Some Notes on the Effect of Slope
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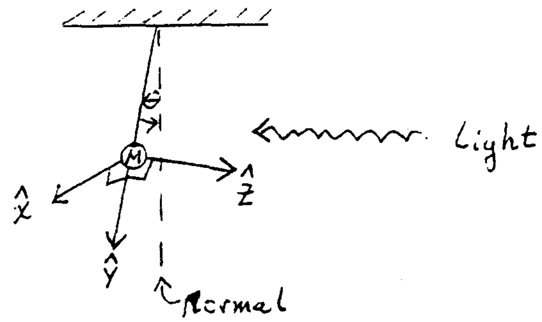
The principal effect of a slope in the LIGO arms is to allow both vibrational and thermal noise motion of the masses along the local vertical to feed through into motion along the beam axis in the interferometers. For this discussion refer to the diagram below:



Let the spectral density of motion be given by $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ which have dimensions of $m/\sqrt{Hz}$. $\hat{y}$ is measured along the suspension wire, which is assumed to hang along the local vertical direction. θ is the dangle angle between the suspension wire and the normal to the light beam axis. Assume the light is incident from the right, nearly along the $\hat{z}$ axis. The interferometers seek to measure $\Delta\tilde{L}$, the motion of the mass along the light beam. Now $\tilde{x}$ will only affect the interferometer in higher order and the lowest order contribution to $\Delta\tilde{L}$ is:

$$\Delta\tilde{L} = \left[\tilde{z}^2 + (\theta\tilde{y})^2\right]^{1/2}$$

Thus the slope is irrelevant as long as $\theta\tilde{y} < \tilde{z}$.

Seismic Noise:

Assume a four layer vibration isolation stack with vertical resonance, f_V, and horizontal resonance, f_H, followed by a wire suspension with pendulum resonance, f_P, and spring resonance, f_S. Since the amplitude of ground motion is nearly isotropic, we have $\tilde{x}_G \approx \tilde{y}_G \approx \tilde{z}_G$.

Let f be the signal frequency. Typically we have anisotropic stacks and suspensions, that is, $f_H < f_V$ and $f_P < f_S$. Also we expect typically to be limited by seismic noise in LIGO only at very low frequencies. As long as $f < Qf_H$, we can analyze the stacks in the high Q limit where the transfer function falls off as $1/f^2$. For $f_H \approx 5$Hz, this applies to f between 50 and 100 Hz, below which we expect seismic effects to dominate over thermal noise. The wire suspensions are clearly operating in the $1/f^2$ regime. We therefore have

$$\tilde{y}(f) \approx \tilde{z}_G(f)\left(\frac{f_V}{f}\right)^8\left(\frac{f_S}{f}\right)^2$$

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It is useful to parametrize the effect of slope in terms of a correction to the numbers one would calculate if the slope were ignored. In the case of seismic noise or vibration, one would normally calculate the effect on $\Delta\tilde{L}$ due to $\tilde{z}$ alone:

$$\Delta\tilde{L}(\theta = 0) \approx \tilde{z}_G(f) \left(\frac{f_H^8 f_P^2}{f^{10}} \right)$$

Thus we obtain

$$\Delta\tilde{L}_{seismic}(\theta) \approx \Delta\tilde{L}_{seismic}(\theta = 0) \left[1 + \theta^2 \left(\frac{f_V}{f_H} \right)^{16} \left(\frac{f_S}{f_P} \right)^4 \right]^{1/2}$$

Since we expect that the second term in the bracket is generally much larger than unity, we can approximate as follows:

$$\Delta\tilde{L}_{seismic}(\theta) \approx \Delta\tilde{L}_{seismic}(\theta = 0)\ \theta\ \left(\frac{f_V}{f_H} \right)^8 \left(\frac{f_S}{f_P} \right)^2$$

Example:

Let $f_V = 3 \times f_H$, and $f_S = 5 \times f_P$, which is typical of current performance in stacks and suspensions. For $\theta = .01$ (1% slope) we have:

$$\Delta\tilde{L}_{seismic} \approx 1600\ \times\ \Delta\tilde{L}_{seismic}(\theta = 0)$$

However even in a level layout the test masses dangle at a small angle relative to the normal to the beam axis, due to the curvature of the earth. This angle is $\theta = 0.3$mrad. Thus in a level layout, the vertical motion would still dominate motion along the light beam and we have:

$$\Delta\tilde{L}_{seismic} \approx 50\ \times\ \Delta\tilde{L}_{seismic}(\theta = 0)$$

We could of course alleviate this effect by making more isotropic stacks and suspensions. The vertical stack resonance is the most leveraged quantity. For example making $f_V = 2 \times f_H$ would reduce the feedthrough of vertical motion by a factor of 25. While one could also reduce the vertical spring resonance, it probably will not come down by much and it is less leveraged. Another solution is to simply add more vertical isolation. One can basically think of the seismic implications of a larger slope as somewhat equivalent to the effect of having a seismically noisier site.

Broadband Thermal Noise:

The thermally induced motion of the test masses can be estimated as:

$$\tilde{z}(f) = \left[\frac{4kT\gamma_P}{M}\right]^{1/2} \frac{1}{4\pi^2 f^2}$$

$$\tilde{y}(f) = \left[\frac{4kT\gamma_S}{M}\right]^{1/2} \frac{1}{4\pi^2 f^2}$$

where γ_P is the damping rate for the pendulum mode and γ_S is the damping rate for the spring mode of the suspension. In general $\gamma = 2/\tau$ where τ is the damping time of a oscillator as measured at the signal frequency. If we again define the motion along the beam in terms of the result we would get by ignoring the slope, we obtain:

$$\Delta\tilde{L}_{thermal} = \Delta\tilde{L}_{thermal}(\theta = 0) \left[1 + \theta^2 \frac{\gamma_S}{\gamma_P}\right]^{1/2}$$

This implies that the slope compromises thermal noise performance unless we have:

$$\gamma_S \leq \frac{\gamma_P}{\theta^2}$$

Example:

A nominally level layout requires $\gamma_S \leq 10^7 \gamma_P$, which is easily met. On the other hand, a 1% slope requires $\gamma_S \leq 10^4 \gamma_P$. The latter specification may or may not be achievable with our current W suspension wires.

We expect thermal noise in W suspension wires to start limiting LIGO performance in level layouts below a few hundred Hz. The layout slope significantly complicates the problem of improving the thermal noise limited performance of LIGO. The reason for this is that a clever idea which could lower the damping of the pendulum mode will not significantly improve the noise unless a comparable decrease of losses in the spring mode is achieved at the same time.

Notes of Explanation for Thermal Noise:

1. The relation between $\tilde{z}(f)$ and $\gamma_z(f)$ results from the Fluctuation Dissipation Theorem. The underlying principle is that energy flow (heat flow) between an oscillator and a thermal reservoir at different temperatures is symmetric under reversal of the temperature differential for every frequency interval (detailed balance).
2. $\tilde{z}(f)$ and $\tilde{y}(f)$ are independent of the resonance frequency of the oscillator in the simple model because the rms energy of the oscillator in equilibrium with its reservoir is fixed by the Equipartition Theorem.
3. As a result of (2) there remains a $M^{1/2}$ dependence in $\tilde{z}$ and $\tilde{y}$, but this is the same for both modes.

Notes on the Possibility of Removing Slope Problems:

A possible solution to slope induced compromises to LIGO performance is to apply a constant force to the test masses so that the suspensions again hang normal to the beam axis. However this approach while possible in principle looks practically impossible. The most stringent specifications are on the constancy of this force, and the requirement that this force be applied through an effective spring whose losses must be smaller by $1/\theta^2$ than the losses in the vertical spring. We have so far considered mechanical, electrostatic and magnetostatic springs, all of which have practical shortcomings. While we cannot prove that such a solution cannot be found, it is likely that coming up with a solution will at least require a major research and development effort and may well not pay off.