New Folder Name Displacement & Sensitivity

Appendix 3: comparison of shotnoise formulas.

Phase, Displacement, h Sensitivity

for Recombined Fabry-Perot Interferometer

(Fomulae here apply below knee frequency)

1. Phase sensitivity (no modulation):

$$\varphi(f) = \left[\frac{2h\nu}{\eta P}\right]^{1/2}$$

P - total power

φ - phasedifference betw the two arms.

Appendix 1

2. Displacement sensitivity:

$$\ell(f) = \varphi(f)\frac{d\ell}{d\varphi} = \varphi(f)\frac{d\ell_1}{d\varphi_1}$$

ℓ_1 : displacement in one arm

φ_1 : phase shift imparted to light by interaction with one arm.

Appendix 2: $\frac{d\ell_1}{d\varphi_1} = -\frac{T_1}{8k}$ T_1: input coupler transmission $k = \frac{2\pi}{\lambda}$

$$\ell(f) = \frac{T_1}{8k}\left[\frac{2h\nu}{\eta P}\right]^{1/2}$$

In terms of RES* definitions: $\frac{\eta P}{h\nu} = \dot{N}$, $k = \frac{1}{\lambda\!\!\!^-}$, $T_1 = \mathcal{L}$

*) RES = R.E. Spero: Comparison of Shot Noise Formulas, 10-Feb-9
(Appendix 3)

$$\boxed{\ell(f) = \lambda \mathcal{L} \frac{1}{\sqrt{\dot{N}}} \frac{\sqrt{2}}{8}}$$

3. h Sensitivity:

$\ell = h L$ $\quad$ L - length of one arm

$$\boxed{h(f) = \frac{\lambda \mathcal{L}}{L} \frac{1}{\sqrt{\dot{N}}} \frac{\sqrt{2}}{8}} \qquad x(f) = \frac{\lambda \mathcal{L}}{\sqrt{\dot{N}}} \frac{\sqrt{2}}{8}$$

- For: $\eta P = 2W$ $\longrightarrow$ $\dot{N} = 5.20 \times 10^{18} \frac{\text{photons}}{s}$

Recycling factor 30 $\longrightarrow$ $\dot{N} = 1.56 \times 10^{20} \frac{\text{photons}}{s}$

- For $\mathcal{L} = 3\%$

$L = 4{,}000$ m

$$\longrightarrow \boxed{h(f) = 8.7 \times 10^{-24} / Hz^{1/2}}$$

Again, in terms of RES: $x_0 = \frac{\lambda \mathcal{L}}{\sqrt{\dot{N}}} \cdot \frac{\sqrt{3}}{4}$

and: $\boxed{x(f) = x_0 / \sqrt{6}}$

Appendix 1

Mid-Fringe/Dark Fringe Comparison for Simple Michelson Interferometer

1. – No modulation

2. – Photocurrent at symmetric/antisymmetric port:

efficiency

$$i_\pm = \frac{\eta P e}{2 h\nu}\left[1 \pm \cos\phi\right] = \Gamma\left[1 \pm \cos\phi\right]$$

- P : total incident power
- ϕ : Phase difference between light beams from the two arms

3. – Small signal φ : $\phi = \phi_0 + \varphi$; ϕ_0 : static phase difference.

- $\cos\phi = \cos\phi_0 - \varphi \sin\phi_0$
- $i_\pm = \Gamma\left[1 \pm \cos\phi_0 \mp \varphi \sin\phi_0\right]$
- $i_\pm^{signal} = \mp \varphi\, \Gamma \sin\phi_0$
- $i_\pm^{DC} = \Gamma\left(1 \pm \cos\phi_0\right)$ – enters shot noise calculation, according to $i^{shot}(f) = \left[2 i^{DC} e\right]^{1/2}$

4. – when only the antisymmetric port is used:

- $i^{signal}(f) = \varphi(f)\,\Gamma \sin\phi_0$
- $i^{shot}(f) = \left[\,2\Gamma\left(1 - \cos\phi_0\right) e\,\right]^{1/2}$

• Equating $i^{signal}(f)$, $i^{shot}(f)$:

$$\varphi(f) = \left[\frac{2e}{\Gamma} \cdot \frac{1-\cos\phi_0}{\sin^2\phi_0}\right]^{1/2} = \frac{1}{\cos\frac{\phi_0}{2}}\left[\frac{e}{\Gamma}\right]^{1/2}$$

• Minimum value for $\varphi(f)$ is reached when $\cos\frac{\phi_0}{2} = 1$, i.e. $\phi_0 = 0$, that is for a dark fringe, in which case:

$$\boxed{\varphi(f) = \left[\frac{e}{\Gamma}\right]^{1/2}} \quad \text{or: } \varphi(f) = \left[\frac{2h\nu}{\eta P}\right]^{1/2}$$

Derivation used above (with slight differences):
W. A. Edelstein, J. Hough, J. R. Pugh, W. Martin: Limits to the measurement of displacement in an interferometric gravitational radiation detector, J. Phys. E, 11, 710 (1978)

5.— Interferometer at mid-fringe, signal from both ports used:

• $i^{signal} = i_-^{signal} - i_+^{signal} = 2\varphi\Gamma$ (since $\phi_0 = \frac{\pi}{2}$ for mid-fringe operat.

• $i^{shot} = [2 \times 2e\Gamma]^{1/2}$

• $\boxed{\varphi(f) = \left[\frac{e}{\Gamma}\right]^{1/2} = \left[\frac{2h\nu}{\eta P}\right]^{1/2}}$

6.— CONCLUSION:

Michelson interferometer shot noise limited sensitivity, is the same, for the two cases:

a) Signal from antisymmetric port only, which is kept on a dark fringe

b) Interferometer at mid-fri[illegible] signal from the two ports combined

NOTE: This result was first communicated to me by R. Schilling, in 1982

ALEX

Appendix 2

See following pages for details

1. Phase Sensitivity of a Fabry-Perot Cavity

(for the total field coming back from a cavity with high transmission input coupler)

$$\frac{d\phi}{dx} = 2k\sqrt{R_2}\,\frac{T_1}{\left[\sqrt{R_1} - (1-L_1)\sqrt{R_2}\,\right]\left[1-\sqrt{R_1R_2}\,\right]} \quad ; \quad k = \frac{2\pi}{\lambda}$$

For $R_1 = 1 - T_1 - L_1$, $R_2 = 1 - L_2$ $(T_2 = 0)$

and $T_1, L_1, L_2 \ll 1$:

$$\frac{d\phi}{dx} = \frac{8kT_1}{(L_1+L_2-T_1)(L_1+L_2+T_1)} = \frac{4kT_1}{\pi}\cdot\frac{\mathcal{F}}{L_1+L_2-T_1}$$

$$\boxed{\frac{d\phi}{dx} = -\frac{4k\mathcal{F}}{\pi}\cdot\frac{1}{1-\frac{L_1+L_2}{T_1}}} = \frac{-8k}{T_1\left(1+\frac{L_1+L_2}{T_1}\right)\left(1-\frac{L_1+L_2}{T_1}\right)}$$

$$\frac{d\phi}{dx} = 4k\,\frac{1}{1-r_1} = \frac{8k}{T_1} \text{ ok}$$

$$\frac{1}{1-\sqrt{1-T_1}} = \frac{1}{1-\left(1-\frac{1}{2}T_1\right)} = \cancel{8k}$$

For high transmission: $T_1 \gg L_1, L_2$:

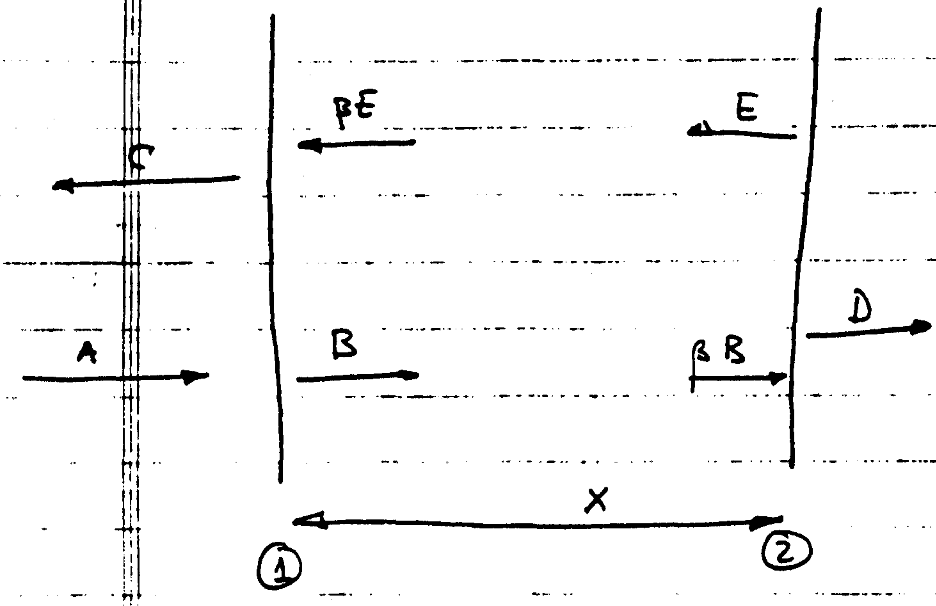
$$\boxed{\frac{d\phi}{dx} = -\frac{8k}{T_1}} = -\frac{4k}{\pi}\mathcal{F}$$

$$\boxed{\frac{d\phi}{dh} = \frac{\ell}{2}\frac{d\phi}{dx} = -\frac{2k\ell}{\pi}\mathcal{F}} = -\frac{4k\ell}{T_1}$$

$$\mathcal{F} = \frac{\pi c}{\ell}\tau_{st} \Rightarrow \frac{d\phi}{dh} = \frac{c}{\lambda}4\pi\tau_{st} = \frac{c}{\lambda f_0} \quad ; \quad f_0 \equiv \frac{1}{4\pi\tau_{st}}$$

All the above is valid for $f \ll f_0$, otherwise there is an additional factor $\left[1+\left(\frac{f_0}{f_0}\right)^2\right]^{-1/2}$

$\beta = e^{ikx}$



r_i – real (negative

t_i – purely imaginary

At each mirror : $r_i^2 - t_i^2 = 1$

Losses: fractional loss L_i – in power.

At each contact with the mirror, the amplitude decreases : $a \to s_i a$

$\Rightarrow \quad L_i = 1 - s_i^2, \quad s_i^2 = 1 - L_i$

Power reflectance: $R_i = s_i^2 r_i^2$

Power transmittance: $T_i = -s_i^2 t_i^2$

Thus, energy conservation:

$\boxed{R_i + T_i + L_i = 1} \Rightarrow$ O.K.

At steady state:

$$\begin{cases} C = t_1 s_1 \beta E + r_1 s_1 A \\ B = t_1 s_1 A + r_1 s_1 \beta E \\ E = r_2 s_2 \beta B \\ D = t_2 s_2 \beta B \end{cases} \Rightarrow \begin{cases} C = t_1 r_2 s_1 s_2 \beta^2 B + r_1 s_1 A \\ B = t_1 s_1 A + r_1 r_2 s_1 s_2 \beta^2 B \\ D = t_2 s_2 \beta B \end{cases}$$

$$B = A \frac{t_1 s_1}{1 - r_1 r_2 s_1 s_2 \beta^2}$$

$$\frac{C}{A} = \frac{t_1^2 r_2 s_1^2 s_2 \beta^2}{1 - r_1 r_2 s_1 s_2 \beta^2} + r_1 s_1$$ Denote $\beta^2 = \alpha$

$$\frac{C}{A} = \frac{t_1^2 r_2 s_1^2 s_2 \alpha + r_1 s_1 - r_1^2 r_2 s_1^2 s_2 \alpha}{1 - r_1 r_2 s_1 s_2 \alpha}$$

$$\frac{C}{A} = \frac{r_1 s_1 + r_2 s_1^2 s_2 \alpha \overbrace{(t_1^2 - r_1^2)}^{=-1}}{1 - r_1 r_2 s_1 s_2 \alpha}$$

$$\boxed{\frac{C}{A} = s_1 \frac{r_1 - r_2 s_1 s_2 \alpha}{1 - r_1 r_2 s_1 s_2 \alpha}}$$

$$\boxed{\frac{D}{A} = s_1 \frac{t_1 t_2 s_2 \beta}{1 - r_1 r_2 s_1 s_2 \alpha}}$$

At resonance, $\alpha = 1$, $\beta = \pm 1$ (only resonance will be considered below)

$$\frac{C}{A} = s_1 \frac{r_1 - r_2 s_1 s_2}{1 - r_1 r_2 s_1 s_2}$$

$$\frac{D}{A} = \frac{t_1 t_2 s_1 s_2}{1 - r_1 r_2 s_1 s_2}$$ → or multiplied by i

AT RESONANCE:

Cavity reflection coefficient $\rho = \left|\frac{C}{A}\right|^2$:

$$\rho = \left[\frac{\sqrt{R_1} - (1-L_1)\sqrt{R_2}}{1 - \sqrt{R_1 R_2}}\right]^2$$

Cavity transmission coefficient $\theta = \left|\frac{D}{A}\right|^2$:

$$\theta = \frac{T_1 T_2}{(1 - \sqrt{R_1 R_2})^2}$$

9 September 87

Sensitivity of Reflected Field Phase to Change in Cavity Length

Reflected field amplitude:

(1) $$C = A s_1 \frac{r_1 - r_2 s_1 s_2 \alpha}{1 - r_1 r_2 s_1 s_2 \alpha}$$ where $\alpha = a + ib$, $\alpha = e^{2ikx}$

Phase of C relative to $s_1 A$:

(2) $$\phi = \arg(r_1 - r_2 s_1 s_2 \alpha) - \arg(1 - r_1 r_2 s_1 s_2 \alpha)$$

$$= -\operatorname{arctg} \frac{r_2 s_1 s_2 b}{r_1 - r_2 s_1 s_2 a} + \operatorname{arctg} \frac{r_1 r_2 s_1 s_2 b}{1 - r_1 r_2 s_1 s_2 a}$$

At resonance ($\alpha = 1$)

$$\frac{d\phi}{dx} = 2k r_2 s_1 s_2 \frac{1 - r_1^2}{(r_1 - r_2 s_1 s_2)(1 - r_1 r_2 s_1 s_2)}, \quad k = \frac{2\pi}{\lambda}$$

Remember that. $r_1 s_1 = \sqrt{R_1}$ $\quad r_2 s_2 = \sqrt{R_2}$

$$\frac{d\phi}{dx} = 2k s_1 \sqrt{R_2} \frac{1 - r_1^2}{(r_1 - s_1\sqrt{R_2})(1 - \sqrt{R_1 R_2})}$$

$$\frac{d\phi}{dx} = 2k \sqrt{R_2} \frac{s_1^2 - r_1^2 s_1^2}{(r_1 s_1 - s_1^2 \sqrt{R_2})(1 - \sqrt{R_1 R_2})}$$

(3) $$\frac{d\phi}{dx} = 2k \sqrt{R_2} \frac{1 - L_1 - R_1}{\left[\sqrt{R_1} - (1 - L_1)\sqrt{R_2}\right](1 - \sqrt{R_1 R_2})}$$

(3) $$\frac{d\phi}{dx} = 2k \sqrt{R_2} \frac{T_1}{\left[\sqrt{R_1} - (1 - L_1)\sqrt{R_2}\right]\left[1 - \sqrt{R_1 R_2}\right]}$$