence of residual gas in the vacuum chambers is damping of the test masses through test mass/molecule collisions. Assume that at each collision with the test mass a molecule carries away a momentum μv_0 . Then, the quality factor Q_{gas} corresponding to the damping of a mass m and cross sectional area S by the residual gas is⁶:

$$Q_{gas} = \frac{m \omega_0 p_0 \mathcal{N}^{\frac{1}{2}}}{S \eta_0 (kT)^{\frac{1}{2}} \rho M^{\frac{1}{2}}} \quad (7)$$

where $\omega_0 = 2\pi f_0$ with f_0 the resonant frequency of the pendulum. The spectral density of gravity wave signal for unit signal to noise ratio is:

$$\delta h_{gas}^2(f) = \frac{16kT\omega_0}{L^2 m Q_{gas}} \frac{1}{(2\pi f)^4} \quad (\text{Hz}^{-1}) \quad (8)$$

⁵corresponding to a periodic signal with $h = 5.5 \times 10^{-29}$, at 215 Hz or higher, and an integration time of 10^7 s .

⁶slightly different values result for different assumptions concerning the momentum transfer. See also V. B. Braginski, A. B. Manukin: *Measurement of small forces in physics experiments*, (U. of Chicago Press, 1977), p 59, and also The Blue Book (*A study of a long baseline gravitational wave antenna system*, 1984), p. V-26. As it will be pointed out in the Summary, the differences between Q_{gas} values corresponding to the various assumptions lead to little variation in the vacuum specification.

Combining Eqs. (7,8) yields:

$$\delta h_{gas}^2(f) = \frac{\eta_0(kT)^{\frac{3}{2}} SM^{\frac{1}{2}}}{\pi^4 p_0 \mathcal{N}^{\frac{1}{2}} L^2 m^2 f^4 p} \quad (9)$$

This noise contribution decreases steeply with increasing frequency; it is therefore more critical at the lowest frequencies of interest.

2.1.1 Wide Band Signal

In this case, h_{gas}^2 is obtained by integrating over the appropriate frequency band, from f_1 to f_2 . Noting that:

$$\int_{f_1}^{f_2} \frac{df}{f^4} \sim \frac{1}{3f_1^3} \quad (10)$$

(for $\Delta f = f_2 - f_1 \sim f_1$) and replacing the values of the parameters and constants in Eq. (9) yields:

$$h_{gas}^2 = 8.2 \times 10^{-32} \frac{SM^{\frac{1}{2}}}{m^2 f_1^3 p} \quad (11)$$

2.1.2 Narrow Band Signal

In this case, one has to determine the desired spectral density of strain by dividing the desired strain sensitivity by the square root of the signal band width and proceed by calculating the admissible pressure from Eq. (9).

2.2 Pressure Specification for the Chambers

As Eqs. (9,11) show, noise due to collisions between the test mass and residual gas molecules is rapidly decreasing with frequency. Therefore, the pressure specification has to be derived for the lowest frequency f_1 of interest, in the following way:

1. Determine the desired strain sensitivity h_d (wide band signal) or the corresponding spectral density of strain $\delta h_d^2(f)$ (narrow band signal).
2. Determine the dominant molecular species and the appropriate value for M .

3. For the wide band case, set $h_{gas} = h_d$ in Eq. (11) and determine p :

$$p = 1.22 \times 10^{31} \frac{m^2 f_1^3}{SM^{\frac{1}{2}}} h_d^2 \quad (12)$$

For the narrow band case, set $\delta h_{gas}^2(f) = \delta h_d^2(f)$ and determine the pressure specification from Eq. (9).

If the pressure specification has been calculated for hydrogen, then for a gas G :

$$p_G = D p_{H_2} \quad ; \quad D = \frac{M_{H_2}^{\frac{1}{2}}}{M_G^{\frac{1}{2}}} \quad (13)$$

The values of D for several gasses are given in Table 1 (column 5).

For example, for $h_d = 10^{-28}$ in a band of 1.6×10^{-8} Hz (from a periodic source) at 10 Hz, and for hydrogen dominating the residual gas, unit signal to noise ratio corresponds⁷ to $p = 3.2 \times 10^{-8}$ torr. The corresponding pressure for water is $p_{H_2O} = 1.1 \times 10^{-8}$ torr.

3 Summary of Pressure Specifications

The present note outlines the procedure for deriving the vacuum specifications for LIGO pipes and vacuum chambers from the desired strain sensitivity and from knowledge of the residual gas composition. The pressure specifications derived for hydrogen and water are summarized in Tables 2,5 below.

The following comments apply:

1. "Ultimate sensitivities" in Tables 2,5 are the bottom lines in the sensitivity curves shown in the December 1987 Proposal (Appendix A).
2. All other parameters being equal, the pressure specification scales with the desired sensitivity as $p \propto h^2$.
3. The most demanding pressure specifications (Table 4) correspond to the highest sensitivities expected to be achieved with interferometers

⁷for a 1 ton fused silica test mass, 0.8 m long, 0.85 m in diameter

Table 2: Specification for pressure in the pipes (bursts)

GW frequency (Hz)	Ultimate h	p_{H_2} (torr)	p_{H_2O} (torr)
10	1.3×10^{-23}	6.1×10^{-7}	6.3×10^{-8}
50 ^a	6.5×10^{-24}	3×10^{-8}	3.2×10^{-9}
100	1.1×10^{-23}	4.4×10^{-8}	4.5×10^{-9}
1000	1.1×10^{-22}	4.4×10^{-7}	4.5×10^{-8}
^a best ultimate sensitivity for bursts			

Table 3: Specification for pressure in the chambers^a (bursts)

GW frequency (Hz)	Ultimate h	p_{H_2} (torr)	p_{H_2O} (torr)
10	1.3×10^{-23}	2.6×10^{-6}	8.5×10^{-7}
50 ^b	6.5×10^{-24}	8×10^{-5}	2.7×10^{-5}
100	1.1×10^{-23}	1.8×10^{-3}	6.1×10^{-4}
1000	1.1×10^{-22}	1.8×10^2	6.1×10^1
^a see Point 4 in this Summary			
^b best ultimate sensitivity for bursts			

Table 4: Specification for pressure in the pipes (periodic^a)

GW frequency (Hz)	Ultimate h	p_{H_2} (torr)	p_{H_2O} (torr)
10	10^{-27}	2.3×10^{-6}	2.4×10^{-7}
100	1.1×10^{-28}	2.7×10^{-8}	2.9×10^{-9}
215 ^b	5.5×10^{-29}	6.9×10^{-9}	7.1×10^{-10}
1000	5.5×10^{-29}	6.9×10^{-9}	7.1×10^{-10}
^a an integration time of 10^7 s is assumed			
^b highest sensitivity for this and higher frequencies			

Table 5: Specification for pressure in the chambers^a (periodic^b)

GW frequency (Hz)	Ultimate h	p _{H₂} (torr)	p _{H₂O} (torr)
10	10 ⁻²⁷	3.2 × 10 ⁻⁶	1.1 × 10 ⁻⁶
100	1.1 × 10 ⁻²⁸	3.9 × 10 ⁻⁴	1.3 × 10 ⁻⁴
215 ^c	5.5 × 10 ⁻²⁹	2.1 × 10 ⁻³	6.8 × 10 ⁻⁴
1000	5.5 × 10 ⁻²⁹	1	0.3

^a see Point 4 in this Summary

^b an integration time of 10⁷ s is assumed

^c highest sensitivity for this and higher frequencies

operating close to the quantum limit and looking for periodic sources, with an integration time of 10⁷ s.

4. The tabulated pressure specifications for the chambers correspond to damping of the test mass by residual gas being the dominant noise source related to gas. One should keep in mind, however, that the chamber manifold has nonzero length, ~ 1% the length of the pipes. This means that if the pressure in the chamber manifold exceeds 100 times the pressure in the pipe, index fluctuations in the manifold will become the dominating gas related noise and interferometer performance will degrade. One should therefore require that the pressure in the chambers should not exceed ten times the pressure in the pipes.
5. Once the pressure specification for hydrogen has been determined, the specifications for other gasses are derived by using Eq. (6) for the pipes and Eq. (13) for the chambers.
6. Estimates using slightly different assumptions for the gas/test mass interaction conclude that the resulting noise is somewhat higher, leading to tolerable pressure levels in the chambers up to twice lower than the ones given above.

4 Conclusion

The result of the above estimates is that in order to achieve the ultimate LIGO performance, as defined in the December 1977 proposal, one should maintain the following ceilings on gas pressure:

- For hydrogen, 7×10^{-9} torr in the pipes and 7×10^{-8} torr in the chambers.
- For water, 7×10^{-10} torr in the pipes and 7×10^{-9} torr in the chambers.

Optical Pathlength Fluctuations in an Interferometer Due to Residual Gas

*S. E. Whitcomb
others?*

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Planning is now underway for large gravitational wave antennae to be constructed in the next 5 years. These antennae will use ultra-precise laser interferometers to monitor gravitationally-induced motions of test mass over multi-kilometer baselines with a precision of 10^{-17} m or better. With such a high precision, these interferometers are sensitive to a number of noise sources which are normally ignored in other applications of optical interferometry. This letter discusses one such noise source, the fluctuations in optical pathlength (equivalent to fluctuations in the index of refraction of the residual gas) due to the statistical variation of the number of atoms in the laser beam. Since the cost of the vacuum system is the major expense associated with the large antennae, it is important to estimate the vacuum requirements as accurately as possible.

Assume that the measurement is to be made along the z axis from $z=0$ to $z=L$ using a laser beam with a gaussian intensity profile (in cylindrical coordinates)

$$I(r, z) = \frac{2I_c}{\pi w(z)^2} \exp\left[-\frac{2r^2}{w(z)^2}\right]$$

where $w(z)$ is the waist parameter and I_c is the total energy flowing in the beam. In what follows, it is assumed that $w \ll L$ and $\frac{dw}{dz} \ll 1$.

The most useful formulation of the problem is to calculate the spectral density of the optical pathlength fluctuations. The optical pathlength $l(t)$ fluctuates about its mean $\langle l \rangle$ as molecules move across the beam. These fluctuations can be characterized by the spectral noise density ($f \neq 0$)

$$G_l(f) = \lim_{T \rightarrow \infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} l(t) \exp(i2\pi f t) dt \right|^2$$

where $l(t)$ is the optical pathlength as a function of time. $G_l(f)$ has units of $\text{m}^2 \text{Hz}^{-1}$.

As shown below, the precision needed for gravitational-wave detection requires gas pressures which yield mean free paths long compared to the transverse dimensions of the beam. Thus the molecules can be considered to cross the beam independently and without collisions. Our approach will be to calculate the effect of a single molecule crossing the beam, as a function of its velocity and position of closest approach to the beam axis, and to subsequently integrate over the equilibrium distribution of velocities and positions to get the spectral density of the optical pathlength fluctuations.

The presence of each molecule in the laser beam has two effects: a small distortion of the phase fronts and a small change in the average phase of the light. The net phase shift is proportional to the number of molecules in the beam, while the wave front distortions grow proportional to the square root of the number of molecules and can thus be ignored. The phase shift due to a particular molecule (and the corresponding addition to the optical pathlength) is proportional to the intensity of the laser light at the position of that molecule divided by the total energy flux of the beam; that is, if the position of the i^{th} molecule is (r_i, z_i) , its presence makes a contribution to the optical pathlength

$$\delta L_i = \beta \exp\left[-\frac{2r_i^2}{w(z_i)^2}\right]$$

where β depends among other things on the type of the molecule. Assuming there is only a single molecular species present, β can be calculated by summing the contributions to the optical pathlength from a uniform distribution of molecules in the beam and comparing the result with that predicted using the classical index of refraction. This procedure gives

$$\beta = \frac{2(\bar{n}-1)}{\eta_0 \pi \omega^2} = \frac{2\alpha}{\pi \omega^2}$$

where η_0 is the number density of molecules, \bar{n} is the mean index of refraction at that density, and $\alpha = (\bar{n}-1)/\eta_0$ depends only on the type of molecule.

In fact, the molecules move through the beam randomly and the contribution to the total optical pathlength of the i^{th} molecule is a function of time $\delta L_i(t)$. If the molecule makes its closest approach to the beam axis with impact parameter ρ_i at time t_i and with speed perpendicular to the beam axis v_i , then

$$\delta L_i(t) = \beta \exp\left[-2 \frac{\rho_i^2 + (t-t_i)^2 v_i^2}{\omega(z_i)^2}\right]$$

The total optical pathlength as a function of time is obtained by summing over all molecules

$$L(t) = L + \sum_i \delta L_i(t)$$

where L is the physical pathlength. This yields a spectral density ($f \neq 0$)

$$\begin{aligned} G_1(f) &= \lim_{T \rightarrow \infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} \sum_i \delta L_i(t) \exp(i2\pi f t) dt \right|^2 \\ &= \lim_{T \rightarrow \infty} \frac{2}{T} \sum_i \left| \int_{-T/2}^{T/2} \delta L_i(t) \exp(i2\pi f t) dt \right|^2 \end{aligned}$$

The cross terms in $G_1(f)$ average to zero because they depend on the transit times of unrelated molecules. The individual integrals are straightforward. The remaining sum and limit may be evaluated using the distribution function for ρ and v .

$$G_1(f) = 2 \int \int \int \eta(\rho, v) \left| \int_{-\infty}^{\infty} \delta l(\rho, v) \exp(i2\pi f t) dt \right|^2 d\rho dv dz$$

where $\eta(\rho, v) d\rho dv$ is the number of particles per unit length per second making their closest approach to the beam axis with impact parameter ρ and perpendicular speed v . Assuming a Maxwell-Boltzmann velocity distribution, η is given by

$$\eta(\rho, v) d\rho dv dz = \frac{4\eta_0 v^2}{v_0^3} \exp\left[-\frac{v^2}{v_0^2}\right] d\rho dv dz$$

where $v_0 = (2kT/m)^{1/2}$. Integrating over ρ and v gives the result

$$G_1(f) = \int_0^L \frac{2\eta_0 \alpha^2}{v_0 \omega(z)} \exp\left[-\frac{2\pi f \omega(z)}{v_0}\right] dz \quad (1)$$

As a practical example of the usage of (1), consider a gravitational wave antenna with two 5 km long arms. A reasonable goal for the strain sensitivity is $\Delta L/L \leq 3 \times 10^{-23} \text{ Hz}^{-1/2}$ at a frequency of 1 kHz. One possible configuration for the gravitational wave interferometer uses a pair of optical cavities for the two arms. The strain noise of such an interferometer due to the gas

$$\frac{\Delta L}{L} = \frac{[2G_1(f)]^{1/2}}{L}$$

is found by looking at the (incoherent) difference between the two arms. The most economical design, in terms of total evacuated volume and mirror size, is a cavity configuration near confocal. In this case the beam has a roughly constant diameter $\omega \approx (\lambda L/2\pi)^{1/2}$. For an antenna with 5 km long arms operating at

$\lambda = 500 \text{ nm}$, a sensitivity of $3 \times 10^{-23} \text{ Hz}^{-1/2}$ at 1 kHz requires a density $< 2 \times 10^{11} \text{ cm}^{-3}$ (assuming the residual gas is N_2), corresponding to a pressure $< 6 \times 10^{-6} \text{ torr}$. In some vacuum systems, more than 95% of the residual gas is hydrogen. If this is the case, then the pressure could be higher by a factor of ~ 10 because of the lower refractive index and higher speed of hydrogen.

An alternative configuration for the interferometer is a Michelson interferometer using Herriott delay lines in the two arms. The geometry of the Herriott delay line is such that the b one-way paths through the arms are nearly independent (non-overlapping) with a minimum beam diameter approximately the same as for the optical cavities discussed above. It follows that the strain sensitivity of such an interferometer is given by

$$\frac{\Delta L}{L} = \frac{[2bG_1(f)]^{1/2}}{bL}$$

The pressure required for a delay line is a factor of b higher than for an interferometer using optical cavities. This result is simple to understand: since the delay line uses a b times larger volume of gas than the cavity, it can have a b times higher density to get the same fractional fluctuations in the column density.

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alex_mail

27R1

List of 3/1/89

From wea Sun Nov 20 12:50:39 1988
To: alex
Subject: pressure specification draft
Cc: fjr

Minor corrections/comments/suggestions:

page 1: definition of v naught: change "the mean square speed..."
to "the rms speed..."

page 2: integration variable dz missing from eq. 2

page 2: "For two arms, the amount of noise doubles, but the signal powers
quadruples, so..." -- do you mean 'the signal quadruples..' ?

page 2: footnote: change "...one has to take the average..."
to "... one may take the average ..." (there are clearly
other ways to solve the problem...)

page 3: table 1: suggest re-ordering as below and adding two columns:
p(allowable)/p(allowable for hydrogen) for tubes and for chambers:

gas	M	(n..	tubes	chambers
			p(allow ----- p(h2)	p(allow ----- p(h2)
He	4	0.51	10.5	.71
H2	2	1.66	1	1
H2O	18	5.15	.104	.33
O2	32	6.66	.062	.25
N2	28	6.90	.058	.27
Ar	40	7.04	.056	.22
CO	28	7.82	.045	.27
CO2	44	11.6	.021	.21

page 3: to implement the new column in table 1, suggest adding eq. 6a:

$p(\text{allowable}) = 1.49 \times 10^{41} (h \text{ sub } d)^2 / \text{delta } f$ for hydrogen;
allowable partial pressures for other constituents are obtained
by multiplying the hydrogen allowable pressure by the factor given
in table 1 (clumsy words, but you get the drift...)

page 4: the square root of 15 is 3.87

page 4: 1st paragraph of 2.1: cross sectional area ... don't you mean total
surface area?

page 5: replace "hidrogen" with 'hydrogen' (2 places)

page 5: (summary): replace 'hydrogen' and 'water' statements with one
statement on (total) pressure expressed as "hydrogen-equivalent pressure";
table 1 tells us how to apportion this among various constituents.

With these minor corrections and additions (the suggestions, of course,
are optional), give it a title ("Pressure Specification for the LIGO" would
be fine), incorporate the Whitcomb paper as Appendix I, and send it to Ernie
for circulation and comments. This should be easy to accomplish by Monday
afternoon (11/21).

Now for some comments; some of these may be eventually incorporated, but
I don't want to hold up immediate circulation of this draft.

88/11/20
15:48:56

alex_mail

2

First, section 1 is just what we need. Given a policy decision on sensitivity as a function of time, I know exactly how to specify tube pressure performance.

2nd, I think that section 2 needs more work; if i follow the physics correctly, I would suggest two things:

1. carry eq. 10 one more step: approximately carry out the integration (referred to in 2.2-3) and evaluate for a 1 Hz bandwidth by replacing $1/f^4$ with $1/(3*f(\min)^3)$, and solve for p:

$$p(\text{allowable}) = 6.1 \times 10^{37} h(f(\min))^2 * f(\min)^3 \text{ for hydrogen}$$

of course, maybe this is wrong; it doesn't give the numerical result you get on page 5....

2. keep the mass, surface area (if I'm correct in this) and suspension pendulum frequency of the test mass as parameters of the result, so we can evaluate the result for smaller test masses (and other optical components)

Third, the summary should evaluate hydrogen-equivalent pressures (you have already taken care of other constituents in table 1) for three cases: "possible early LIGO detectors", "possible advanced LIGO detectors" (the one case you do evaluate), and one (or more) arbitrarily but realistically chosen intermediate cases...

Meanwhile, I already know a lot more about this subject than I did a week ago.

Comment on
"Pressure Specification for LIGO"
by Alex Abramovici, 23 November 1988

It would be helpful to those referring to this document if the allowable pressures for each gas species were given explicitly. The values for hydrogen and water are already given as examples. The existing Table I could be expanded to show these values for all the gases.

Boude Moore

12/12/88

LIGO-1000000000
23 November 1988

Pressure Specification for LIGO

by Alex Abramovici

1 The Pressure in the Pipes

1.1 Noise Due to the Presence of Gas in the Pipes

Molecules crossing the beam path inside the Fabry-Perot cavities in a gravity wave detector cause fluctuations in the index of refraction of the medium between the mirrors. This, in turn, leads to fluctuations in the optical path which are perceived by the interferometer as fluctuations in the length of the arms. It can be shown¹ that the spectral density of the equivalent strain (for unit signal to noise ratio) is:

$$\delta h^2(f) = \frac{1}{L^2} \int_0^L \frac{2\eta}{v_0 w(z)} \left(\frac{n_0 - 1}{\eta_0} \right)^2 \exp \left[-2\pi f \frac{w(z)}{v_0} \right] dz \quad (1)$$

where:

- L - the length of the interferometer arms.

¹for the theory, see S. E. Whitcomb *Optical pathlength fluctuations in an interferometer due to residual gas*, provided as an Appendix

- η - the number of gas molecules per unit volume. η_0 is the number of molecules per unit volume at atmospheric pressure, so that $\eta = \eta_0 p/p_0$, with p_0 the standard atmospheric pressure of 760 torr.
- n_0 - the index of refraction of the gas at atmospheric pressure.
- v_0 - the rms speed of molecules at the operating temperature²: $v_0 = (2kT/\mu)^{1/2}$, where k is the Boltzmann constant, T is the absolute temperature and μ is the mass of the molecule. $\mu = M/\mathcal{N}$, where M is the mass number and \mathcal{N} is the Avogadro number.
- $w(z)$ - the gaussian beam radius at coordinate z .
- f - the frequency of the fluctuations.

Replacing all the above into Eq.(1) yields:

$$\delta h^2(f) = \frac{2}{L^2} \int_0^L \frac{p}{p_0} \frac{(n_0 - 1)^2}{\eta_0 w(z)} \left(\frac{M}{2kT\mathcal{N}} \right)^{\frac{1}{2}} \exp \left[-\frac{2\pi f w(z) M^{\frac{1}{2}}}{(2kT\mathcal{N})^{\frac{1}{2}}} \right] dz \quad (2)$$

For a LIGO type cavity, e. g. consisting of a flat mirror and a mirror with $R = 5 \text{ km}$, 4 km apart, $w(z)$ is practically a constant. If the pressure is constant, the integral can be readily evaluated³. It is reasonable to replace $w(z)$ in Eq. (2) by 2.28 cm , the geometric mean of the largest and smallest value of $w(z)$ for the 4 km cavity. Eqs. (1,2) refer to a single arm. For two arms, the amount of noise doubles, but the signal power quadruples, so that a unit signal to noise ratio is obtained for half the $\delta h^2(f)$ given by Eqs. (1,2). One notes that, for molecules with mass numbers in the lower tens, the exponential in Eq. (2) is very very close to one for frequencies below 1 kHz . Replacing all the constants by their numerical values yields:

$$\delta h^2(f) = 2.43 \times 10^{-34} (n_0 - 1)^2 p M^{\frac{1}{2}} \quad (\text{Hz}^{-\frac{1}{2}}) \quad (3)$$

where the pressure p is to be expressed in torr. Values of $n_0 - 1$ and M for several gasses of interest are given in Table 1.

²room temperature, for the LIGO

³if p is a function of z , one has to take the average value, over the length of the beam

Table 1: $n_0 - 1$, M , C and D for several gasses

Gas	$(n_0 - 1) \times 10^4$	M	C	D
H ₂	1.4	2	1.0	1.0
He	0.36	4	10.7	0.71
O ₂	2.8	32	0.062	0.25
N ₂	3.0	28	0.058	0.27
CO ₂	4.5	44	0.021	0.21
CO	3.4	28	0.045	0.27
Ar	2.8	40	0.056	0.22
Water	2.5	18	0.104	0.33

If several molecular species are present, the combined power spectral density of equivalent strain noise is evaluated by using:

$$\delta h^2(f) = \sum_i \delta h_i^2(f) \quad (4)$$

1.2 Pressure Specification for the Pipes

The specification for the tolerable pressure in the pipes is obtained as follows:

1. Determine the desired strain sensitivity h_d .
2. Divide h_d by the square root of the corresponding band width $(\Delta f)^{1/2}$ in order to obtain the spectral density of the signal, $\delta h_d(f)$.
3. Assume that a certain molecular species dominates the composition of the gas in the pipes. Evaluate the corresponding pressure specification by setting $\delta h_d(f) = \delta h(f)$ in Eq. (3):

$$p = \frac{4.11 \times 10^{33} h_d^2}{(n_0 - 1)^2 M^{\frac{1}{2}} \Delta f} \quad (\text{torr}) \quad (5)$$

and by using Table 1. If the pressure specification p_{H_2} for hydrogen has been calculated, than for another gas G :

$$p_G = C p_{H_2} \quad ; \quad C = \frac{(n_{0H_2} - 1)^2 M_{H_2}^{\frac{1}{2}}}{(n_{0G} - 1)^2 M_G^{\frac{1}{2}}} \quad (6)$$

The values of C for several gasses are given in Table 1 (column 4).

For example⁴ for $h_d = 10^{-23}$, $\Delta f = 100$ Hz and a residual gas dominated by hydrogen, the admissible residual pressure is 1.5×10^{-7} torr, for unit signal to noise ratio. For a residual hydrogen pressure $p_{H_2} = 10^{-8}$ torr, the signal to noise ratio would be $\sqrt{15} = 3.87 \sim 4$, according to Eq. (4). If the dominant molecular species were water, a similar signal to noise ratio would require the pressure not to exceed $p_{H_2O} = 10^{-9}$ torr.

2 The Pressure in the Chambers

2.1 Damping of the Test Mass by Residual Gas

The main effect of the presence of residual gas in the vacuum chambers is damping of the test masses through test mass/molecule collisions. Assume that at each collision with the test mass a molecule carries away a momentum μv_0 . Then, the quality factor Q_{gas} corresponding to the damping of a mass m and cross sectional area S by the residual gas is⁵:

$$Q_{gas} = \frac{m\omega_0 p_0 \mathcal{N}^{\frac{1}{2}}}{S\eta_0 (kT)^{\frac{1}{2}} p M^{\frac{1}{2}}} \quad 1 \quad (7)$$

where $\omega_0 = 2\pi f_0$ with f_0 the resonant frequency of the pendulum. The spectral density of strain for unit signal to noise ratio is:

$$\delta h_{gas}^2(f) = \frac{4kT\omega_0}{L^2 m Q_{gas} (2\pi f)^4} \quad (Hz^{-\frac{1}{2}}) \quad (8)$$

⁴corresponding to a burst signal with $h = 10^{-23}$, at 100 Hz

⁵slightly different values result for different assumptions concerning the momentum transfer. See also V. B. Braginski, A. B. Manukin: *Measurement of small forces in physics experiments*, (U. of Chicago Press, 1977), p 59, and also The Blue Book (*A study of a long baseline gravitational wave antenna system*, 1984), p. V-26. As it will be pointed out in the Summary, the differences between Q_{gas} values corresponding to the various assumptions are irrelevant as far as the vacuum specification is concerned.

Combining Eqs. (7,8) yields:

$$\delta h_{gas}^2(f) = \frac{\eta_0(kT)^{\frac{3}{2}} SM^{\frac{1}{2}}}{4\pi^4 p_0 N^{\frac{1}{2}} L^2 m^2 f^4 P} \quad (9)$$

h_{gas}^2 is obtained by integrating over the appropriate frequency band, from f_1 to f_2 . Noting that:

$$\int_{f_1}^{f_2} \frac{df}{f^4} \sim \frac{1}{3f_1^3} \quad (10)$$

(for $\Delta f = f_2 - f_1 \sim f_1$) and replacing the values of the parameters and constants in Eq. (9) yields:

$$h_{gas}^2 = 2.05 \times 10^{-32} \frac{SM^{\frac{1}{2}}}{m^2 f_1^3 P} \quad (11)$$

2.2 Pressure Specification for the Chambers

As Eq. (11) shows, noise due to collisions between the test mass and residual gas molecules is rapidly decreasing with frequency. Therefore, the pressure specification has to be derived for the lowest frequency f_1 of interest, in the following way:

1. Determine the desired strain sensitivity h_d .
2. Determine the dominant molecular species and replace the appropriate value for M in Eq. (11).
3. Set $h_{gas} = h_d$ in Eq. (11) and determine p :

$$p = 4.88 \times 10^{31} \frac{m^2 f_1^3}{SM^{\frac{1}{2}}} h_d^2 \quad (12)$$

If the pressure specification has been calculated for hydrogen, then for a gas G :

$$p_G = D p_{H_2} \quad ; \quad D = \frac{M_{H_2}^{\frac{1}{2}}}{M_G^{\frac{1}{2}}} \quad (13)$$

The values of D for several gasses are given in Table 1 (column 5).

For example, for $h_d = 10^{-23}$ (from a burst source) in the band from 10 Hz to 20 Hz, and for hydrogen dominating the residual gas, unit signal to noise corresponds⁶ to $p = 6.1 \times 10^{-6}$ torr. A signal to noise ratio of 4 corresponds to $p_{H_2} = 3.8 \times 10^{-7}$ torr. The corresponding pressure for water is $p_{H_2O} = 1.3 \times 10^{-7}$ torr.

3 Summary

The present note outlines the procedure for deriving the vacuum specifications for LIGO pipes and vacuum chambers from the desired strain sensitivity and from knowledge of the residual gas composition. The following specifications are derived for hydrogen and water:

- Hydrogen pressure should not exceed 10^{-8} torr in the pipes and 3.8×10^{-7} torr in the chambers.
- Water pressure should not exceed 10^{-9} torr in the pipes and 1.3×10^{-7} torr in the chambers.

The following comments apply:

1. The pressure specifications above correspond to the highest strain sensitivities expected to be achieved with interferometers operating close to the quantum limit ($h = 10^{-23}$). For the sensitivity of $h = 10^{-21}$ expected from the first LIGO interferometers, all the above pressure specifications should be multiplied by 10^4 (Eqs. (5,12)).
2. Once the pressure specification for hydrogen has been determined, the specifications for other gasses are derived by using Eq. (6) for the pipes and Eq. (13) for the chambers.
3. The pressure specifications above correspond to a signal to noise ratio of 4.
4. Estimates using slightly different assumptions for the gas/test mass interaction conclude that the resulting noise is somewhat higher, leading to tolerable pressure levels in the chambers up to twice lower

⁶for a 1 ton fused silica test mass, 0.8 m long, 0.85 m in diameter

than the ones given above. This is not significant, however, since a small increase in the operating frequency (from 10 Hz to 12.6 Hz) will make the above specification valid again (see Eq. (11)).

Optical Pathlength Fluctuations in an Interferometer Due to Residual Gas

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Planning is now underway for large gravitational wave antennae to be constructed in the next 5 years. These antennae will use ultra-precise laser interferometers to monitor gravitationally-induced motions of test mass over multi-kilometer baselines with a precision of 10^{-17} m or better. With such a high precision, these interferometers are sensitive to a number of noise sources which are normally ignored in other applications of optical interferometry. This letter discusses one such noise source, the fluctuations in optical pathlength (equivalent to fluctuations in the index of refraction of the residual gas) due to the statistical variation of the number of atoms in the laser beam. Since the cost of the vacuum system is the major expense associated with the large antennae, it is important to estimate the vacuum requirements as accurately as possible.

Assume that the measurement is to be made along the z axis from $z=0$ to $z=L$ using a laser beam with a gaussian intensity profile (in cylindrical coordinates)

$$I(r, z) = \frac{2I_0}{\pi w(z)^2} \exp\left[-\frac{2r^2}{w(z)^2}\right]$$

where $w(z)$ is the waist parameter and I_0 is the total energy flowing in the beam. In what follows, it is assumed that $w \ll L$ and $\frac{dw}{dz} \ll 1$.

The most useful formulation of the problem is to calculate the spectral density of the optical pathlength fluctuations. The optical pathlength $l(t)$ fluctuates about its mean $\langle l \rangle$ as molecules move across the beam. These fluctuations can be characterized by the spectral noise density ($f \neq 0$)

$$G_l(f) = \lim_{T \rightarrow \infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} l(t) \exp(i2\pi f t) dt \right|^2$$

where $l(t)$ is the optical pathlength as a function of time. $G_l(f)$ has units of $\text{m}^2 \text{Hz}^{-1}$.

As shown below, the precision needed for gravitational-wave detection requires gas pressures which yield mean free paths long compared to the transverse dimensions of the beam. Thus the molecules can be considered to cross the beam independently and without collisions. Our approach will be to calculate the effect of a single molecule crossing the beam, as a function of its velocity and position of closest approach to the beam axis, and to subsequently integrate over the equilibrium distribution of velocities and positions to get the spectral density of the optical pathlength fluctuations.

The presence of each molecule in the laser beam has two effects: a small distortion of the phase fronts and a small change in the average phase of the light. The net phase shift is proportional to the number of molecules in the beam, while the wave front distortions grow proportional to the square root of the number of molecules and can thus be ignored. The phase shift due to a particular molecule (and the corresponding addition to the optical pathlength) is proportional to the intensity of the laser light at the position of that molecule divided by the total energy flux of the beam; that is, if the position of the i^{th} molecule is (r_i, z_i) , its presence makes a contribution to the optical pathlength

$$\delta L_i = \beta \exp \left[-\frac{2r_i^2}{w(z_i)^2} \right]$$

where β depends among other things on the type of the molecule. Assuming there is only a single molecular species present, β can be calculated by summing the contributions to the optical pathlength from a uniform distribution of molecules in the beam and comparing the result with that predicted using the classical index of refraction. This procedure gives

$$\beta = \frac{2(\bar{n}-1)}{\eta_0 \pi w^2} = \frac{2\alpha}{\pi w^2}$$

where η_0 is the number density of molecules, \bar{n} is the mean index of refraction at that density, and $\alpha = (\bar{n}-1)/\eta_0$ depends only on the type of molecule.

In fact, the molecules move through the beam randomly and the contribution to the total optical pathlength of the i^{th} molecule is a function of time $\delta L_i(t)$. If the molecule makes its closest approach to the beam axis with impact parameter ρ_i at time t_i and with speed perpendicular to the beam axis v_i , then

$$\delta L_i(t) = \beta \exp \left[-2 \frac{\rho_i^2 + (t-t_i)^2 v_i^2}{w(z_i)^2} \right]$$

The total optical pathlength as a function of time is obtained by summing over all molecules

$$l(t) = L + \sum_i \delta L_i(t)$$

where L is the physical pathlength. This yields a spectral density ($f \neq 0$)

$$\begin{aligned} G_l(f) &= \lim_{T \rightarrow \infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} \sum_i \delta L_i(t) \exp(i2\pi f t) dt \right|^2 \\ &= \lim_{T \rightarrow \infty} \frac{2}{T} \sum_i \left| \int_{-T/2}^{T/2} \delta L_i(t) \exp(i2\pi f t) dt \right|^2 \end{aligned}$$

The cross terms in $G_1(f)$ average to zero because they depend on the transit times of unrelated molecules. The individual integrals are straightforward. The remaining sum and limit may be evaluated using the distribution function for ρ and v .

$$G_1(f) = 2 \int \int \int \eta(\rho, v) \left| \int_{-\infty}^{\infty} \delta l(\rho, v) \exp(i2\pi f t) dt \right|^2 d\rho dv dz$$

where $\eta(\rho, v) d\rho dv$ is the number of particles per unit length per second making their closest approach to the beam axis with impact parameter ρ and perpendicular speed v . Assuming a Maxwell-Boltzmann velocity distribution, η is given by

$$\eta(\rho, v) d\rho dv dz = \frac{4\eta_0 v^2}{v_0^2} \exp\left[-\frac{v^2}{v_0^2}\right] d\rho dv dz$$

where $v_0 = (2kT/m)^{1/2}$. Integrating over ρ and v gives the result

$$G_1(f) = \int_0^L \frac{2\eta_0 \alpha^2}{v_0 w(z)} \exp\left[-\frac{2\pi f w(z)}{v_0}\right] dz \quad (1)$$

As a practical example of the usage of (1), consider a gravitational wave antenna with two 5 km long arms. A reasonable goal for the strain sensitivity is $\Delta L/L \leq 3 \times 10^{-23} \text{ Hz}^{-1/2}$ at a frequency of 1 kHz. One possible configuration for the gravitational wave interferometer uses a pair of optical cavities for the two arms. The strain noise of such an interferometer due to the gas

$$\frac{\Delta L}{L} = \frac{[2G_1(f)]^{1/2}}{L}$$

is found by looking at the (incoherent) difference between the two arms. The most economical design, in terms of total evacuated volume and mirror size, is a cavity configuration near confocal. In this case the beam has a roughly constant diameter $w \approx (\lambda L/2\pi)^{1/2}$. For an antenna with 5 km long arms operating at

$\lambda = 500 \text{ nm}$, a sensitivity of $3 \times 10^{-23} \text{ Hz}^{-1/2}$ at 1 kHz requires a density $< 2 \times 10^{11} \text{ cm}^{-3}$ (assuming the residual gas is N_2), corresponding to a pressure $< 6 \times 10^{-6} \text{ torr}$. In some vacuum systems, more than 95% of the residual gas is hydrogen. If this is the case, then the pressure could be higher by a factor of ~ 10 because of the lower refractive index and higher speed of hydrogen.

An alternative configuration for the interferometer is a Michelson interferometer using Herriott delay lines in the two arms. The geometry of the Herriott delay line is such that the b one-way paths through the arms are nearly independent (non-overlapping) with a minimum beam diameter approximately the same as for the optical cavities discussed above. It follows that the strain sensitivity of such an interferometer is given by

$$\frac{\Delta L}{L} = \frac{[2bG_r(f)]^{1/2}}{bL}$$

The pressure required for a delay line is a factor of b higher than for an interferometer using optical cavities. This result is simple to understand: since the delay line uses a b times larger volume of gas than the cavity, it can have a b times higher density to get the same fractional fluctuations in the column density.