New Folder Name Noise Characterization

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NOISE CHARACTERIZATION OF OSCILLATORS
FREQUENCY VARIANCE OF MARK I

1. Oscillator signal: $A(t)\cos[\Omega_0 t + \phi(t)]$

- $\phi(t)$: slowly varying function of time
- Frequency deviation:

$$\Omega_{inst} \equiv \Omega_0 + \Omega(t)$$

$$\Omega(t) = \dot{\phi}(t)$$

2. Definition of τ-average of function $f(t)$:

$$f_\tau(t) = \frac{1}{\tau}\int_0^\tau f(t'+t)\,dt'$$

3. Definition of time average of $f(t)$:

$$\langle f(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\,dt$$

4. The clock/oscillator problem:

a) frequency used for practical purposes is the τ-average of Ω_{inst}:

$$\Omega_{inst;\tau} = \Omega_0 + \overbrace{\frac{1}{\tau}[\phi(t+\tau)-\phi(t)]}^{\Omega_\tau(t)}$$

b) If $\phi(t)$ is statistical process, define - derive statistical properties of $\Omega_\tau(t)$

(5) Assuming $\phi(t)$ is a zero-average statistical process, i.e. $<\phi(t)> = 0$:

$\Rightarrow \quad <\Omega_\tau(t)> = 0$

(6) variance/standard deviation of $\Omega_\tau(t)$:

- variance $\sigma^2_{\Omega_\tau}$, or, simplified, σ^2_τ:

$$\sigma^2_\tau \equiv <[\Omega_\tau(t) - <\Omega_\tau(t)>]^2> = <\Omega^2_\tau(t)>$$

$$\sigma^2_\tau = \frac{1}{\tau^2}<\phi^2(t+\tau) - 2\phi(t+\tau)\phi(t) + \phi^2(t)> = \frac{2}{\tau^2}\left[<\phi^2(t)> - <\phi(t+\tau)\phi(t)>\right]$$

Because $<\phi^2(t+\tau)> = <\phi^2(t)>$ ← stationary process

By definition, $<F(t+\tau)F(t)> \equiv R_F(\tau)$ ← the autocorrelation function of $F(t)$.

Thus:

$$\sigma^2_\tau = \frac{2}{\tau^2}[R_\phi(0) - R_\phi(\tau)]$$

(7) Autocorrelation function is Fourier transform of power spectral density:

$$- R_\phi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_\phi(\omega)e^{i\omega\tau}d\omega = \frac{1}{\pi}\int_0^\infty S_\phi(\omega)\cos\omega\tau\,d\omega$$

ω is the rate of fluctuation of $\phi(t)$

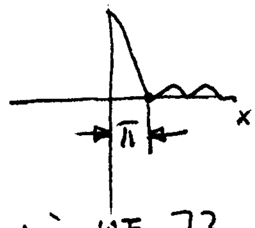
$$- S_\phi(\omega) = \frac{1}{\omega^2}S_{\dot\phi}(\omega) \equiv \frac{1}{\omega^2}S_\Omega(\omega)$$

(8). $$\sigma_\tau^2 = \frac{2}{\pi \tau^2} \int_0^\infty \frac{1}{\omega^2}\left[S_\Omega(\omega) - S_\Omega(\omega)\cos\omega\tau \right] d\omega$$

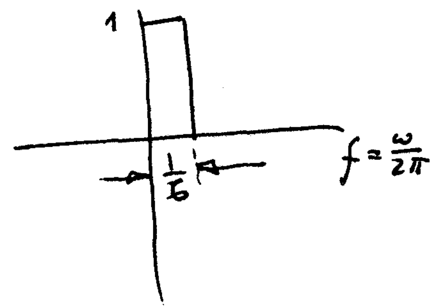
$$\sigma_\tau^2 = \frac{1}{\pi}\int_0^\infty S_\Omega(\omega)\left[\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right]^2 d\omega$$

(9) Approximate formula:

$$\frac{\sin^2 x}{x^2}$$



Approximate $\left[\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right]^2$ with:



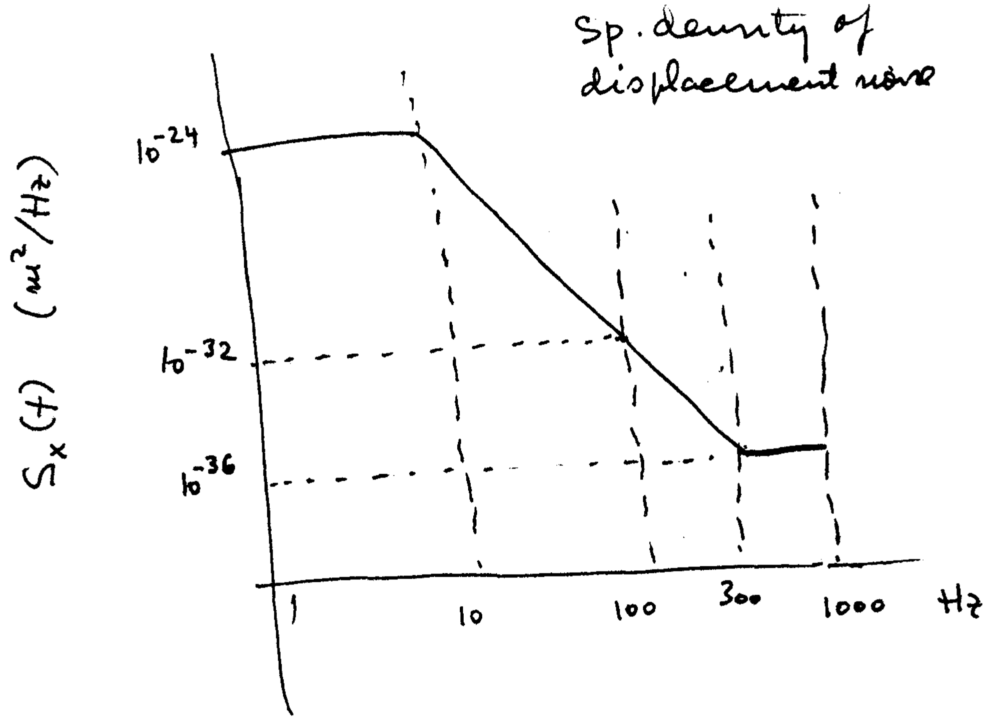
Therefore: $$\sigma_\tau^2 \cong \frac{1}{\pi}\int_0^{\frac{1}{\tau}} S_\Omega(2\pi f) \cdot 2\pi \, df$$

$$\sigma_\tau^2 \cong 2\int_0^{1/\tau} S_\Omega(2\pi f)\, df$$

Replacing $\sigma^2_{\tau,\Omega} \to \sigma^2_{\tau,\nu}$, $S_\Omega \to S_\nu$:

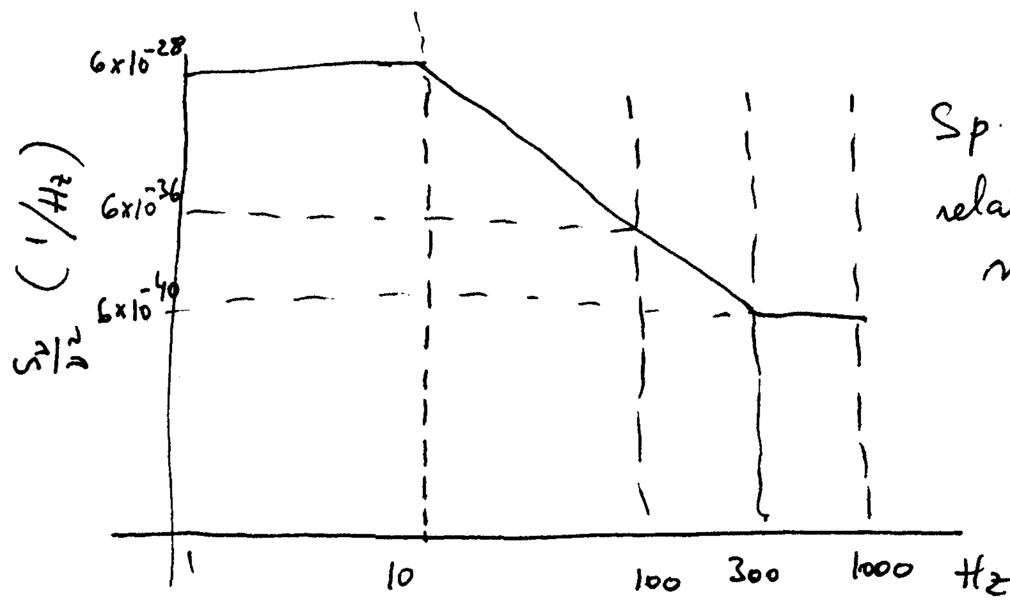
$$\sigma^2_{\tau,\nu} = 2\int_0^{1/\tau} S_\nu(f)\, df$$

40 m - MARK I



$$S_\nu(f) = \frac{\nu^2}{L^2} S_x(f) \, , \quad \frac{S_\nu(f)}{\nu^2} = \frac{S_x(f)}{L^2}$$

$L = 40\,m$,

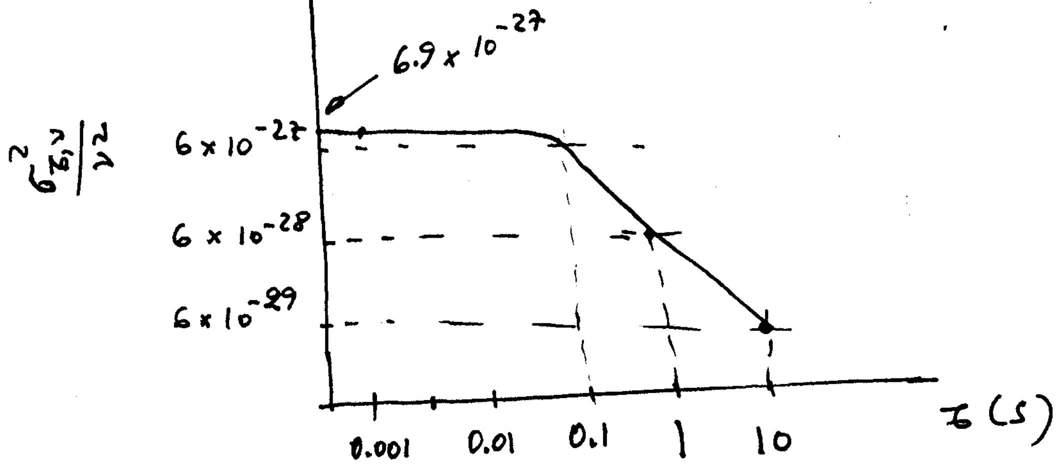


Sp. density of relative frequency noise.

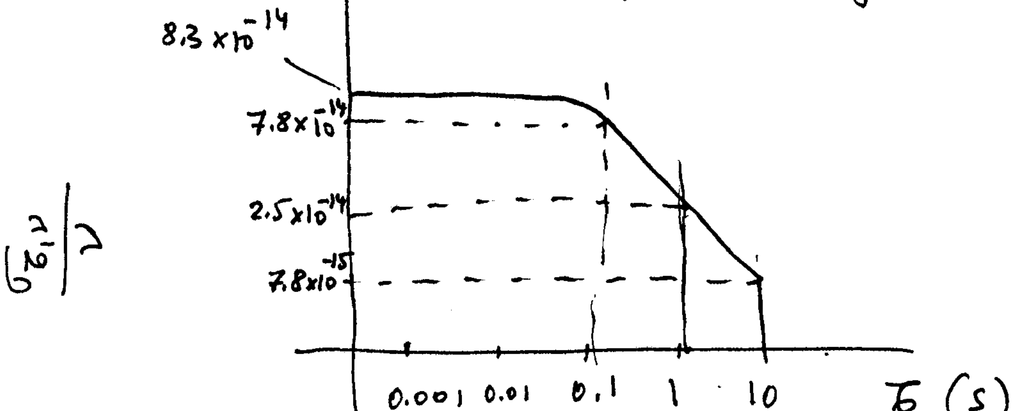
At 1kHz, in a band of 1kHz : $\left[\frac{\int_{\Delta f} S_\nu(f)\,df}{\nu^2} \right]^{1/2} = 7.5 \times 10^{-19}$

40m — MARK I

~ Relative variance of frequency, versus averaging time.



~ Relative standard deviation of frequency, versus averaging time.



2c

The oscillator problem:

- The frequency used for practical purposes is the instantaneous frequency, averaged over a time interval τ.

$\omega \quad \cancel{\frac{2\pi f \tau}{2}} = \cancel{\pi} \quad f = \frac{1}{\tau}$

10	6×10^{-29}
1	6×10^{-28}
0.1	6×10^{-27}
0.01	6.9×10^{-27}
0.003	. . . .
0.001	

$\frac{df}{f^8} = \frac{1}{f^7}$

$$\frac{6 \times 10^{-28}}{7} \left(\frac{10^8}{f^7} \right)$$

NOISE CHARACTERIZATION OF OSCILLATORS

1. Oscillator signal:

 - $A(t)\cos[\Omega_0 t + \phi(t)]$ $\qquad \Omega(t) = \dot{\phi}(t)$

 - $\phi(t)$: not a function of time, but a statistical fluctuation

 - τ-Average frequency noise:

$$\Omega = \qquad \Omega_\tau(t) = \frac{1}{\tau}[\phi(t+\tau) - \phi(t)]$$

with $\Omega_\infty(t) = 0 = \langle \Omega(t) \rangle$

Ω_τ is the average over the interval τ

2. Definition of time average of function $f(t)$:

$$\langle f(t) \rangle = \lim_{T\to\infty} \int_{-T/2}^{+T/2} f(t)\,dt$$

3. Definition of time-variance of $f(t)$ (called, simply, variance):

$$\sigma^2 \equiv \langle [f(t) - \langle f(t) \rangle]^2 \rangle$$

$$\sigma^2 = \langle [f^2(t) - 2f(t)\langle f(t)\rangle + \langle f(t)\rangle^2] \rangle =$$

$$= \langle f^2(t) \rangle - \langle f(t) \rangle^2$$

4. Standard deviation: is the square root of σ^2

5. Variance of frequency :

$$\sigma_\tau^2 \equiv \sigma_{\Omega,\tau}^2 = \langle \Omega_\tau^2 (t) \rangle - \langle \Omega_\tau (t) \rangle^2$$