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LIGO PROJECT

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Requirements for LIGO servo gains:

Allowed deviations for the differential-mode sensing

The allowed deviation from resonance (for Fabry-Perots), dark fringe (for the Michelson), and the optimum phase (Mach-Zehnder) is calculated as determined by various noise sources and imperfections. In summary, for an LIGO-scale interferometer using external modulation, with a contrast of $C = 1 - 10^{-3}$, and balance in the storage times of the arms of $\beta = 10^{-2}$, the three systems should not exceed the following deviations:

Cavities: 2×10^{-5} rad (non-linearity of $d\phi/dx$)

Michelson path length: 3×10^{-2} rad (changes in dark fringe intensity)

Mach-Zehnder error signal: 2×10^{-3} rad (mixer saturation from lf motion)

Mach-Zehnder path length: 0.14 rad (sensitivity to strain)

Since all three systems (cavities, Michelson, Mach-Zehnder) influence the final detected phase difference, the gain as a function of frequency in the various branches of the servo-loops can be balanced to take advantage of the configuration.

Yet to be addressed are the allowed common-mode deviations (sum of cavity lengths, recycling cavity lock); the physical mirror motions before 'servoing'; and the servo topologies and gains.

Fabry-Perot cavities

1) *FM sensitivity from cavity differences.* If the two Michelson arms do not have the same storage time there will be a sensitivity to frequency noise on the laser light. This could be due to the arm cavities having different storage times, or the inside interferometer (made up of the beamsplitter and the near mirrors) being asymmetric, or due to the cavity mirrors having different masses. The fractional imbalance $\Delta\tau = \tau_1 - \tau 2$ leads to a sensitivity

$\beta = \Delta\tau/[(\tau_1 + \tau_2)/2]$, making an apparent length change δl due to a frequency input spectrum $\nu(f)$ equal to

$$\delta l(f) = \beta\, l \frac{\nu(f)}{\nu}$$

for arms of length l. For a β of 10^{-2}, this requires that the frequency noise be less than $\nu(f) = 1.6\times10^{-6}$ Hz/$\sqrt{\text{Hz}}$ to contribute less than $\delta l = 10\times10^{-19}$ m/$\sqrt{\text{Hz}}$. The recycling cavity will act as a temporal filter of a single pole at about 10 Hz, so that the input light to the interferometer should have roughly 1.6×10^{-5} Hz/$\sqrt{\text{Hz}}$ at 100 Hz.

2) *FM sensitivity from non-linearity of $d\phi/dx$ in Fabry-Perot cavities.* The phase change per length change of a Fabry-Perot cavity is a quadratic function of the deviation from resonance around the resonance. The slope of the phase $d\phi/dx$ near resonance is

$$\frac{d\phi}{dx} = -\frac{(1-A_2)^{\frac{1}{2}}(1-A_1-T_1)^{\frac{1}{2}}\left[-(1-A_2)^{\frac{1}{2}}(1-A_1-T_1)^{\frac{1}{2}} + \cos(x)\right]}{1 + (1-A_2)(1-A_1-T_1) - 2(1-A_2)^{\frac{1}{2}}(1-A_1-T_1)^{\frac{1}{2}}\cos(x)}$$
$$+ \frac{(1-A_1)(1-A_2)\left[-(1-A_1)(1-A_2) + (1-A_1-T_1)^{\frac{1}{2}}\cos(x)\right]}{(1-A_1)^2(1-A_2)^2 + (1-A_1-T_1) - 2(1-A_1)(1-A_2)(1-A_1-T_1)^{\frac{1}{2}}\cos(x)}$$

where T_1 is the power transmission of the input mirror, A_1 and A_2 are the power losses in the mirrors, and $x = 4\pi(v - v_0)l/c$ is the deviation from resonance. An expansion around the null point (x=0) shows a quadratic decrease in the slope. For the sample values $T_1 = 0.03$, A_1=A_2=100 ppm, the slope (normalized to the slope at $x = 0$) is approximately $d\phi/dx \approx= -1 + 4.35\times10^3 x^2 - 2\times10^7 x^4$. To put the numbers in context, if we have a cavity storage time matching of $\beta = 10^{-2}$ and do not want the deviation from resonance of the cavities to change the effective storage time by more than $\beta\tau$, the difference of the deviations of the two cavities $x_1 - x_2$ should not exceed $x = 1.5\times10^{-3} \hat{=} 2\times10^{-5}$ rad $\hat{=}$ 0.05 linewidths $\hat{=} 6\times10^{-11}$ m, or about $\phi = 6° \hat{=}$ 0.1 rad of phase shift on reflection from the cavity.

A different, and much smaller effect of motions away from the resonance point of a Fabry-Perot is the upconversion (to higher odd harmonics) as a result of the non-linearity of $\phi(x)$. Expanding $\phi(x)$ around $x = 0$ for the sample cavity parameters used above gives $\phi(x) \approx -x + 1440x^3 + \ldots$. For example, motion at f =33 Hz will be upconverted to $3f$ =100 Hz, resulting in a phase shift due to the nonlinearity of $\phi_{99\text{Hz}} = (1440/4)x^2_{33\text{Hz}}$. The test mass motion at 33 Hz, filtered by the stack, will be no greater than 10^{-14} m/$\sqrt{\text{Hz}}$. This leads to a $\phi_{99\text{Hz}}$ due to upconversion that is some 6 orders of magnitude less than the ϕ due to direct mass motion. Higher harmonic upconversion leads to even smaller contributions.

3) *Mirror motions due to photon pressure.* While this is not a servo gain question, it is a phenomenon due to the intensity fluctuations and fits in well here. Intensity fluctuations exert a time-varying force on the Fabry-Perot mirrors. There may be imbalances in the interferometer either in the storage time (the β above) or in intensity (e.g., a 48-52 beam-splitter or additional losses in one cavity) resulting in a fractional imbalance for intensities

β_{int}. The resulting forces will not be the same in the two cavities, leading to real motions of the mirrors:

$$\delta x(f) = \frac{\delta I}{I} \frac{2P_{\text{circ}}(\beta + \beta_{\text{int}})}{mc(2\pi f)^2} \frac{1}{\sqrt{1+(4\pi\tau f)^2}}$$

where P_{circ} is the circulating power in the Fabry-Perot cavities. The last term accounts for the reduction of the intensity fluctuations coupled into the cavity above its knee frequency $1/4\pi\tau$. Assuming a $(\beta + \beta_{\text{int}}) = 10^{-2}$, $P_{\text{circ}} = 10$ kW, a mirror mass of m=10 kg, and $1/4\pi\tau = 100$ Hz, we demand an intensity noise of $\frac{\delta I}{I} < 10^{-6}$ to keep $\delta x(f) < 10\times10^{-19}$ at 100 Hz. The recycling cavity filters the intensity noise, as above, but the laser for this example must have $\frac{\delta I}{I} < 10^{-5}$ at 100 Hz. This noise source is insignificant compared to residual seismic noise at very low frequencies (1 Hz).

Michelson Interferometer

Changes in the Michelson output intensity. Deviations from the Michelson dark fringe lead to a quadratic increase in the intensity at the interferometer output. This makes changes in the shot noise level in the detection system, and ultimately leads to power dissipation problems in the photodetector. The intensity is given by

$$I = \frac{I_{\text{max}}}{2}[1 - CJ_0(m)\cos(x)]$$

where I_{max} is the (equivalent, in the case of recycling) output intensity on the bright fringe, $C = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})$ is the contrast due to all sources (mode mismatch and intensity differences) but without modulation, and $J_0(m)$ and m the first Bessel function and the differential modulation depth applied. One reasonable criterion is that the intensity due to the rms deviations in the interferometer differential phase $x = 2\pi\delta l_{\text{mich}}/\lambda$ be smaller than the intensity due to the limited contrast and the modulation depth m of the interferometer. For small deviations x, this gives $x < \sqrt{1 - CJ_0(m)}$. For small modulation depths ($m < .01$) and good contrasts ($C > (1 - 10^{-3})$) this gives $x \approx 3\times10^{-2}$ rad rms.

1.2×10^{-9} m 0.1 Å for arms

Sensitivity of Michelson to frequency noise. This is addressed under Fabry-Perot cavities 1) above.

Mach-Zehnder Interferometer

1) *Deviation of the Mach-Zehnder itself from the null point.*

Sensitivity to mirror motion in the Mach-Zehnder. The phase of the two beams which are brought to interference on the Mach-Zehnder beamsplitter is

$$\phi_o^{\rm MZ} = \Phi_{\rm ref} - k(l_1 + l_2)/2 - \frac{E_1 + E_2}{E_1 - E_2}\frac{k\Delta l}{2}$$

where $\Phi_{\rm ref} = l_{1\rm ref} - l_{2\rm ref}$ is the difference of the length of the Mach-Zehnder reference arms, l_1 and l_2 are the lengths of the Michelson arms, including the effective 'length' (or phase change) due to the Fabry-Perot cavities, E_1 and E_2 are the electric fields in the two Michelson arms, $\Delta l = l_1 - l_2$, and $k = 2\pi/\lambda$ is the wavenumber. The phase differences in the Michelson are 'amplified' by the factor $(E_1 + E_2)/(E_1 - E_2)$. This factor can be related to the Michelson contrast for the $\rm TEM_{00}$ component of the antisymmetric output beam C_{00}, so that the recovered signal at the modulation frequency Ω and depth m is

$$I_\Omega = 2\sqrt{I_{\rm ref} I_0} J_1(m)\left[\frac{k\Delta l}{2} + \sqrt{\frac{1 - C_{00}}{2}}\left(\frac{\Phi_{\rm ref} - k(l_1 + l_2)}{2}\right)\right]$$

where $I_{\rm ref}$ and I_0 are the two Mach-Zehnder beam intensities. The sensitivity of the Mach-Zehnder to mirror motion in the Mach-Zehnder as compared with mirror motion in the inside Michelson (e.g., the Michelson beamsplitter) is reduced by a ratio of roughly $\approx \sqrt{1 - C_{00}/2}$, where the Michelson contrast C_{00} is the contrast as limited by the $\rm TEM_{00}$ beam interference. Higher order modes do not degrade this ratio. For a contrast $C_{00} = 1 - 10^{-3}$, the ratio is 2.2×10^{-2}. The implication is that component motion in the Mach-Zehnder proper, and phase noise in general introduced in the Mach-Zehnder (e.g., Pockels cells), is much reduced in importance.

Sensitivity to differential signals (GWs). The sensitivity $\mathcal{S}$ to differential strain in the main interferometer is cosinusoidal around the correct phase for the Mach-Zehnder: $\mathcal{S} \sim \cos\Phi_{\rm ref} - k(l_1 + l_2)$. If we choose not to have a reduction of say 1% in sensitivity, the Mach-Zehnder phase is constrained to $\Phi_{\rm ref} - k(l_1 + l_2) \; < 0.14$ rad.

Constraints from the Michelson. The signal to hold the Michelson accurately on the dark fringe comes from the Mach-Zehnder. If the Mach-Zehnder strays from the correct phase ($\Phi_{\rm ref} - k(l_1 + l_2 \neq \pi$), the Michelson must correct for it, pulling it away from the Michelson dark fringe. The ratio of phase sensitivities in the Mach-Zehnder to the Michelson is the same as mentioned in 2) above, thus $1/\sqrt{(1 - C_{00})/2}$, or about 1/50 for a $\rm TEM_{00}$ contrast of $1 - 10^{-3}$. Otherwise said, the Mach-Zehnder should be held to $<\phi_{\rm MZ}> \, \leq \, <\phi_{\rm MI}> \times\sqrt{(1 - C_{00})/2}$. For our example, if the Michelson is to be held to within 10^-2 rad, the Mach-Zehnder path length difference must be held to within some 0.5 rad. The expected damped, but not servo-controlled rms motion of the components is expected to be much less than this limit.

2) *Sensitivity to frequency noise.* The sensitivity of the Mach-Zehnder to frequency noise depends on the TEM_{00} contrast C_{00} of the Michelson and the symmetry of the Mach-Zehnder arms. Rewriting the bracketed term in the expression above for the Mach-Zehnder signal, but now including the frequency noise, gives

$$\left[\frac{k\Delta l}{2} + 2\pi\nu(f)\Delta\tau + \sqrt{\frac{1-C_{00}}{2}}\left(\Phi_{\text{ref}} - \frac{k(l_1+l_2)}{2} + 2\pi\nu(f)\Delta\tau_{\text{ext}}\right)\right]$$

where $\nu(f)$ is the spectral density of frequency noise and $\Delta\tau_{\text{ext}}$ is the imbalance in the storage time of the Mach-Zehnder arms (presumably primarily due to length differences). It should be possible to make $\Delta\tau_{\text{ext}} \leq \Delta\tau$, and there is an additional reduction in the importance from the Michelson contrast.

3) *AM sensitivity due to deviation from the MZ error signal null point.* Deviations from the error signal null point lead to first-order sensitivity to intensity fluctuations in the laser beam at the GW measurement frequency. This noise term is proportional to the deviation from the null point of the servo error signal, whether it is a Mach-Zehnder or a Michelson path length difference or a Fabry-Perot cavity, or a combination of the three, that produces the net deviation. A (small) deviation in an arm cavity can be corrected by a (larger) change in the Michelson path length difference, and then this purturbation can be reduced by the servo-loop which uses the Mach-Zehnder error signal with the correct servo topology.

Given this, deviations of the error signal from the null point allow power fluctuations in the laser power to appear as a signal at the error signal output. This should be less than the shot-noise limited sensitivity, thus putting an upper limit on the convolution in frequency of the deviations from the null point $\delta\phi(f)$ and the power fluctuations $\widetilde{\delta I}(f)/I$ at the measurement frequency f. A reasonable approximation to this, given the very small bandwidth of the deviations from the null point, is given by the product of the root mean square (rms) deviations $\delta\phi_{\text{rms}}$ and the power fluctuations. For a system which is optimally modulated,

$$\delta\phi_{\text{rms}}\frac{\widetilde{\delta I}(f)}{I} < \sqrt{\frac{h\nu}{\eta P}} \ \ \text{rad}/\sqrt{\text{Hz}}$$

In the case of the cavity length difference sensed by the external modulation scheme, this puts, strictly speaking, a condition on the Mach-Zehnder interferometer deviations. However, the Michelson path length differences make, by far, the biggest contribution to the phase differences sensed by the Mach-Zehnder (as seen above).

As an example, the shot noise limited sensitivity for the main interferometer is 1×10^{-10} rad/$\sqrt{\text{Hz}}$ ($h \approx 10^{-21}$ in a 1 kHz band). A typical Argon laser (Garching small frame Coherent) shows $\delta I(f)/I$ $\text{Hz}^{-\frac{1}{2}}$ of 1×10^{-4} at 100 Hz. If there were no temporal filtering by the recycling cavity, and no active power stabilization, this would require that the Mach-Zehnder make deviations of less than 1×10^{-6} rad rms from the null point. In fact, the power fluctuations of the pre-stabilized laser are filtered by the recycling cavity transfer function which behaves like a pole at about 10 Hz, and active intensity prestabilization by a factor of 1000 at 100 Hz is straightforward. This gives an maximum allowed deviation from the null point of $\approx 10^{-2}$ rad rms, with a tradeoff between laser intensity prestabilization and loop gain.

5) *Mixer saturation.* The dynamic range of the signal recovery system from the photodiode to the voltage output signal will probably be limited by the RF mixer which is used to demodulate the RF ($\approx$ 10 MHz) phase-modulated signal. These devices can be crudely characterized by an output noise N V/$\sqrt{\mathrm{Hz}}$ and a maximum linear output voltage $V_{\mathbf{max}}$. If we choose an RF preamplifier gain such that the shot noise $\phi_{\mathbf{shot}}$ at the mixer output exceeds the mixer noise by a certain safety factor $\mathcal{A}$, then the output limit gives a corresponding Mach-Zehnder phase deviation ϕ_{MZ} of

$$\phi_{MZ} = \frac{\phi_{\mathbf{shot}} V_{\mathbf{max}}}{\mathcal{A} N} \; .$$

The mixers currently used show less than 3×10^{-8} V/$\sqrt{\mathrm{Hz}}$ of noise at the IF output, and have about 1 V rms maximum linear output; the shot noise limited phase is $\phi_{\mathbf{shot}} = 10^{-10}$ rad/$\sqrt{\mathrm{Hz}}$. For a safety factor of 2, this demands deviations from the null point of the Mach-Zehnder phase of less than $\phi_{MZ} < 1.7\times10^{-3}$. A closer look at the mixer performance may relax this figure.