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# 49a

List of 5/01/89

From: R. Weiss

To: Alex, Boude, Bill

Concerning: Vacuum in fore optics

Reference: R.W. Scattering by Residual Gas LIGO file #49

Phase noise due to forward scattering

$$\psi(f) = \frac{(2\pi)^2 \alpha (\rho_{\#} l)^{1/2}}{\lambda (\omega_0 v_0)^{1/2}} \; e^{-\sqrt{2\pi}\frac{f\omega_0}{v_0}} \qquad (1) RW page 8$$

[Note: Expression for a single pass through gas column]

$\alpha$ = molecular polarizability Example: $\alpha(N_2) = 1.6 \times 10^{-24} cm^3$

$\rho_{\#}$ = particle density Example: at 1 atm $\rho_{\#} = 3 \times 10^{19}/cm^3$

$\lambda$ = light wave length

$v_0$ = thermal velocity of gas with polarizability $\alpha$
Example: $v_0(N_2, 300K) = 4.2 \times 10^4 cm/sec$

$\omega_0$ = Gaussian radius at focus of beam Example: if close to confocal $\omega_0 = \left(\frac{\lambda l}{\pi}\right)^{1/2}$

$l$ = length of gas column

Neglect expontential factor in all ensuing estimates

Assume $\frac{\sqrt{2\pi} f \omega_0}{v_0} < 1$

Statistical fluctuation limit due to forward scattering expressed as a limit on $h(f)$, the gravitational wave sensitivity

$$\underset{\text{forward scat}}{h(f)} = \frac{\sqrt{2}(2\pi)\alpha(\rho_{\#})^{1/2}}{(\omega_0 v_0 l)^{1/2}} = \left(\frac{dh}{d\psi}\right)\psi(f)$$

$l$ = length of antenna = 4km

$\rho_{\#}$ = number density averaged over antenna arm

Expression includes both arms

Condition for other parts of vacuum system is that phase noise in 4km arm should dominate.

A) In those paths with phase sensitivity / after split

   a) main beam tubes

   b) test mass→ splitter

   Scaling goes as (EQ 1)

$$\psi^2(f) \quad \propto \quad \frac{\rho_{\#} l}{\omega_0}$$

So phase noise from different parts of path $\psi(f) \quad \propto \quad \left( \sum_{i=1}^{n} \frac{<\rho_{\# i}> l_i}{\omega_{0i}} \right)^{1/2}$

sum over different segments

After beam splitter $\omega_{0i} \sim \omega_0 \cong \left( \frac{\lambda l_{\text{main tube}}}{\pi} \right)^{1/2}$

So scaling goes as $\quad < \rho_{\#} > l_{\text{section}}$

Or to make contributions for phase noise comparable in short path vs main tube

$$\frac{< \rho_{\#} >_{\text{short}}}{< \rho_{\#} >_{\text{main tube}}} = \frac{< P(torr) >_{\text{short}}}{< P(torr) >_{\text{long}}} = \frac{l_{\text{long}}}{l_{\text{short}}}$$

Conditions in beam path before beam split

1) Phase noise is common mode to both interferometer arms so scaling is not the same as after split.

2) Must worry about mode mixing (angular jitter noise) imparted by residual gas

A) Common mode→differential mode conversion in interferometer

Primary source is through unbalance in arm storage time and frequency fluctuations of input light

Assume that frequency noise is determined by input mode filter cavity

Frequency noise sensitivity

$$h(f) = \frac{\nu(f)}{\nu_0}\frac{\Delta\tau_{\mathrm{st}}}{\tau_{\mathrm{st}}}$$

$\nu_0$ = base frequency $\sim 6\times10^{14} Hz$

$\nu(f)$ = frequency noise $Hz/Hz^{1/2}$

$h(f)$ = gravitational wave strain noise strain$/Hz^{1/2}$

$\Delta\tau_{\mathrm{st}}/\tau_{\mathrm{st}} = \left(\frac{\Delta F}{F}+\frac{\Delta l}{l}\right)$ fractional storage time unbalance in the two 4km arms

Frequency noise due to forward scattering in input frequency reference cavity

$$\underset{\text{forwardscat}}{\nu(f)} = \left(\frac{\Delta f}{\Delta\psi}\right)_{\text{cavity}}\psi(f) = \left(\frac{c}{16\pi l_{\mathrm{cav}}}\right)\left[\frac{(2\pi)^2\alpha}{\lambda v_0^{1/2}\omega_{0_{\mathrm{cav}}}^{1/2}}\left(\rho_{\#}l_{\mathrm{cav}}\right)^{1/2}\right]$$

(Note: Finesse of cavity is absorbed in phase noise for multiple path through the gas column)

Convert to limit on $h(f)$ in full interferometer

$$\underset{\substack{\text{freq noise}\\\text{from forward scat}\\\text{in input cavity}}}{h(f)} = \frac{\Delta\tau_{\mathrm{st}}}{\tau_{\mathrm{st}}}\left[\frac{\pi}{4}\frac{\alpha}{(v_0\omega_0)^{1/2}}\left(\frac{\rho_{\#_{\mathrm{cav}}}}{l_{\mathrm{cav}}}\right)^{1/2}\right]$$

For estimation

$$\underset{\substack{\text{forward}\\\text{scat}\\\text{beam tubes}}}{h(f)} \quad = \quad \underset{\substack{\text{freq noise}\\\text{from forward scat}\\\text{in input cavity}}}{h(f)}$$

$$\left[\frac{<\rho_{\#}>}{\omega_0 l}\right]_{\text{main}} \sim \left[\frac{\Delta\tau_{st}}{\tau_{st}}\right]^2 \left[\frac{<\rho_{\#}>}{\omega_0 l}\right]_{\text{cav}}$$

If minimum beam size in either system

$$\omega_0 = \left(\frac{\lambda l}{\pi}\right)^{1/2}$$

$$\left[\frac{<\rho_{\#}>}{l^{3/2}}\right]_{\text{main}} \quad \sim \quad \left[\frac{\Delta\tau_{st}}{\tau_{st}}\right]^2 \left[\frac{<\rho_{\#}>}{l^{3/2}}\right]_{\text{cav}}$$

Scaling

$$\frac{\underset{\text{cavity}}{<P(torr)>}}{\underset{\text{main}}{<P(torr)>}} \quad = \quad \left(\frac{l_{\text{cavity}}}{l_{\text{main}}}\right)^{3/2} \left[\frac{\tau_{st}}{\Delta\tau_{st}}\right]^2$$

Example:

$$<P>_{\text{main}} \quad = \quad 10^{-9} torr \qquad N_2$$

$$\frac{l_{\text{cavity}}}{l_{\text{main}}} = \frac{1.2\times10^3}{4\times10^5} = 3\times10^{-3}$$

$$\frac{\Delta\tau_{st}}{\tau} = 1\times10^{-3}$$

$$<P(torr)>_{\text{cavity}} \quad \leq \quad 1.6\times10^2 \quad <P(torr)>_{\text{main}}$$

$$\leq 1.6\times10^{-7}\begin{pmatrix}N_2\\H_2O\end{pmatrix} \qquad \text{for ultimate sensitivity}$$

$$h(f) = 3\times10^{-25} \quad \text{strain}/Hz^{1/2}$$

This assumes entire frequency reference derived from input cavity, no trim from main cavity.

Mode mixing (beam angular jiggle) due to statistical fluctuations in gas column

mixing of $TE_{11}$ mode from $TE_{00}$ mode by residual gas forward scattering fluctuations

RW p16, 17
Note: mistake
p17
$\alpha = 1.6 \times 10^{-24} cm^3$

$$\frac{E_{11}}{E_{00}} \sim \alpha \frac{32}{\lambda} \frac{(l\rho_{\#})^{1/2}}{\omega_0} \left[ \int \text{overlap} \right]$$

$\Uparrow$
0.1

Does not set a serious limit

$l$ = length of region with average number density $\rho_{\#}$

$l\rho_{\#}$ = column density

Overlap integral $\leq 1$

Other condition that sets pressure is damping of pendulum in TM chambers, this is satisfied for $P(torr) < 10^{-6}$

$N_2, H_2O$

# BATCH START

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STAPLE
OR
DIVIDER

DRAFT: SCATTERING BY RESIDUAL GAS

Some of the calculations in this document had previously been made by Stan Whitcomb in an unpublished document.

OUTLINE OF CALCULATION

1) Definition of mode amplitudes – complex amplitude
2) Mode amplitude of point scatterer
3) Model of point scatterer polarizability, electric dipole approximation
4) Time dependence of mode amplitude
5) Fourier transform of mode amplitude
6) Integration over Maxwell velocity distribution
7) Power spectrum of noise in mode amplitude
8) Phase and amplitude fluctuations in the mode amplitude
9) Mode mixing by residual gas

THE MODE FIELDS

Propagation along z

$$E_{m,n}(x,y,z,t) = E_o \frac{\omega_o}{\omega(z)} \cdot H_m\left(\frac{\sqrt{2}x}{\omega(z)}\right) \cdot H_n\left(\frac{\sqrt{2}y}{\omega(z)}\right) e^{-\frac{x^2+y^2}{\omega^2(z)}}$$

$$e^{-ikz - ik\left(\frac{x^2+y^2}{2R(z)}\right) + i(m+n+1)\tan^{-1}\left(\frac{\lambda z}{\pi\omega_o^2}\right) + i\omega t}$$

$H_j\left(\frac{\sqrt{2}x}{\omega(z)}\right)$ are the hermite polynomials

$$H_o(\psi) = 1, \quad H_1(\psi) = 2\psi, \quad H_2(\psi) = 4\psi^2 - 2 \ \ldots$$

$$\omega^2(z) = \omega_o^2\left[1 + \left(\frac{\lambda z}{\pi\omega_o^2}\right)^2\right] \qquad \text{beam radius } 1/e$$

$$R(z) = z\left[1 + \left(\frac{\pi\omega_o^2}{\lambda z}\right)^2\right] \qquad \text{wave front radius}$$

For the confocal cavity the beam radius at the waist is

$$\omega_o\,(R \to \infty) = \sqrt{\frac{\lambda \ell}{2\pi}}$$

## THE MODE AMPLITUDES

Let the exit pupil of the optical system be defined at a point where the modes have wavefront radius $R(o) \to \infty$. This is a convenient place to define the mode amplitude associated with an arbitrary field.

The mode amplitude is complex and given by

$$A_{m,n} = \underset{z=0}{\int_x \int_y} u_{m,n}(x,y) \cdot E(x,y) \cdot dxdy$$

$u_{m,n}(x,y)$ are normalized so that

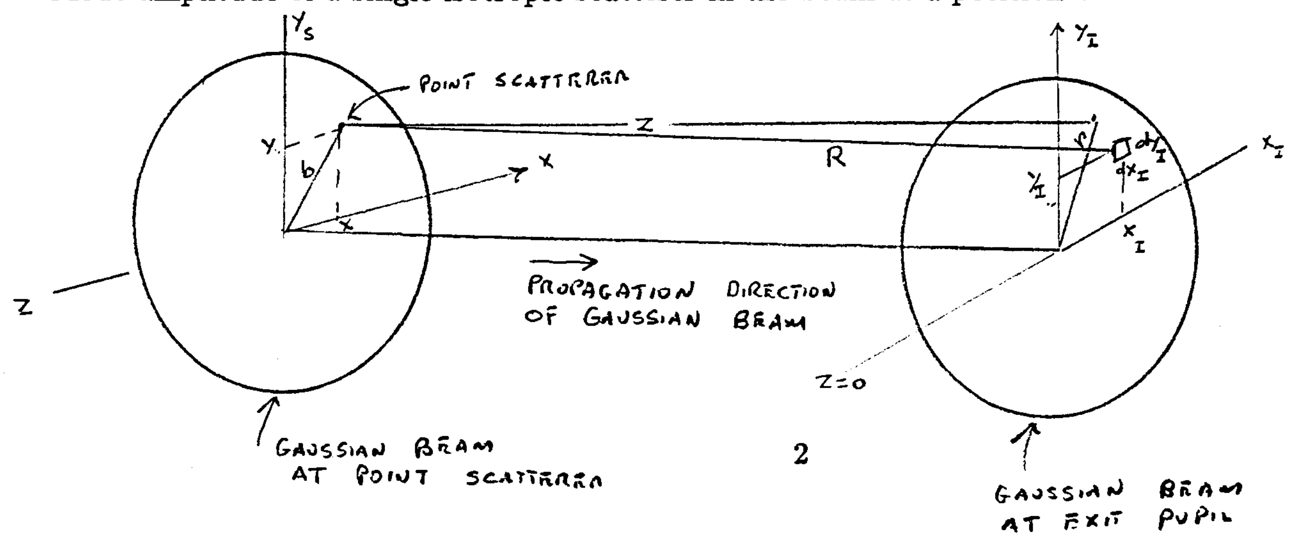
$$\int_x \int_y u_{m,n}(x,y) u^*_{j,k}(x,y) \cdot dxdy = \delta_{m,j}\delta_{n,k}$$

For Example

$$u_{o,o}(x,y) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega_o} \quad e^{-\frac{x^2+y^2}{\omega_o^2}} \qquad u_{1,o}(x,y) = 2\sqrt{\frac{2}{\pi}} \frac{x}{\omega_o^2} \quad e^{-\frac{x^2+y^2}{\omega_o^2}}$$

$$u_{11}(x,y) = 4\sqrt{\frac{2}{\pi}} \frac{xy}{\omega_o^3} \quad e^{-\frac{x^2+y^2}{\omega_o^2}} \qquad u_{o,1}(x,y) = 2\sqrt{\frac{2}{\pi}} \frac{y}{\omega_o^2} e^{-\frac{x^2+y^2}{\omega_o^2}}$$

Mode amplitude of a single isotropic scatterer in the beam at a position z



Assume a point scatterer (molecule) at field position x,y,z in a Gaussian beam traveling along -z direction and coming to a focus at the exit pupil. We want to determine the mode amplitude of the scattered field.

ssumption

Assuming weak scattering – single scattering – so that the wave incident on the scatterer is unperturbed by previous scatterings, the scattered field is isotropic and given by

$$\vec{E}_{sc}\left(\overset{\text{exit pupil}}{x_1, y_1, 0} \mid \overset{\text{source}}{x, y, z}\right) = \frac{k^2 e^{ikR}}{R}\left[\left(\hat{k}\times \vec{P}\right)\times \hat{k}\right]$$

where $\hat{k}$ is a unit vector along the scattered wave propagation direction and $\vec{P}$ is the induced electric dipole in the scatterer due to the incident wave

$$\vec{P} = \alpha(\omega)\vec{E}_{inc}$$

$\alpha(\omega)$ is the molecular polarizability at frequency $\omega$ in units of $\mathrm{cm}^3$/molecule.

The molecular polarizability is best determined from bulk measurements of the refractive index at $\omega$.

$$\alpha(\omega) = \frac{n(\omega) - 1}{2\pi\rho_{\#}}$$

$n(\omega)$ index of refraction of a gas composed of number density $\rho_{\#}$ molecules/$\mathrm{cm}^3$.

We will be concerned initially with scattering into the exit pupil so that the polarization of the scattered field is the same as the incident field.

$$\vec{E}_{sc}(x_I, y_I, 0 \mid x, y, z) = k^2\alpha(\omega)\cdot\frac{e^{ikR}}{R}\cdot\vec{E}_{inc}(x, y, z)$$

$$R = \left[(x_I - x)^2 + (y_I - y)^2 + z^2\right]$$

*note*: $R$ is directed from scatterer to the exit pupil

The mode amplitudes of the scattered field are given by the integrals

$$A_{m,n}\,(\text{scat}) = \int\limits_{x_I}\int\limits_{y_I}\limits_{z=0} u_{m,n}\,(x_I, y_I)\, E_{sc}\,(x_I, y_I, 0 \mid x, y, z)\, dx_I dy_I$$

$$A_{m,n}\,(\text{scat}) = \int\limits_{x_I}\int\limits_{y_I}\limits_{z=0} u_{m,n}\,(x_I, y_I)\, k^2 \alpha(\omega) \frac{e^{ikR}}{R} \cdot E_{inc}\,(x, y, z)\, dx_I dy_I$$

The integral is evaluated by the method of stationary phase

see: Jackson, J. D. "Classical Electrodynamics" 2nd edition, 1975, p 454-459.

or Ishimaru, A. "Wave Propagation and Scattering in Random Media" Academic Press, 1978, p 287-292.

Method of Stationary Phase

$$I = \int_{-\infty}^{+\infty} dx_I \int_{-\infty}^{+\infty} dy_I A\,(x_I, y_I) \cdot \frac{e^{ik|r_I - r_s|}}{|\, r_I - r_s \,|}$$

If $A(x_I, y_I)$ is not changing rapidly over integration and the point in the exit pupil which is stationary in the phase is $x_{IO}, y_{IO}$, the integral becomes

$$I = A\,(x_{IO}, y_{IO}) \cdot \frac{2\pi i}{k} \quad e^{ik(z_I - z_s)} \qquad \text{( spatial delta function)}$$

$$x_{IO} = x, \quad y_{IO} = y \qquad \text{the coordinates of the particle}$$

$$A_{m,n}(\text{scat}) = i \cdot 2\pi k \alpha(\omega) u_{m,n}(x, y) E_{inc}(x, y, z) \quad e^{-ikz}$$

The mode amplitude of the incident field in first order (neglecting the change in the incident field due to the scattering) is

$$A_{m,n}(\text{incid}) = \int_{x_I}\int_{y_2} u_{m,n} \cdot E_{inc}(x_I, y_I, 0)\, dx_I dy_I \qquad z=0$$

The total mode amplitude

$$A_{m,n}(\text{total}) = A_{m,n}(\text{incid}) + A_{m,n}(\text{scat})$$

SPECIAL CASES

Determine mode amplitude of $TE_{00}$ at exit pupil and furthermore let the incident beam be a $TE_{00}$ with the same mode dimensions at the exit pupil

$$E_{inc}(x,y,x,t) = E_o \frac{\omega_o}{\omega(z)} \quad e^{-\frac{x^2+y^2}{\omega^2(z)}} \quad e^{\left[ikz + ik\frac{(x^2+y^2)}{2R(z)} - i \tan^{-1}\left(\frac{\lambda z}{\pi\omega_o^2}\right) + i\omega t\right]}$$

$$A_{oo}(\text{scat}) = i \cdot 2\pi k\alpha(\omega)\sqrt{\frac{2}{\pi}\frac{1}{\omega_o}} \quad e^{-\frac{x^2+y^2}{\omega_o^2}} \quad E_o\frac{\omega_o}{\omega(z)} \quad e^{-\frac{x^2+y^2}{\omega^2(z)}}$$
$$e^{i\left[k\frac{x^2+y^2}{2R(z)} - \tan^{-1}\left(\frac{\lambda z}{\pi\omega_o^2}\right)\right]}$$

$$A_{oo}(\text{incid}) = \int_{x_I}\int_{y_I} E_o\sqrt{\frac{2}{\pi}\frac{1}{\omega_o}} \quad e^{-\frac{2\left(x_I^2+y_I^2\right)}{\omega_o^2}} \quad dx_I dy_I$$

$$= E_o\omega_o\sqrt{\frac{\pi}{2}}$$

$\overset{\text{approx}}{\Longrightarrow}$ The dominant effect of the scattered beam is to cause a phase shift in the mode amplitude

For first estimate let $\omega^2(z) \sim \omega_o^2$ — large diameter beam

$$e^{i\left(k\frac{\left(x^2+y^2\right)}{2R(z)} - \tan^{-1}\left(\frac{\lambda z}{\pi\omega_0^2}\right)\right)} \longrightarrow 1$$

neglect curvature of phase fronts and phase shift along beam due to caustic at $z = 0$

$$A_{oo}\,(\text{scat}) \sim i2\pi\sqrt{\frac{2}{\pi}} \cdot \frac{k\alpha(\omega)}{\omega_o} \cdot E_o \quad e^{-\frac{2\left(x^2+y^2\right)}{\omega_o^2}}$$

Total mode amplitude becomes

$$A_{oo}\,(\text{total}) = E_o \cdot \omega_o \cdot \sqrt{\frac{\pi}{2}} \left[ 1 + i\, \overbrace{\frac{4k\alpha(\omega)}{\omega_o^2} \quad e^{-2\frac{\left(x^2+y^2\right)}{\omega_o^2}}}^{\psi} \right]$$

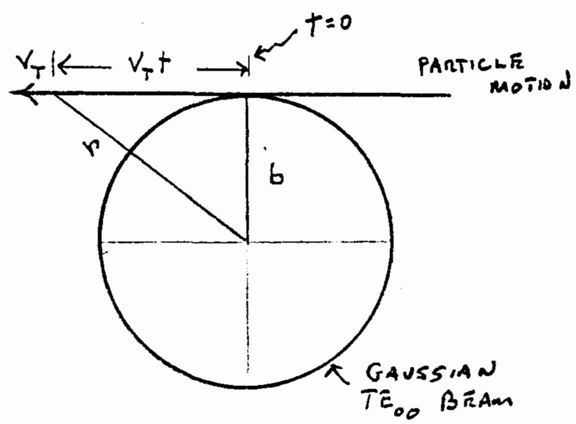
The scattering causes a phase shift but not a first order amplitude change in this approximation – an index of refraction change

The first order phase shift

$$1 + i\psi \cong e^{i\psi} \qquad \psi \ll 1$$

TIME DEPENDENCE OF THE PHASE SHIFT

Since the scattered field at the exit pupil depends on the position of the particle in the incident field the time dependence of the mode amplitude phase shift, $\psi$, is determined by the particle's velocity in the direction transverse to the incident beam.




The particle encounters the beam with impact parameter b

$$r^2 = b^2 + \mathrm{v}_\mathrm{T}^2 \mathrm{t}^2$$

$$\psi(\mathrm{t}) = \overbrace{\frac{4k\alpha(\omega)}{\omega_o^2} \quad e^{-2b^2/\omega_o^2}}^{A(b)} \cdot \overbrace{e^{-2v_T^2 t^2/\omega_o^2}}^{u(\mathrm{t},\mathrm{v_T})}$$

$\psi(t)$ has an amplitude $A(b)$ which only depends on the impact parameter

and a pulse shape $u\,(\mathrm{t},\mathrm{v_T})$ which depends on the transverse velocity but not on the position of the particle

$$\psi(t) = A(b)u\,(t, v_T)$$

The Fourier transform of the pulse shape

$$\mathrm{S_u}\,(\mathrm{f},\mathrm{v_T}) = \int_{-\infty}^{+\infty} u\,(t, v_T)\, e^{-i\omega t} \delta t = 2 \int_0^{\infty} u\,(t, v_T) \cos \omega t dt$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{\omega_0}{v_T} \quad e^{-\frac{\omega^2 \omega_0^2}{8 v_T^2}}$$

To determine the power spectrum of the phase fluctuations due to the pulses we use the result given by

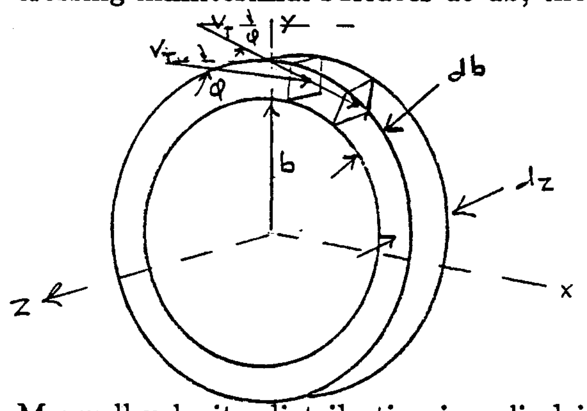
Middleton, D. *Introduction to Statistical Communication Theory* McGraw Hill 1960, p 235-236.

For pulses that overlap and have an average rate of occurance $\langle \dot{n}_0 \rangle$

$$\psi^2\,(\mathrm{f}) = 2\langle \dot{\mathrm{n}}_0 \rangle \langle |A(b) S_u\,(\mathrm{f}, \mathrm{v_T})\,| \rangle^2_{v_T b}$$

The averages are taken over the Maxwellian velocity distribution and the impact parameter

A way to do the counting is to use the Maxwell distribution in cylindrical coordinates and count the particles crossing infinitesimal surfaces *db dz*, the surfaces have normals along $\varphi$, angle between the x axis and the normal. All surfaces have the same impact parameter.




Maxwell velocity distribution in cylindrical coordinates

$$P\,(v_z, v_T)\, v_T dv_T dv_z d\varphi = \left(\frac{m}{2\pi kT}\right)^{3/2} \quad e^{-m\left(v_z^2+v_T^2\right)/2kT} v_T dv_T dv_z d\varphi$$

The pulse rate generated in a small section of d$\varphi$ of the ring at the impact parameter b is

$$d\dot{n} = \rho_{\#} \left(\frac{m}{2\pi kT}\right)^{3/2} \quad e^{-mv_z^2/2kT} e^{-mv_T^2/kT} v_T^2 dv_T d_z db d\varphi$$

m=particle mass
$\rho_{\#}$=average particle density particles/cm$^3$

The averages of the amplitude and spectra of the pulses are taken along with the pulse counting. The integral for the phase power spectrum

$$\psi^2(\mathrm{f}) = 2 \int_{\text{all pulses}} A^2(b) |\mathrm{S_u}\,(\mathrm{f}, \mathrm{v_T})\,|^2 \cdot \mathrm{d}\dot{\mathrm{n}}$$

$$= \frac{2 \times 16\mathrm{k}^2 \alpha^2(\omega)}{\omega_o^4} \cdot \frac{\pi}{2} \omega_o^2 \cdot \rho_{\#} \left(\frac{m}{2\pi kT}\right)^{3/2} \quad *$$

$$\int_{z=0}^{\ell} dz \int_{\varphi=0}^{2\pi} d\varphi \int_{b=0}^{\infty} e^{-4b^2/\omega_o^2}\, db \int_{v_2=-\infty}^{+\infty} e^{-mv_z^2/2kT}\, dv_z \int_{v_T=o}^{\infty} e^{-\frac{\omega^2\omega_o^2}{8v_T^2}-\frac{mv_T^2}{2kT}}\, dv_T$$

Integration is taken over a gas column of length $\ell$

$$\psi^2(f) = \frac{(2\pi)^2\, \mathrm{k}^2 \alpha^2 (\omega_{\mathrm{opt}}) \langle \rho_{\#} \rangle \ell}{\omega_0 \left(2\mathrm{kT}/\mathrm{m}\right)^{1/2}} \quad e^{-\pi f \omega_o \Big/ \left(\frac{kT}{m}\right)^{1/2}}$$

This is the first order result of the calculation

The average velocity of a molecule

$$\langle \mathrm{v} \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{2\mathrm{kT}}{\mathrm{m}} \right)^{1/2}$$

The most probable velocity

$$v_0 = \left( \frac{2kT}{m} \right)^{1/2}$$

$$\psi^2(f) = \frac{(2\pi)^4\, \alpha^2 (\omega_{\mathrm{opt}}) \langle \rho_{\#} \rangle \ell}{\lambda^2 \omega_o v_0} \quad e^{-\frac{\sqrt{2}\pi f \omega_o}{v_o}}$$

Evaluation of strain noise due to phase fluctuations in a simple multi pass interferometer with independent paths

The total column with b passes $\qquad L_{\mathrm{total}} = b\ell$

The position sensitivity

$$x^2(f) = \left( \frac{\lambda}{2\pi} \right)^2 \left( \frac{1}{b^2} \right) \psi^2(f)$$

The strain sensitivity becomes

$$\underset{\text{arm}}{h^2(f)} = \frac{x^2(f)}{\ell^2} = \frac{(2\pi)^2 \alpha^2 (\omega_{opt}) \langle \rho_\# \rangle}{b\omega_o v_o \ell} \quad e^{-\sqrt{2}\pi f \omega_o / v_0}$$

But there are two independent paths, one in each arm, with independent phase fluctuations

$$h^2_{\text{total}}(f) = \frac{2(2\pi)^2 \alpha^2 (\omega_{opt}) \langle \rho_\# \rangle}{b\omega_o v_o \ell} \quad e^{-\sqrt{2}\pi f \omega_o / v_0}$$

Using the confocal beam size at the mirrors $\qquad \omega_o = \left(\frac{\lambda \ell}{\pi}\right)^{1/2}$

$$h_{\text{total}}(f) = \frac{2^{3/2} \pi^{5/4} \alpha (\omega_{opt}) \rho_\#^{1/2}}{b^{1/2} v_o^{1/2} \lambda^{1/4} \ell^{3/4}} \quad e^{-\sqrt{2\pi} \frac{f (\lambda \ell)^{1/2}}{v_o}}$$

To compare with old result (bluebook and graphs in proposals)

$$\frac{\alpha_{\text{old}}(\omega)}{2\pi} = \alpha_{\text{new}}(\omega)$$

Comparing with old result

$$\frac{h_{\text{new}}(f)}{h_{\text{old}}(f)} = 2^{1/4} \qquad \text{larger in amplitude}$$

spectral slightly $\qquad f(1/\ell) = \dfrac{v_o}{\omega_o \sqrt{2}\pi} \qquad$ new

different $\qquad f(1/\ell) = \dfrac{v_o}{\omega_o 2\pi} \qquad$ old $\qquad \sqrt{2}$ different

<u>INCLUSION OF GAUSSIAN WAVEFRONT CURVATURE</u>

The new effect is amplitude fluctuation of the mode amplitude due to the time varying diffraction pattern of the moving scatters.

Again use the $TE_{00}$ mode only

The scattered mode amplitude is no longer purely in quadrature to the mode amplitude of the incident beam

$$\mathrm{A}_{00}\,(\mathrm{scat}) = \mathrm{i}2\pi\mathrm{k}\frac{\alpha(\omega)}{\omega_o}\sqrt{\frac{2}{\pi}} \quad e^{-\frac{x^2+y^2}{\omega_0^2}} \cdot \mathrm{E}_o \frac{\omega_0}{\omega(z)} \quad e^{-\frac{x^2+y^2}{\omega^2(z)}} \ \mathrm{x}$$

$$e^{i\frac{k\left(x^2+y^2\right)}{2R(z)} - i\tan^{-1}\frac{\lambda z}{\pi\omega_o^2}} \quad = \quad A(x,y)e^{i\varphi}$$

$\overset{\text{approx}}{\Longrightarrow}$ Let $\omega_0 \cong \omega(z)$ in the amplitude

Look at the phase of the exponential $e^{i\varphi}$

$$\varphi = \frac{\pi}{2} + \frac{k\left(x^2+y^2\right)}{2R(z)} - \tan^{-1}\frac{\lambda z}{\pi\omega_0^2}$$

Gaussian waist $\omega_0$ at z=0 $\quad \omega_0^2 = \frac{\lambda\ell}{2\pi}$ Where $\ell$ is the distance between confocal mirrors

$$2R(z) = 2z\left[1 + \left(\frac{\pi\omega_0^2}{\lambda z}\right)^2\right] = 2z\left[1 + \left(\frac{\ell}{2z}\right)^2\right] = \ell\left[\left(\frac{2z}{\ell}\right) + \left(\frac{\ell}{2z}\right)\right]$$

$$\tan^{-1}\frac{\lambda z}{\pi\omega_o^2} = \tan^{-1}\frac{2z}{\ell}$$

$$A_{00}\,(\mathrm{scat}) = 2\pi\frac{k\alpha(\omega)}{\omega_0}\sqrt{\frac{2}{\pi}} \quad e^{-2r^2/\omega_o^2} \cdot E_o \quad e^{i\left(\frac{\pi}{2} + \frac{kr^2}{\ell\left(\frac{2z}{\ell}+\frac{\ell}{2z}\right)} - \tan^{-1}\frac{2z}{\ell}\right)}$$

at $z/\ell \ll 1$ the result will be as before but further away from the exit pupil the diffraction pattern of the individual scatterers causes new phase shifts to occur. The distant scatters cause both phase and amplitude fluctuations in the mode amplitude

The total mode amplitude has both real and imaginary fluctuation terms.

$$\mathrm{A}_{00}\,(\mathrm{total}) = \mathrm{E}_0 w_0\sqrt{\frac{\pi}{2}}\left[1 + \frac{4k\alpha(w)}{w_0^2} \quad e^{-2r^2/w_0^2} \cos\left(\frac{\pi}{2} + \frac{kr^2}{\ell\left(\frac{2z}{\ell}+\frac{\ell}{2z}\right)} - \tan^{-1}\left(\frac{2z}{\ell}\right)\right)\right.$$
$$\left. + i\,\frac{4k\alpha(w)}{w_0^2} \quad e^{-2r^2/w_0^2} \sin\left(\frac{\pi}{2} + \frac{kr^2}{\ell\left(\frac{2z}{\ell}+\frac{\ell}{2z}\right)} - \tan^{-1}\left(\frac{2z}{\ell}\right)\right)\right]$$

Without further calculation, it is worth noting that the phase fluctuations in the mode amplitude will be comparable with fractional $\left(|A|/A\right)$mode amplitude fluctuations when integrated over the column of gas.

Looking at the structure of the above equation, it is plausible to argue that the phase fluctuation power spectrum is comparable with the fractional amplitude power spectrum and that furthermore they will be partially correlated.

$$\text{hunch is :} \quad \psi^2(\mathrm{f}) \approx \frac{|\mathrm{A(f)}|^2}{|\mathrm{A}|^2}$$

The detailed calculation is left as an exercise for the reader with nothing better to do.

I believe it is enough to show how a representative term looks upto the point of averaging over the impact parameter and pulse rate to see this.

For either the in phase or quadrature term in $A_{00}$(total) the Fourier transform of the pulse will involve integrals of the following type.

Again using $r^2 = b^2 + v_T^2 t^2$, the individual in phase and quadrature components can be divided into an amplitude and phase determined by the impact parameter and a time dependent phase and amplitude determined by the particle velocity.

A typical term in setting up the Fourier transform is

$$\underset{(\bar{0},\frac{\pi}{2})}{A_{00}(t)} = \frac{4k\alpha(\omega)}{\omega_0^2} \quad e^{-2b^2/\omega_0^2}\, \overbrace{\left[e^{-2v_T^2 t^2/\omega_0^2}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left(\frac{\pi}{2}+\frac{kb^2}{\ell(f(z))}-\tan^{-1}\left(\frac{2z}{\ell}\right)+\frac{kv_T^2t^2}{\ell(f(z))}\right)\right.}$$

$$f(z) = \left(\frac{2z}{\ell}+\frac{\ell}{2z}\right)$$

For a fixed value of z, the argument of the sinusoid can be expressed as a constant plus a time dependent term.

The Fourier transform of the time dependent part will have integrals of the form

$$\mathrm{S_u(f)} = 2\int_0^{\infty} \mathrm{e}^{-2\mathrm{v}_T^2\mathrm{t}^2/\omega_0^2}\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}\left(\frac{\mathrm{kv_T^2t^2}}{\ell\,(\mathrm{f(z)})}+\varphi\,(\mathrm{b,z})\right)\cos\omega\mathrm{tdt}$$

$$\text{where } \varphi\,(\mathrm{b,z}) = \frac{\pi}{2}+\frac{\mathrm{kb^2}}{\ell\,(\mathrm{f(z)})}-\tan^{-1}\left(\frac{2\mathrm{z}}{\ell}\right)$$

The integrals are of the form

$$\int_0^\infty e^{-\beta x^2} \left\{ \begin{matrix} \sin ax^2 \\ \cos ax^2 \end{matrix} \right\} \cos bx$$

$$= \frac{\sqrt{\pi}}{2\left(\beta^2 + a^2\right)^{\frac{1}{4}}} \quad e^{-\beta b^2 / 4\left(\beta^2+a^2\right)} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \left( \frac{1}{2} \tan^{-1} \frac{a}{\beta} - \frac{ab^2}{4\left(\beta^2 + a^2\right)} \right)$$

A typical term in the Fourier transform will look like

$$\mathrm{S_u}\,(\mathrm{f,z}) \sim \sqrt{\frac{\pi}{2}} \cdot \frac{\omega_0}{\mathrm{v_T}} \frac{1}{\left(1 + \frac{1}{4(f(z))^2}\right)^{\frac{1}{4}}} \quad e^{-\frac{\omega_0^2 \omega^2}{8 v_T^2} \cdot \frac{1}{\left(1+\frac{1}{4f(z)}\right)}}$$

$$\left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} \left( \frac{1}{2} \tan^{-1} \cdot \frac{1}{2f(z)} - \frac{\omega^2 l}{16 k v_T^2 f(z)} \cdot \frac{1}{\left(1 + \frac{1}{4(f(z))^2}\right)} \right)$$

f(z) runs from $\infty$ to 2.5 for z going 0 to $\ell$.

The spectrum at z→0 is exactly as before. The change in the amplitude of the spectrum at z=l is at most 10%.

The phase of the sinusoid changes slowly with z and the large phase shifts at high frequency are not noticed because of the rapid reduction in overall amplitude by the exponential term.

I will assume that the hunch is right to a factor of 2

$$\frac{|\mathrm{A(f)}|^2}{|\mathrm{A}|^2} = \psi^2(\mathrm{f})$$

Is this a problem for the LIGO?

A good reference is the mode amplitude fluctuations due to the Poisson (shot) noise in the beam.

The power spectrum of the power fluctuations, if the beam matches the $\mathrm{TE_{00}}$ mode at the exist pupil, is

$P^2(f)_{\mathrm{Poisson}} = 2h\nu < P >$ — h = Planck's constant

The rms power fluctuation is a band $\Delta$f — $\nu$ = photon frequency

$P_{rms} = (2h\nu\Delta f)^{1/2} < P >^{1/2}$ — $< P >$ average power in the mode, $P^2(f)$

power spectrum of the power fluctuations

In terms of the mode amplitude, the Poisson noise is given by

$$A^2_{rms_{Poisson}} = (2h\nu\Delta f)^2 |A|$$

The ratio of the scattering amplitude noise to the Poisson noise depends on the mode amplitude and the bandwidth. For a narrow bandwidth

$$\frac{A^2_{rms\ scattering}}{A^2_{rms\ Poisson}} = \frac{(2\pi)^4 \alpha^2 (\omega_{opt}) \langle\rho_{\#}\rangle \ell}{\lambda^2 \omega_0 v_0 (2h\nu)^{\frac{1}{2}}} \quad e^{-\frac{\sqrt{2\pi} f \omega_0}{v_0}} \cdot |A| (\Delta f)^{1/2}$$

Sample calculation $\quad f < \frac{v_0}{\omega_0 \sqrt{2\pi}} = 2.6\text{KHz}$

$$\begin{aligned} \alpha_{N_2} &= 1.54 \times 10^{-24} \text{cm}^3/\text{atom} \\ \lambda &= 1.06 \times 10^{-4} cm \\ \ell &= 4 \times 10^5 cm \\ \omega_0 &= 3.67\text{cm} \\ v_0 (N_2) &= 4.3 \times 10^4 \text{cm/sec} \\ \rho\# &= 3 \times 10^{10} \text{mol/cm}^3 \rightarrow 10^{-6} \text{torr} \\ \Delta f &= 1\text{KHz} \\ P_{watts} &= 100\text{Watts} \end{aligned}$$

$$\frac{A^2\text{rms scat}}{A^2\text{rms Poisson}} = 1.5 \times 10^{-27} \langle\rho_{\#}\rangle \sqrt{P(\text{watts})\Delta f} = 1.5 \times 10^{-14}$$

CONCLUSION

Amplitude fluctuations due to gas scattering will never be larger than shot noise for any stored power or residual gas density that might be contemplated. The glow due to Rayleigh scattering may be more significant. This will be addressed later when the scattering by mirrors and tube walls is considered. The reduction of the average forward intensity could also be significant with very good mirrors (discussion later)

Mode mixing by residual gas

Let the incident beam be in the $TE_{00}$ mode. Determine the coupling into other modes by the scatterers.

The mode amplitude from one scatterer

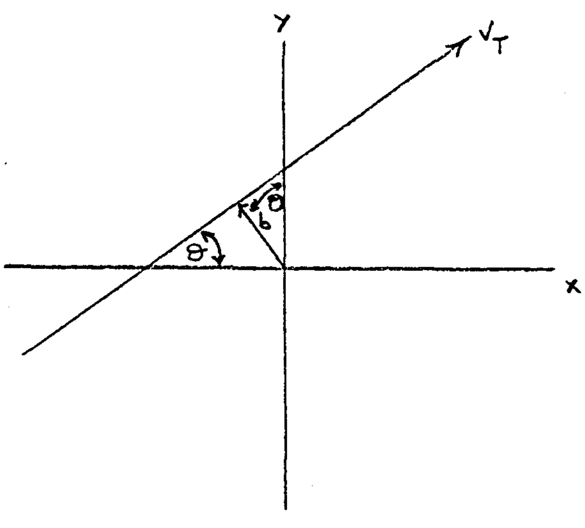
$$
\begin{aligned}
\mathrm{A}_{\mathrm{m,n}} &= \int\limits_x \int\limits_y u_{m,n}(x,y) E_{\text{scattered}}\left(x,y,o|x',y',z'\right) dxdy \\
&= \int\limits_x \int\limits_y u_{m,n}(x,y) \frac{k^2 e^{ikR}}{R} \cdot \alpha(w) E_0 \frac{\omega_0}{\omega(z')} \quad e^{-\frac{x^{2'}+y^{2'}}{\omega^2(z')}} \\
&\quad e^{ikz' + i\frac{k\left(x'^2+y'^2\right)}{2R(z')} - i\tan^{-1}\frac{\lambda z'}{\pi\omega_0^2}} dxdy
\end{aligned}
$$

Using stationary phase integral

Position of the scattered is $\mathrm{x}', \mathrm{y}'$

$$
\mathrm{A}_{\mathrm{m,n}} = \mathrm{i}\alpha(\omega)\mathrm{E}_0 2\pi\mathrm{K}\frac{\omega_0}{\omega(\mathrm{z}')}\left[\underbrace{u_{m,n}\left(x',y'\right) e^{-\frac{x'^2+y'^2}{\omega^2(z')}}}_{\beta(\mathrm{x}',\mathrm{y}')} \quad \mathrm{e}^{\mathrm{i}\frac{k\left(x'^2+y'^2\right)}{2\mathrm{R}(z')} - \mathrm{i}\tan^{-1}\left(\frac{\lambda z'}{\pi\omega_0^2}\right)}\right]
$$

Since $\mathrm{x}'$ and $\mathrm{y}'$ change with time due to the motion of the molecules, the molecules cause pulses in the mode amplitude $\mathrm{A}_{m,n}$. Much as before, the mode amplitude will be complex but in this case there is no average amplitude from the incident field, so that the quantity of interest is the modulus of the mode amplitude. The intergals are more difficult to carry out and as an example, to gain intuition, I will calculate the power spectrum of the excitation into the $u_{11}$ mode.




Position of the molecule with impact parameter b and transverse velocity $\mathrm{v}_T$ making angle $\vartheta$ with x axis

$$
\begin{aligned}
\mathrm{v_x} &= \mathrm{v_T}\cos\vartheta \\
\mathrm{v_T} &= \mathrm{v}\sin\vartheta
\end{aligned}
$$

Position of particle at t=0 is the impact parameter

$$
\begin{aligned}
x(0) &= \sin\vartheta b \\
y(0) &= \cos\vartheta b
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{x(t)} &= -\sin\vartheta\mathrm{b} + \mathrm{t} \;\; \mathrm{v_T}\cos\vartheta \\
y(t) &= \cos\vartheta b + v_T \;\; t \;\; \sin\vartheta \\
r^2(t) &= b^2 + v_T^2 t^2
\end{aligned}
$$

The excitation of the 1,1, mode as a function of time from this one molecule is

$$A_{11}(t) = \left[ \underbrace{i\frac{\alpha(\omega)E_0 k}{\omega_0^3} 8\frac{\sqrt{2}}{\pi} \quad e^{-2b^2/\omega_0^2}}_{A(b)} \right] \underbrace{(-\sin\vartheta b + v_T t\cos\vartheta)(\cos\vartheta b + v_T + \sin\vartheta)\, e^{-2v_T^2 t^2/\omega_0^2}}_{u(t,b)}$$

Take Fourier transform of pulse shape $u$(t,b)

Get three integrals

$$\begin{aligned}
\mathrm{S_u}(\mathrm{f}, \mathrm{v_T}, \mathrm{b}) =& 2\int_0^\infty (-\sin\vartheta\cos\vartheta)\, b^2 \cos\omega t\; e^{-2v_T^2 t^2/\omega_o^2}\, dt \\
&+ 2\int_0^\infty b v_T t \left(2\cos^2\vartheta - 1\right)\cos\omega t\; e^{-2v_T t^2/\omega_o^2}\, dt \\
&+ 2\int_0^\infty v_T^2 t^2 (\sin\vartheta\cos\vartheta)\cos\omega t\; e^{-2v_T^2 t^2/\omega_o^2}\, dt \\
=& 2\left\{ \frac{(\sin\vartheta\cos\vartheta)}{2} \cdot \sqrt{\frac{\pi\omega_0}{2\mathrm{v_T}}} \left[ \frac{\left(\frac{4v_T^2}{\omega_0^2} - \omega^2\right)\omega_0^4}{16 v_T^2} - b^2 \right] \quad e^{-\frac{\omega^2\omega_0^2}{8v_T^2}} \right. \\
&\left. + \left(\cos^2\vartheta - 1\right)\frac{b\omega_o^2}{4v_T}\left(1 - \frac{\omega\omega_o}{2^{3/2}v_T}\sum_{k=o}^{\infty}\frac{(-1)^k k!}{(2k+1)!}\left(\frac{\omega\omega_o}{\sqrt{2}v_T}\right)^{2k+1}\right.\right\}
\end{aligned}$$

The power spectrum of the mode amplitude becomes

$$A_{11}^2(f) = 2\langle \mathrm{n_O}\rangle \langle |A(b) S_u(f, v_T, b)|^2 \rangle_{v_T,\vartheta,b}$$

With the averages taken over the impact parameter, angles, and Maxwell velocity distribution as before.

The integral over the pulses becomes

$$A^2_{\left(\frac{11}{00}\right)}(f) = 2 \int\limits_{all\ pulses} A^2(b) | S_{u\,\left(\frac{11}{10}\right)} (f, v_T, b) |^2 d\dot{n}$$

$$= \frac{2^8}{\pi} \frac{\alpha^2(\omega) E_0^2 k^2}{\omega_0^4} \langle q \rangle \left(\frac{m}{2\pi kt}\right)^{\frac{3}{2}} \int\limits_{z=0}^{\ell} dz \int\limits_{-\infty}^{+\infty} e^{-\frac{mv_z^2}{2kT}} dv_z$$

$$\int\limits_{\vartheta=0}^{2\pi} \int\limits_{v_T^0}^{\infty} \int\limits_{b=0}^{\infty} e^{-4b^2/\omega_0^2} \left[ \sin\vartheta \cos\vartheta \sqrt{\frac{\pi}{2}} \left[ \frac{\left(\frac{4v_T^2}{\omega_0^2} - \omega^2\right)}{16 v_T^2} - b^2 \right] e^{-\frac{\omega^2 \omega_0^2}{8 v_T^2}} \right.$$

$$\left. + \left(\cos^2\vartheta - 1\right) \frac{b\omega_0}{2} \left(1 - \frac{\omega\omega_0}{2^{\frac{3}{2}} v_T} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)} \left(\frac{\omega\omega_0}{\sqrt{2} v_T}\right)^{2k+1}\right)\right]^2 e^{-mv_T^2 t/2kT} dv_T d\vartheta$$

$$= \frac{2^8}{\pi^2} \frac{\alpha^2(\omega) E_0^2 k^2 \ell \langle q_\# \rangle}{\omega_0^4} \left(\frac{m}{2kT}\right) \int\limits_{\vartheta} \int\limits_{v_T} \int\limits_{b} (\qquad)$$

Put integral into dimensionless form by using

$$v_T/v_0 = \beta \qquad\qquad v_0 = \left(2kT/m\right)^{\frac{1}{2}}$$

$$b/\omega_0 = \gamma$$

$$\omega/\left(v_0/\omega_0\right) = \Omega$$

The power spectrum of the $TE_{11}$ mode amplitude normalized to the square of the $TE_{00}$ mode amplitude becomes

$$A^2_{00} = E_0^2 \omega_0^2 \pi/2$$

$$\frac{A^2_{11}(f)}{A^2_{00}} = \frac{2^7}{\pi^3} \frac{\alpha^2(\omega) k^2 l \langle \rho_\# \rangle}{v_0 \omega_0} \int\limits_{\theta=0}^{2\pi} \int\limits_{\beta=0}^{\infty} \int\limits_{\gamma=0}^{\infty} e^{-4\gamma^2} \left[ \frac{\sin\theta\cos\theta}{2} \sqrt{\frac{\pi}{2}} \left(1 - \frac{1}{4}\frac{\Omega^2}{\beta^2} - 4\gamma^2\right) e^{-\frac{\Omega^2}{8\beta^2}} \right.$$

$$\left. + \left(\cos^2\vartheta - 1\right)\gamma^2 \left(1 - \frac{\Omega}{\beta 2^{\frac{3}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{\Omega}{\sqrt{2}\beta}\right)^{2k+1}\right]^2 e^{-\beta^2} d\beta d\gamma d\theta$$

The dimensionless integral calculated numerically shows a flat spectrum to $\Omega \sim 1$ and has a value $\sim 0.1$.

The power excited in the $TE_{11}$ mode/$TE_{00}$ mode

$$\frac{P_{11}}{P_{00}} = \frac{\int A_{11}^2(f)}{A_{00}^2} df \simeq \frac{2^8}{\pi^2} \frac{\alpha^2(\omega) k^2 \ell \langle \rho_{\#} \rangle}{\omega_0^2} \cdot (0.1)$$

↑

value of integral

$\omega \leq v_0/\omega_0$

Could the mode mixing be seen in the LIGO?

sample numbers

$\ell = 4 \times 10^5 cm$

$\alpha(N_2) = 1.5 \times 10^{-4} cm$

$\omega_0 = 3.67 cm$

$\lambda = 1.06 \times 10^{-4} cm$

$v_0(N_2) = 4.3 \times 10^4 cm/sec$

$\rho_{\#} = 3 \times 10^{10} mol/cm^3 \Rightarrow 10^{-6} torr$

$\frac{P_{11}}{P_{oo}} \sim 1.8 \times 10^{-22}$ for $P_{oo} \sim 100$ *Watts*

$P_{11} \sim 10^{-20}$ *Watts* of no consequence

The power into the other modes is probably comparable. I would expect the frequency spectrum to extend to higher frequencies as the order of the mode increases since the molecule crosses more lobes in the modes in its transit across the beam.

If the mode coupling could ever be seen, it would appear as a slight broadening of the beam associated with a time varying speckle. The mirrors and mirror motion are bound to be a far more serious source of mode mixing.

Scattering by the residual gas into other than the forward direction

Scattering into other than the forward direction is incoherent. Both due to the random distribution of the scatterers as well as the doppler shift due to their motion. To evaluate this scattering, it is sufficient to sum intensities.

The direct backscatter is looked at separately since for stationary scatterers it is coherent but smaller than the forward scattering in the ratio of the wave length to the column length.

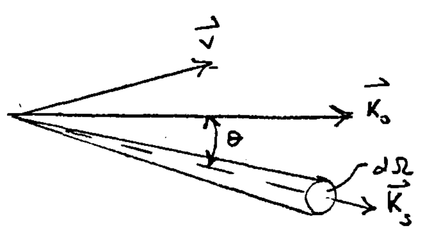
The major effect of the incoherent scattering is to reduce the forward intensity and to produce a "glow" in the tube which will become part of a random scattered field, reflected, scattered and diffracted by the tube walls.

The calculation of the incoherent scattering requires less care and is discussed under Rayleigh Scattering by:

| | | | |
|---|---|---|---|
| Jackson, D. | p | 422 - 423 | Static Molecules |
| Ishimaru | p | 80 - 84 | Moving Molecules |

The differential scattering cross section per unit volume with a particle density $\rho$ #/cc is given as

$$\frac{d\sigma}{d\Omega} = k_s^4 \alpha^2(w)\ \rho_{\#}$$




$k_0$ is the direction of the incident light

$k_s$ is the direction of the scattered light

$\alpha(w)$ is the molecular polarizability at $w$, assumed to be slowly varying with $w$ (not at a resonance in the molecule) so $\alpha(w_s) \sim \alpha(w_0)$

$v$ = velocity of the molecule

Total scattering cross section per unit volume integrated over the Doppler width is:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi k_s^4 \alpha^2(w) \rho_{\#}$$

The intensity is reduced as

$$I(z) = I(0)e^{-\sigma z} \qquad \frac{I(\ell)}{I(0)} \sim 1 - \sigma \ell b$$

The attenuation in the gas column is then

$$A = \sigma \ell b$$

Typical numbers

$b = \#$ of passes $= 50$

$\lambda = 1.06 \times 10^{-4} cm$
$\rho_{\#} = 3 \times 10^{10} mol/cm^3 \rightarrow 10^{-6} torr$
$\ell = 4 \times 10^5 cm$
$\alpha(N_2) \sim 1.5 \times 10^{24} cm^3/mol$

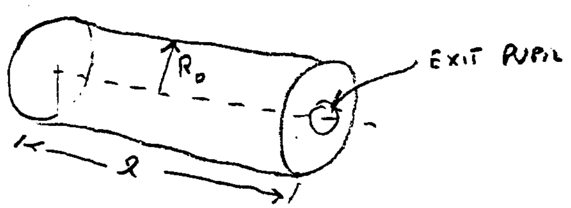
Attenuation $= 2 \times 10^{-10}$

Which is negligible relative to the mirror losses

The power sctttered into $4\pi$ Steradians is

$$P_{scat} = P_{incident} A$$

The scattered radiation becomes isotropized and somewhat depolarized by reflection, scattering and diffraction at the tube walls and baffles. A simple model to determine the brightness, defined as the intensity per solid angle, at the exit pupil is to model the tube as a randomizing cavity.




The energy density of the scattered light in the tube is $u$

Equilibrium demands

Power into the tube walls = power scattered by residual gas (neglect ends which is ok if 1-R>0)

$$(1-R) \frac{cu}{4} A_T = P_{scat/pass} b$$

$$\text{Brightness} = \text{B} = \frac{cu}{4\pi} = \frac{P_{in} A b}{(I-R) A-T}$$

b = # of passes of light
R = Power reflection of tube walls
$A_T$ = area of tube walls
$= 2\pi R_o \ell$

The area x solid angle product (etandue) of the exit pupil, if one couples out only one mode, is

$$A\Omega = \lambda^2$$

Neglecting the depolarization and the Doppler shift

The power coupled out due to the random scttering of the residual gas isotropized by the tube becomes

$$\frac{P_{oo}}{P_{in}} = \frac{\lambda^2 Ab}{(1-R)A_T} = \frac{2(2\pi)^3\alpha(\omega)\rho_{\#}b}{\lambda^2(1-R)R_o}$$

This is an upper estimate since it neglects the randomization of the phase, depolarization and the frequency width due to the Doppler shift by the molecular scatterers.

Sample #

$$R = 1/2 \qquad b = 50$$
$$R_o = 24'' = 61_{cm}$$
$$\rho_{\#} = 3 \times 10^{10} mol/cm^3 \Rightarrow 1 \times 10^{-6} torr$$
$$\alpha(N_2) = 1.5 \times 10^(-24) cm^3/mol$$

$P_{oo}/P_{in} =\sim 5 \times 10^{-27}$

This is completely negligible, the contribution form the mirrors will be much larger

## SPECIAL CASE OF BACKSCATTERING

Back scattering is similar to forward scattering in that coherence plays a role.

For stationary backscatters the average back scattered field is small. From simple arguments one can see that the scatterers in cross sections at different values of $z$ in an incident beam propagating in the $+z$ direction will radiate waves back to an exit pupil with phase shifts $2kz$. The back scattered field at $z$ and $z+\lambda/4$ will cancel and the only net contribution will come from a layer about $\lambda/2$ in thickness rather than the total column length as is the case in forward scattering. This is the basis of the "obliguity factor" in the Kirchoff-Huyghens diffraction theory.

The analysis changes with moving scatterers

Take a Gaussian beam in the $TE_{oo}$ propagating in the $+z$ direction with the exit pupil at $z = 0$.

Neglect the beam curvature and phase shift. Because of the caustic at the exit pupil

$$E(z, r') = E_o e^{-r^2/w_o^2} e^{ikz'}$$

The scattered field is again given by

$$E_{sc}(x,y,o|x',y',z') = \frac{k_s^2 e^{ik_s R}}{R}\alpha(\omega)E_{\rm inc}(x'y'z')$$

Because of the Doppler shift in the backscattered direction

$$k_s = k_o(1 - 2v_{z/c}) \qquad \text{where } v_z \text{ is the velocity of the scatterer in the z direction}$$

The stationary phase intergration over the exit pupil gives the mode amplitude

$$A_{oo}(v_z) = i\sqrt{\frac{2}{\pi}}\frac{\alpha(\omega)}{\omega_o}E_o\ e^{-2r'^2/\omega_o^2}2\pi k_s e^{i(2k_o z' - \frac{2k_o v_z}{c}z')}$$

The scatterer moves

$$r'^2 = b^2 + v_T^2 t^2 \qquad z' = z_o' + v_z t$$

$$A_{od}(z_o', v_z, v_T, t) \cong i\sqrt{\frac{2}{\pi}}\frac{\alpha(\omega)}{\omega_o}\ E_o 2\pi k_o e^{-2b^2/\omega_o^2}\ e^{i2k_o z_o'(1-v_z/c)}$$

$$e^{-2v_T^2t^2/\omega_o^2}e^{i2k_o v_z t} \quad = \quad A(b, z_o, v_z)u(t, v_T, v_z)$$

Neglecting second order terms in $v_z/c$

The Fourier transform of the pulse shape is

$$S_u(f, v_T, v_z) = \int_{-\infty}^{\infty} e^{-2v_T^2\ t^2/\omega_o^2}\ e^{i(2k_o v_z t - \omega t)}\ dt$$

$$= 2\int_{-0}^{+\infty} e^{-2v_T^2t^2/\omega_o^2} \quad \cos(2k_o v_z - \omega)t\ dt$$

$$S_u(f, v_T, v_z) = \sqrt{\frac{\pi}{2}}\left(\frac{\omega_o}{v_T}\right)\ e^{-(2k_o v_z - \omega)^2}\ \frac{\omega_o^2}{8v_T^2}$$

The power spectrum of the pulse over all $v_z$ in the Maxwell distribution gives

$$\langle (S_u(f, v_T))^2 \rangle_{v_z} = \frac{\pi}{2} \frac{\omega_o^2}{v_T^2} \left( \frac{m}{2\pi kT} \right)^{1/2} \int_{-\infty}^{+\infty} e^{-mv_z^2/kT} \quad e^{-\frac{(2k_o v_z - \omega)^2 \omega_o^2}{4 v_T^2}} dv_z$$

The integral is done by completing the square in the exponent

$$\langle (S_u(f, v_T))^2 \rangle_{v_z} = \frac{\pi}{2} \frac{\omega_o^2}{v_T^2} \left( \frac{m}{2\pi kT} \right)^{1/2} \quad e^{-\frac{\omega^2 \omega_o^2}{4 v_T^2} \frac{1}{(1 + \frac{kT}{m v_T^2} k_o^2 \omega_o^2)}}$$

$$\int_{-\infty}^{+\infty} e - \left[ \left( \frac{m}{kT} + \frac{k_o^2 \omega_o^2}{v_T^2} \right)^{1/2} v_z - \frac{k_o \omega \omega_o^2}{2 v_T^2 \left( \frac{m}{2kT} + \frac{k_o^2 \omega_o^2}{v_T^2} \right)^{1/2}} \right]^2 dv_z$$

$$\langle |S_u(f, v_T)|^2 \rangle_{v_z} = \frac{\frac{\pi}{2} \frac{\omega_o^2}{v_T^2} \left( \frac{m}{2kt} \right)^{1/2}}{\left( \frac{m}{2kt} + \frac{k_o^2 \omega_o^2}{v_T^2} \right)^{1/2}} \quad e^{-\frac{\omega^2 \omega_o^2}{4 v_T^2} \frac{1}{(1 + \frac{2kT}{m v_T^2} k_o^2 \omega_o^2)}}$$

In terms of the most probable velocity

$$\langle |S_u(f, v_T)|^2 \rangle_{v_z} = \frac{\frac{\pi}{2^{3/2}} \frac{\omega_o^o}{v_T^2}}{\left( 1 + \left( \frac{v_o}{v_r} \right)^2 \frac{k_o^2 \omega_o^2}{2} \right)^{1/2}} \quad e^{-\frac{\omega^2 \omega_o^2}{4 v_T^2} \frac{1}{(1 + (\frac{v_o}{v_T})^2 \frac{k_o^2 \omega_o^2}{2})}}$$

$$\simeq \frac{\pi}{2^{3/2} v_o^2 k_o^2} \quad e^{-\frac{\omega^2}{2 v_o^2 k_o^2}}$$

The pulse power spectrum in the mode averaged over all transverse velocities

$$A_{oo}^2(f) = 2 \int_{all\ pulses} A^*(b, z_o, v_z)\ A(b, z_o, v_z)\ |S_u(f, v_T)|^2 d\dot{n}$$

$$= \frac{8\pi^2}{\sqrt{2}} \frac{\alpha^2(\omega)}{v_o^2 \omega_o^2} E_o^2\ \rho \# \left( \frac{m}{2\pi kT} \right) \quad e^{-\frac{\omega^2}{2 v_o^2 k_o^2}} \quad x$$

$$\int_o^\ell d_z \int_\varphi^{2\pi} d\varphi \int_{b=o}^{\infty} e^{-4b^2/\omega_o^2}\, db \int_{v_T=o}^{\infty} e^{-mv_T^2/kT}\ v_T^2\ dv_T$$

$$A^2_{oo}(f) = \frac{\pi^3}{2}\frac{\alpha^2(\omega)E_o^2\rho_{\#}\ell}{v_o\omega_o} \quad e^{-\omega^2/2v_o^2k_o^2}$$

The spectrum is distributed around $\omega_o$, the light frequency, the frequency $\omega$ is measurable in the fringe phase directly. The spectrum has the Doppler width extending from $f = 0$ to $f(1/\ell) \sim \omega_o v_{o/c}$

Is the backscatter a problem for the LIGO?

The phase fluctuations and the fractional amplitude fluctuations in the mode must be considered. In a Fabry-Perot system the backscattered field becomes part of the cavity field. In a Michelson system without recycling the backscattering is not part of the interferometer output.

The ratio of the phase fluctuations from backscatter to forward scatter is

$$\frac{\psi^2_{\text{back}}(f)}{\psi^2_{\text{forward}}(f)} = \frac{\frac{\pi^2\alpha^2(\omega)\rho_{\#}\ell}{v_o\omega_o^3} \quad e^{-(2\pi f)^2/2v_ok_o}}{\frac{(2\pi)^2k^2\alpha^2(\omega)\rho_{\#}\ell}{\omega_o v_o} \quad e^{-\frac{\sqrt{2}\pi f\omega_o}{v_o}}}$$

$$\cong \frac{1}{k^2\omega_o^2} \quad \sim 2\times 10^{-11} \quad \text{for } \lambda = 1.06\text{u}$$

$$f < v_o/\omega_o, f < v_ok_o$$

The backscattering is negligible relative to the forward scattering.

The relative amplitude fluctuations have the same ratio, backscattering is not a problem.