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Internal Scattering in Fabry-Perot Interferometers

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Abstract: Fabry-Perot interferometers are the most promising sensors for large gravity-wave detectors. The vibrations in the enclosure of the interferometer are considered as a possible noise source. It is shown, that by providing a sufficient number of random scatterers on the walls, this noise should not limit the expected sensitivity of gravity wave detectors.

1. Introduction

Fabry-Perot interferometers are currently used in gravity wave detectors as extremely high quality resonators monitoring the length of interferometer arms. The fundamental limits to the precision of these measurements come from the photon counting noise (photon shott noise), from the thermal noise displacing the mirror surfaces and from the quantum noise (Heisenberg uncertainty). These limitations are well understood (Drever et all 1983) and the existing gravity wave detectors are designed so that this noise will be below the sensitivity threshold. However, the development of present gravity wave detectors with sensitivity of the order 10^{-18} m/$\sqrt{\mathrm{Hz}}$ has wittnessed a constant struggle with non-fundamental noise sources, i.e. noise sources which could in principle be elliminated and are present only due to one's inability to produce ideal components. In this article I will discuss one of the many "non-fundamental" noise sorces - the scattering of light off the vacuum tank walls. It comes about as follows: the two mirrors in the interferometer are not perfect, so that some light is scattered off their surface and hits the walls of the vacuum tank. This light may again find it's way into the interfering path if it either escapes through the other mirror and hits the photodiode or if it is scattered again in the main beam. These effects only add to the interferometer noise if the vacuum tank walls are moving and modulate the phase of the scattered light. I will estimate the size of this effect and I will show that it is not difficult to provide sufficient isolation to make this noise negligible.

2. The model

A Fabry-Perot interferometer may be considered as an electromagnetic cavity enclosed by two end mirrors and the walls of a vacuum tank which is usually a long cylinder with the axis coinciding with the optical axis common to the two mirrors. The two mirrors are the main component of the interferometer, since they provide boundary conditions for a gaussian beam to form inbetween. If the mirrors are perfect, the intensity in the Gaussian beam decays exponentially with the distance from the symmetry axis. This makes the coupling of the beam to the vacuum tank wall decay exponentially with the radius of the vacuum pipe. In this way one usually justifies the neglect of the the vacuum tank boundary in solving the field equations inside the cavity. However, actual mirrors are not perfect and they always scatter a small fraction of incident light. I consider the effect of this stray light by way of a perturbation analysis which starts with a perfect cavity enclosed by smooth nonabsorbing, (almost) nontransmitting mirrors and a nonabsorbing but possibly rough wall. I also assume that for the purpose of present estimates it suffices to describe the EM field inside the interferometer cavity by a scalar field Ψ obeying the wave equation:

$$\Delta\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \qquad (1)$$

Since the perfect cavity is chosen to have nonabsorbing and nontransmitting mirrors, the field Ψ obeys the boundary condition:

$$\Psi\Big|_{\text{boundary}} = 0 \qquad (1a)$$

Here boundary includes the mirror surfaces and the walls of the vacuum tank. I will denote the stationary state solutions to this equation as:

$$\Psi = \bar{\varphi}_\sigma(\boldsymbol{r})\, e^{i\omega_\sigma t} \qquad (2)$$

where φ_σ are eigenfunctions of:

$$\Delta\bar{\varphi}_\sigma + \bar{k}_\sigma^2\,\bar{\varphi}_\sigma = 0 \qquad (3)$$

The spatial dependence of $\bar{\varphi}_\sigma(\boldsymbol{r})$ is, of course, a function of the geometry of the cavity. I assume that the mirrors are spherical with appropriate curvature, so that the modes $\bar{\varphi}_\sigma$ are Gaussian modes. The boundary condition $\Psi=0$ on the wall is not very restricting for the main modes, since they decay exponentially away from the symmetry axis. Therefore, I expect that it is possible to start a meaningfull perturbation analysis with the "perfect" cavity as defined above.

A more realistic model of the Fabry-Perot cavity must include:

a) the nonzero transmisivity of the mirrors.

b) scattering off the mirrors

c) scattering off the moving walls

The nonzero transmisivity of the mirrors can be modelled by adding an imaginary part to the eigenfrequencies [$\omega_\sigma \to \omega_\sigma + \frac{i}{\tau}$] , and the resulting losses are compensated with a weak driving field, which adds a small inhomogeneus term to the boundary conditions. Note that absorption in the mirrors has the same effect at this level, so that it need not be discussed.

I assume that the scattering off the mirrors occurs due to microroughness. In this model the actual cavity differs from the ideal cavity only through a slight perturbation of the boundary where the field Ψ vanishes. The new eigenfunctions and eigenvalues of eq. (1) governing the field Ψ inside the cavity are computed via a perturbation analysis developed in the next chapter.

The scattering off the walls is time dependent due to vibrations produced by external accoustical noise. But note that the field inside the cavity with moving walls is again governed by eq. (1) and, again, only the boundary is perturbed as a function of time. The peturbation theory of the next chapter is applicable for this case also.

3. The perturbation theory

In this chapeter I develop a perturbation theory for the following problem: if the solutions ($\overline{\varphi}_\sigma$) of equation (1) with boundary conditions (1a) on the boundary $\partial\Sigma_0$ are known, find the solutions of eq. (1) with bondary conditions (1a) on a perturbed boundary $\partial\Sigma$ (fig. 1). A convenient way to do this is to introduce two systems of coordinates - one for the unperturbed problem and one for the perturbed problem - so that the values of the coordinates on the boundary are the same for both problems. In this way the perturbation leaves the boundary conditions unchanged, and only the form of the Laplacian operator changes. I start the unperturbed problem in cartesian coordinates (x, y, z). The perturbed coordinates (ξ, η, ζ) are introduced so that the cartesian coordinates (x, y, z) of points inside the perturbed cavity are:

$$x = \xi + U_1(\xi, \eta, \zeta)$$

$$y = \eta + U_2(\xi, \eta, \zeta) \qquad (4)$$

$$z = \zeta + U_3(\xi, \eta, \zeta) .$$

If the point (ξ, η, ζ) belongs to the boundary, then the vector $\mathbf{U}(\xi, \eta, \zeta)$ is the displacement of the boundary (mirors or walls) with respect to the perfect cavity. Note that appart from this restriction the choice of $\mathbf{U}$ ($\mathbf{U} \equiv \{U_1, U_2, U_3\}$) is still quite free. However, in order to make the calculations easy, I will choose a particular gauge. Let us first compute the metric (to first order) with respect to the new coordinates ($x^1 = \xi$, $x^2 = \eta$, $x^3 = \zeta$). The result is:

$$g_{ij} = \delta_{ij} + U_{i,j} + U_{j,i} \qquad (i, j = 1, 2, 3) \qquad (5)$$

Here an index after a comma means differentiation with respect to the coordinate with the given index (for example $U_{3,2} = \partial U_3/\partial\eta$). The Laplacian Δ in the new coordinates is (to first order also):

$$\Delta\Psi = \nabla^2\Psi - \nabla^2\mathbf{U}\cdot\nabla\Psi - 2\, U_{i,j}\Psi_{i,j} \qquad (6)$$

with the notation:

$$\nabla = \left\{\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right\}$$

$$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2} \qquad (7)$$

bold letters indicate vectors
summation over repeated indeces is implied
scalar product is indicated by the dot.

The operator Δ is now obviously the sum of the old unperturbed term (∇^2) and the perturbation $\delta\Delta$, which depends on the perturbing field $\mathbf{U}$. The unperturbed Laplacian is certainly hermitian. It is desirable that the new Laplacian be split in such a way that both parts ∇^2 and $\delta\Delta$ are also hermitian to first order. This requirement will only be met if the divergence of the field $\mathbf{U}$ is a constant, as can be seen, if one remembers that the scalar product in the Hilbert space spanned by Δ is:

$$(X,\Psi) = \int X\,\Psi\,\sqrt{g}\; d^3x \quad , \qquad (8)$$

where $g = 1 + 2\,\nabla\cdot\mathbf{U}$ is the determinant of the metric tensor g_{ij}. Finally, we can write the vector field $\mathbf{U}$ as a sum of a gradient and a curl ($\mathbf{U} = \nabla\Phi$ + **curl** $\mathbf{A}$). If we limit ourselves to perturbed surfaces $\partial\Sigma$ such that the normal to any point in $\partial\Sigma_o$ pierces only one point in $\partial\Sigma$, then we can choose $\mathbf{A} = 0$ (fig. 1) and: $\mathbf{U} = \nabla\Phi$. The hermiticity of the operator ∇^2 requires:

$$\nabla^2\Phi = \text{const} \qquad (9)$$

and the boundary condition is that $(\partial\Phi/\partial n)\mathbf{n}_o$ at P_o in $\partial\Sigma_o$ brings us to a point P in $\partial\Sigma$; here $\partial/\partial n$ means the derivative with respect to the boundary surface and $\mathbf{n}_o$ is the normal to the unperturbed boundary at the point P_o (fig. 1). Note that the coordinate gauge in $\partial\Sigma$ is fixed with respect to the coordinates x, y, z and depends only on the geometry of $\partial\Sigma$.

In the system of coordinates just defined, the perturbation $\delta\Delta$ has the simple form:

$$\delta\Delta(\Psi) = -2\,\Phi_{,i,j}\Psi_{,i,j} \quad . \tag{10}$$

It is particlarly appealing to note that the matrix elements of this operator reduce to surface integrals over the boundary of the cavity, and are:

$$(\bar{\varphi}_\alpha, \delta\Delta\bar{\varphi}_\beta) = (\bar{\varphi}_\beta, \delta\Delta\bar{\varphi}_\alpha) = \tag{11}$$

$$= -\frac{1}{4}\oint (\nabla\Phi\cdot\nabla\bar{\varphi}_\alpha)(\nabla\bar{\varphi}_\beta\cdot d\mathbf{S}) - \frac{1}{4}\oint (\nabla\Phi\cdot\nabla\bar{\varphi}_\beta)(\nabla\bar{\varphi}_\alpha\cdot d\mathbf{S})$$

As already mentioned, scattering off the mirrors is modelled as a stationary process due to perturbations in mirror surface. The main excited mode (φ_o) in the cavity can, threrefore, be expressed as a linear combination of ideal cavity modes:

$$\varphi_o = \bar{\varphi}_o + \sum a^{(o)}_\lambda \bar{\varphi}_\lambda \tag{12}$$

Assuming, for simplicity, a nondegenerate spectrum of $\bar{k}_\alpha$, the expansion coefficients $a^{(o)}_\lambda$ are computed according to standard perturbation techniques:

$$a^{(\alpha)}_\lambda = \frac{(\bar{\varphi}_\lambda, \delta\Delta\bar{\varphi}_\alpha)}{\bar{k}^2_\lambda - \bar{k}^2_\alpha} \quad , \tag{13a}$$

and the corrected eigenvalues (k^2_α) are:

$$k^2_\alpha = \bar{k}^2_\alpha + (\varphi_\alpha, \delta\Delta\,\varphi_\alpha) \quad . \tag{13b}$$

Of course, the perturbation of the mirror $\delta\Delta$ is usually not known in practice. In fact, one usually infers the coefficients a_λ from the distribution of scattered light. For example Thorne (1987) used the following probability of scattering the light into a solid angle $d\Omega$ pointing in the direction $\mathbf{\Omega}$ subtending an angle ϑ ($\vartheta \ll 1$ and $\mathbf{\Omega}$ is not far from the normal to the mirror) with the specularly reflected light :

$$P(\vartheta)\,d\Omega \approx \frac{\beta}{\vartheta^2}\,d\Omega \tag{14}$$

The measurements of Elson and Bennett (1979) if extrapolated to small angles are in reasonable agreement with this expression and the coefficient β derived from these measurements is for superpolished mirrors about $1.5\cdot10^{-6}$.

Finally, we must include the perturbation due to moving vacuum tank walls. Let $\partial\Delta(t)$ be the time dependent perturbation of the Laplacian due to moving walls. Then, according to standard perturbation theory, the field Ψ inside the cavity can be expanded in a time dependent series of stationary eigenfunctions φ_α (not $\bar{\varphi}_\alpha$!) as follows:

$$\Psi(\mathbf{r},t) = \left[\varphi_o(\mathbf{r}) + \sum c_{\lambda}(t)\;\varphi_\lambda(\mathbf{r})\right]e^{i\omega_o t} \quad , \tag{15}$$

and the coefficients c_λ obey the equation (the superscript (o) referring to the zero order excited state quantum numbers will be ommitted henceforth):

$$\frac{d^2}{dt^2}\left(c_\lambda e^{i\omega_o t}\right) + \omega_\lambda^2\left(c_\lambda e^{i\omega_o t}\right) = -2c^2 e^{i\omega_o t}(\varphi_\lambda, \partial\Delta(t)\varphi_o) \tag{16}$$

This equation is reminiscent of a forced harmonic oscillator - the field in the main excited mode ($\Psi^{(o)} = \varphi_o\, e^{i\omega_o t}$) is coupled to other modes via the force term on the right hand side of eq. (16). With this interpretation in mind it seems reasonable to model the finite lifetime of modes inside the cavity by adding a damping term to the above equation.

There is another interesting interpretation of equation (16). Let us denote:

$$\zeta_\lambda(t) = c_\lambda(t)\,e^{i\omega_o t} \tag{17}$$

and according to (15) the field Ψ is a sum of the unperturbed part Ψ_o and the perturbation:

$$\Psi(\mathbf{r},t) = \Psi_o(\mathbf{r},t) + \psi(\mathbf{r},t) = \Psi_o(\mathbf{r},t) + \sum_\lambda \zeta_\lambda(t)\,\varphi_\lambda(\mathbf{r}) \tag{18}$$

Multiply (16) by φ_λ, sum over λ, take into account (3), and one obtains:

$$\frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi = -2c^2\sum\varphi_\lambda(\mathbf{r})\left(\varphi_\lambda(\mathbf{r}'),\partial\Delta(\mathbf{r}',t)\Psi_o(\mathbf{r}',t)\right)$$

Finally invoke the completeness of eigenfunctions φ_λ, and one realizes that ψ is the solution of the forced wave equation with the source spread over the surface of the walls as follows:

$$\frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi = \frac{1}{2}\,c^2\left\{\oint\left(\mathbf{u}\cdot\nabla'\delta^3(\mathbf{r}-\mathbf{r}')\right)\left(\nabla'\Psi_o\cdot d\mathbf{S}'\right) + \oint\left(\mathbf{u}\cdot\nabla'\Psi_o\right)\left(\nabla'\delta^3(\mathbf{r}-\mathbf{r}')\cdot d\mathbf{S}'\right)\right\} \tag{19}$$

This result is similar to the current source model discussed by Kroger and Kretschmann (1970), which was used to describe the scattering off rough surfaces. Note, however, that our mathematical approach is closer to the calculations of Elson (1975).

4. Some estimates

Our final task is to estimate the noise measured by the photodetector in the interferometer that is produced by noise in $c_\lambda(t)$, which is in turn generated by the motion of side walls of the interferometer. In order to understand how this noise is detected by the interferometer, let us first consider a noise-free distance measurement. In this case only the fundamental mode $\Psi^{(o)}$ is excided by a laser which has a frequency $\tilde{\omega}_o$ very close to the eigenfrequency of the cavity ω_o. In order to describe the excitation of this mode, write $\Psi^{(o)}$ in the spirit of eq. (15) as $c_o(t)e^{i\omega_o t}\varphi_o(\mathbf{r})$. One expects that $c_o(t)$ obeys a slightly modified equation (16) as follows:

$$\frac{d^2}{dt^2}\left(c_o\,e^{i\omega_o t}\right) + \frac{2}{\tau}\frac{d}{dt}\left(c_o e^{i\omega_o t}\right) + \omega_o^2\left(c_o\,e^{i\omega_o t}\right) = i\,\frac{\alpha}{2}\,c^2\int_{\text{mirror}}(\mathbf{n}.\nabla\varphi_{\text{laser}}e^{i\tilde{\omega}_o t})(\mathbf{n}.\nabla\varphi_o)\,dS \tag{20}$$

I have guessed the right hand side in analogy with (11). Here $\mathbf{n}$ means the unit normal to the mirror, $\mathbf{n}.\nabla\varphi_{\text{laser}}\,e^{i\omega_o t}$ is the amplitude of the field from the laser on the entrance mirror of the cavity, and α is a constant connected to the contrast.

The photodiode in the interferometer detects the interference between the laser light and the light leaking from the cavity. The phase modulation is so arranged that (in the vicinity of the resonance) the demodulated signal is proportional to the phase difference between the laser light and the cavity light. With ideal contrast this is (according to (20)):

$$U_{\text{signal}} \sim \Delta\varphi = tg^{-1}\left(\frac{\tilde{\omega}_o - \omega_o}{\tilde{\omega}^2}\,\tilde{\omega}_o\,\tau\right) \approx \omega_o\tau\,(1 - \frac{\omega_o}{\tilde{\omega}_o}) = \tilde{\omega}_o\tau\left(\frac{\delta L}{\lambda}\right) \tag{21}$$

The last equality took into account the fact that the detuning of the cavity from the laser is usually interpreted as the change in the length of the cavity (δL).

There are two ways in which the moving vacuum tank walls can inject noise in the interferometer signal.

i) the light scattered off the noisy wall is frequency modulated and if it joins the light on the photodiode it produces a phase shift which is proportional to the ratio of the intensity in the modulated scattered beam to the intensity in the main beam.

ii) the frequency modulated scattered light may find it's way into the main beam and modulates it's phase inside the cavity.

i) The direct contribution of the scattered beam

I imagine that the vacuum tank walls are made of a microcrystaline substance. The flat surfaces of the microcrystals forming the walls are randomly oriented. I will also assume (to make the calculations simpler) that the sizes of the crystals in the surface are larger then the wavelength of light. Due to random orientation (and also due to random surface contamination) the light scattered from different crystals is uncorrelated, so that the intensity of light at any given position is the sum of intensities contributed by different scatterers.

The contribution of each scatterer can be computed according to (19). Since each crystalline surface is small (but large with respect to the wavelength), one can describe Ψ_0 on a particular surface (dS) as a specularly reflected monocromatic plane wave coming from the direction $\mathbf{n}$ determined by the line of sight to the surface of the mirror where the scattered light originates (Fig. 2). The source on the right hand side of equation (19) is a dipole layer with the intensity proportional to the amplitude of the oscillation of the wall ($\mathcal{N}(\mathbf{r},\Omega)$) projected on the surface normal ($\mathbf{n}_0$)and multiplied by the cosine $\mathbf{n}\cdot\mathbf{n}_0$. It produces two additional specularly reflected beams- one with frequency $\omega_0+\Omega$, and the other with $\omega_0-\Omega$. If $\mathcal{J}_0$ is the intensity of light in the beam Ψ_0 at dS, then the two additional beams are reflected in the direction of the original reflected beam (To be precise, there is a small discrepancy, corresponding to the aberation of light with respect to the moving wall, but it can be safely disregarded in our case.) and their intensities are :

$$\mathcal{J}_+ = \mathcal{J}_- = \left(\frac{2\pi}{\lambda}\mathcal{N}(\mathbf{r},\Omega)\right)^2 \mathcal{J}_0(\mathbf{r}) \cos^2\delta \cos^2\gamma \qquad (22)$$

The power of radiation with frequency $\omega_0+\Omega$ reaching a small surface ($d\mathcal{S}$) in the mirror M_1 is the sum of fluxes contributed by all the incoherent scatterers in the wall, i.e.:

$$dP_+ = \int dS\ \mathcal{J}_+(\mathbf{r},\Omega)\, \omega(\delta,\xi) \frac{d\mathcal{S}}{R^2 + x^2} \cos^2\vartheta' \qquad (23)$$

Here $\omega(\vartheta,\xi)\, d\mathcal{S} \cos\vartheta'/(R^2+x^2)$ is the probability that the crystallite dS is oriented so that it reflects the modulated beam on the exit pupil of the mirror.

The integral (23) can be evaluated if we insert (22) for $\mathcal{Q}_+$, where $\mathcal{Q}_o$ is computed according to (14). Te result is:

$$\mathcal{Q}_o = P_o \beta / R^2 \qquad (24)$$

Here $P_o \left(= \int |\Psi_o|^2 dS\right)$ is the circulating power in the interferometer. One must also make an assumption for ω; as a matter of illustration I take the simplest possible form $\omega = (2\pi)^{-1}$ ($0<\delta<\pi/2$, $0<\xi<2\pi$). Finally I compute the power (23) assuming that $\mathcal{N}(\mathbf{r},\Omega) = \mathcal{N}(\Omega)$ is normal to the wall and has the same phase everywhere . The result is:

$$dP_+ \sim \left(\frac{2\pi}{\lambda}\mathcal{N}\right)^2 \frac{\beta P_o}{8\pi R^2}\, d\mathcal{S}\, \mathcal{D}(L/R) \qquad (25)$$

$\mathcal{D}(L/R)$ is the result of integration of all the trigonometric functions over the surface of the wall. Numerical integration has shown that $\mathcal{D}$ is essentially a constant with the value of about 0.156 for L>>R. Some numerical results are given in table 1.

Table 1

L/R	$\mathcal{D}$
10	.209293
100	.160077
200	.15778
400	.15669
800	.15616
1600	.15590
3200	.15578

Closer scrutinty of the integrand in $\mathcal{D}$ also shows that the main contribution to the scattered light comes from the last 2R long piece of the vacuum pipe with the center of the distribution at about 2R from the end of the pipe.

We have obtained the intensity of the component with frequency $\omega_o+\Omega$ across the mirror surface. The field ψ from (18) is proportional to:

$$\psi \sim \left(\frac{dP_+}{d\mathcal{S}}\right)^{1/2} e^{i(\omega_o+\Omega)t}\, e^{i\mu} \qquad (26)$$

The phase μ varies across the surface of the mirror due to different path lengths of rays from the source through scattering to the mirror. I see no easy way for determining this phase everywhere on the surface of the mirror. However, one may notice that $\psi.\Psi_o e^{i\Omega t}$ would be the hologram of the wall seen on the mirror surface in the frequency modulated light. Is is reasonable to assume that it would consist of speckles of size λ. I, therefore assume that the coherence length of μ is λ.

The phase of the field Ψ is measured as the averaged phase with respect to Ψ_o over the surface of the mirror; this is:

$$e^{i\Delta\varphi} = \int \Psi_o^* \Psi \, dS \Big/ \left| \int \Psi_o^* \Psi \, dS \right| \qquad (27)$$

In a Fabry-Perot interferometer Ψ_o is a Gaussian beam with the spot size σ (σ is usually chosen to be $\sqrt{\lambda L}$):

$$\Psi_o \Big|_{mirror} = \sqrt{F_o / 2\pi} \; e^{i\omega_o t} e^{-\rho^2/4\sigma^2} \qquad (28)$$

and we get from (27) neglecting terms second order in ψ:

$$\Delta\varphi = \mathrm{Im}\left(\int \Psi_o^* \psi \, ds \right) \Big/ \left(F_o \sigma^2 \right)$$

$$= \frac{1}{\sqrt{2\pi F_o}} \, |\psi| \, \mathrm{Im}\left(e^{i\Omega t} \int e^{-\rho^2/4\sigma^2} e^{i\mu} \frac{dS}{\sigma^2} \right) \qquad (29)$$

I estimate the remaining integral in (29) by noting that there are N speckles on the mentioned hologram where N is of the order $(\sigma/\lambda)^2$. If this number is a Poisson random process then the average excess of positive over negative phase speckles is $\sqrt{N}$, and the integral in (29) becomes $2\pi/\sqrt{N}$ times a random phase factor (ε). Taking into account (29), (24) and (25), we get the following estimate for the phase noise due to the direct contribution:

$$\Delta\varphi \sim 2\pi \sqrt{\beta \mathcal{D}} \, \frac{\mathcal{N}}{R} \sin(\Omega t + \varepsilon) \; . \qquad (30)$$

The corresponding displacement noise follows from (21):

$$\delta L \sim \frac{\lambda^2}{Q\,L} \sqrt{\beta \mathcal{D}} \, \frac{\mathcal{N}}{R} \sin(\Omega t + \varepsilon) \qquad (31)$$

Here Q is the finesse of the cavity usually defined as $Q = c\tau/L$. Note that this contribution is quite small; for $QL = 300\,\mathrm{km}$, $\lambda = 1\,\mu\mathrm{m}$, $\beta = 10^{-5}$ and $\mathcal{N}/R = 10^{-6}$ the displacement noise is 10^{-27}m.

ii) The contribution of phase modulation to the main beam

To get an estimate for the phase modulation on the main beam, let us return to eq. (16) for $\lambda = 0$. Since the right hand side is resonant to this mode, we must add a damping term as in (20), so that we get:

$$\frac{d^2}{dt^2}\left(c_o \, e^{i\omega_o t}\right) + \frac{2}{\tau}\frac{d}{dt}\left(c_o e^{i\omega_o t}\right) + \omega_o^2\left(c_o \, e^{i\omega_o t}\right) = -2c^2 e^{i\omega_o t}(\varphi_o, \partial\Delta(t)\varphi_o) \qquad (32)$$

If the displacement $\mathcal{N}$ is normal to the wall, the source on the right hand side becomes:

$$\mathrm{Rhs} = c^2 e^{i\omega_0 t} \int \mathcal{N}(\mathbf{r},\Omega) \left(\frac{\partial \varphi_0}{\partial n}\right)^2 dS \Big/ \int |\varphi_0|^2 \, dV \tag{33}$$

where $\partial/\partial n$ is the derivative with respect to the normal to the surface dS. This expression for the source was derived with the assumption that the wall is lossless ($\Psi|_{surface} = 0$) which means that no light hitting the wall is lost and, therefore, all photons hitting the wall eventually participate in the interference on the mirrors. In a realistic interferometer the walls are lossy and scatter light, so that only a small part of photons reaching the walls reenters the interfering path. It seems reasonable to assume that the integrand in the souce should be weighted by the probability $p(\mathbf{r})$ that a scattered photon emanating from one mirror reaching the wall is scattered to the second mirror, reflected to the wall and then scattered back to the first mirror. The paths requiring more scatterings to reach the original mirror are much less probable, so that they may be neglected. If the surfaces of the walls are rough with crystallites large compared to the wavelength of light, one may proceed as before to estimate the described probability; thus:

$$p(\mathbf{r}) = a^2 \, w(\vartheta;\vartheta') \, \Delta\Omega_1 \, w(\vartheta';\vartheta) \, \Delta\Omega_2 \tag{34}$$

Here a is the albedo of the walls, ϑ is the angle between the surface normal at $\mathbf{r}$ and the direction of the incoming photon from the mirror M_1, ϑ' a similar angle in the direction of the mirror M_2, $w(\vartheta,\vartheta')\,d\Omega$ is the probability that a photon striking the surface with the incidence angle ϑ is scattered in the solid angle $d\Omega$ in the direction ϑ', $\Delta\Omega_1$ is the solid angle of mirror M_2 as seen from the point $\mathbf{r}$ and $\Delta\Omega_2$ is the solid angle of the mirror M_1 as seen from the point $\mathbf{r}'$ where the photon reflected off the second mirror hits the wall. As before, I assume that scattering is perfectly random, so that $w(\vartheta,\vartheta')=1/(2\pi)$.

It is easy to see that (Eq. 24):

$$\left(\frac{\partial \varphi_0}{\partial n}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 \mathcal{J}_0 \sin^2(\vartheta) \quad \text{and} \quad \int |\varphi_0|^2 \, dV = P_0 L \quad . \tag{35}$$

The modified right hand side (Rhs) can now be computed using (33), (34) and (35) with the result:

$$\mathrm{Rhs} = \beta \, c^2 \frac{\mathcal{N}}{L} \left(\frac{a}{\lambda}\right)^2 e^{i\omega_0 t} \left(\frac{S_{mirror}}{R^2}\right)^2 \mathcal{F}(L/R) \tag{36}$$

and $\mathcal{F}(L/R)$ is the numerical factor:

$$\mathcal{F}(z) = \left(\frac{2\pi}{a}\right)^2 \left(\frac{R^2}{S_{mirror}}\right)^2 \int \sin^2(\vartheta)\, p(r) \frac{dS}{R^2}$$

$$= 2\pi \int_0^L \sin^2(\vartheta') \sin^4(\vartheta) \cos(\vartheta') \cos(\vartheta) \frac{dx}{R} \qquad (37)$$

$$\approx \frac{2\pi}{3} \left(\frac{R}{L}\right)^2 \qquad \left(\frac{L}{R} >> 1\right)$$

Taking into account (32), (36) and (37) we arrive at the expression for c_o. Note that $\Delta\varphi = \mathrm{Im}(c_o)$ and finally compute the displacement noise as in (30-31). The result is:

$$\delta L(\Omega) = \frac{\beta}{12\pi} a^2 \frac{\mathcal{N}(\Omega)\,\lambda}{L} \left(\frac{S_{mirror}}{R^2}\right)^2 \left(\frac{R}{L}\right)^2 \qquad (38)$$

5. Conclusion

I have derived expressions for the sensitivity of a Fabry-Perot interferometer to noise in the enclosing vacuum tank wall. The crucial assumption leading to (31) and (38) is that the wall is ideally rough so that it perfectly randomizes the phase shifts of scattered light. This ideal situation will probably never be reached in real experiments. In longer and longer interferometers it becomes increasingly difficult to avoid specular reflection off the metallic surface of the wall, since the scattering angles get smaller and smaller and the surface as seen by the incident beam gets smoother and smoother. The equivalents of formulas (31) and (36) change drastically if we go from rough to perfectly reflecting walls. The formula (31) describing the direct contribution of the scattered beam changes in two ways; the amount of light reaching the mirror increases by the ratio $2\pi/\Delta\Omega_2$ and we loose the factor $1/\sqrt{N} \sim \lambda/R$, so that the expression (31) is multiplied by the very large factor $\sqrt{2\pi N/\Delta\Omega_2} \sim L/\lambda$; the wonderfully low noise 3.10^{-27} m turns into the not so impressive 3.10^{-18} m. The modulation of the main beam also changes since the probability $p(r)$ increases to close to 1 which is by the factor of the order $(2\pi/\Delta\Omega_2)(2\pi/\Delta\Omega_1) \sim (L/2)^4/S^2_{mirror}$. For perfectly reflecting walls (38) turns approximatelly into :

$$\delta L \sim \frac{\beta}{192\pi} a^2 \frac{L\lambda}{R^2} \mathcal{N}$$

This makes the relative sensitivity independent of the length of the interferometer and with the data from (31) one obtains:

$$\delta L/L \sim \left[1.7 \cdot 10^{-14}\, m^{-1}\right] \mathcal{N} \qquad (38a)$$

In conclusion we note that the most optimistic estimates give very small values 10^{-27} m/$\sqrt{Hz}$ for the displacement noise due to the direct contribution of scattered light if the vibration amplitude is 10^{-6} m/$\sqrt{Hz}$. The most pesimistic estimate is obtained if the walls are reflecting, when the main contribution comes from the phase modulation of the main beam and the relative sensitivity with the same amount of input noise is of the order $2 \cdot 10^{-20}$ /$\sqrt{Hz}$. One should expect that a simple straight vacuum pipe would be a good reflector at angles of scattering in a 4km pipe, but a number of baffles with semi-random edges along the pipe should be able to change the parameters of the system toward the favourable random scattering regime, where this noise would be unobservable.

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References

R.W.P. Drever et all (1983): Developments in Laser Interferometer Gravitational Wave detectors, Proc. of the 3rd Marcel Grossmann Meeting on General Relativiy, Hu Ning (editor), Science Press and North Holland Publ. Co.

J.M. Elson, J.M. Bennett (1979): Opt. Eng. **18**, 116-124

J.M. Elson (1975): Phys. Rev. **B12**,2541

E. Kröger, E. Kretschmann (1970): Z. Physik **237**, 1-15

K.S. Thorne (1987), private notes

Fig. 1

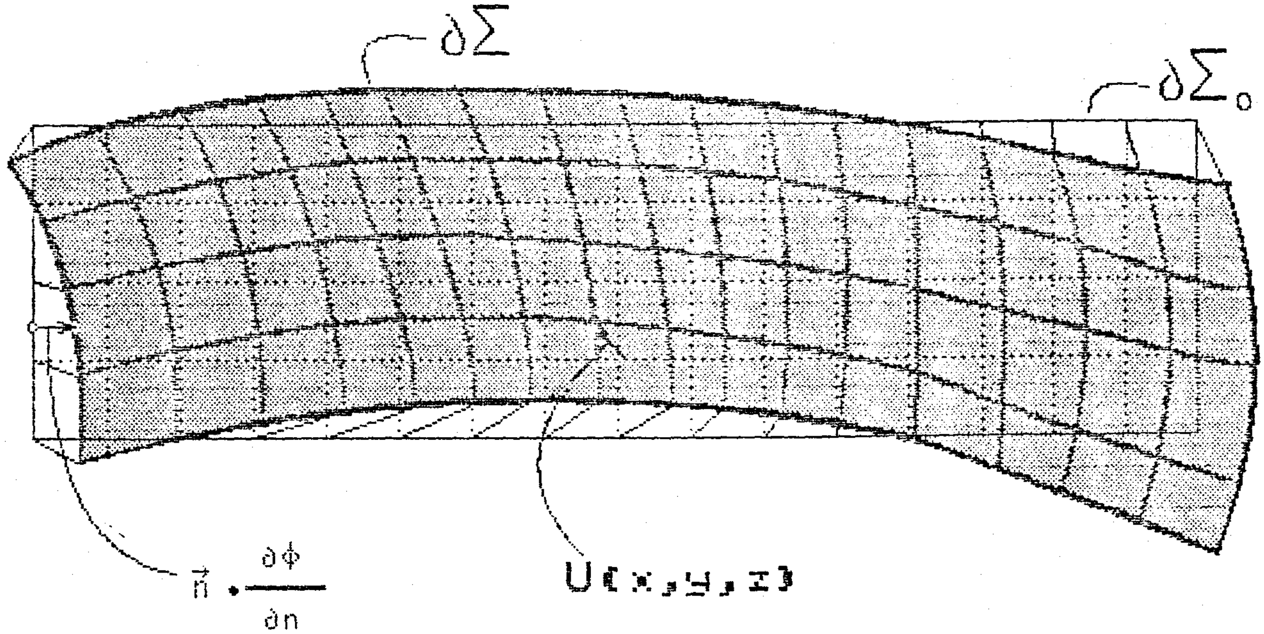

Fig. 2