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- LIGO -
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Derivation of Global and Local Coordinate Axes for the LIGO Sites

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Purpose

Memorandum LIGO-L950128 gives the operational definition of the global and local coordinates to be used at the two LIGO sites. Therein, the arm directions and geographical coordinates of the site vertices are specified. This note documents how the local coordinate systems are defined, relative to the global coordinates. For this analysis, the earth was assumed to be spherical, with R=6,378,164 meters and the LIGO arm lengths were taken to be 4000 meters, with midstations located 2018 meters from the vertex.

Using the measured, as leveled, elevation differences between the vertex and the end of each arm, the Euler angles representing the relative orientations of a set of global axes, tied to the interferometer plane, and a set of local axes tied to the "plumb" and local horizontal directions are obtained. The global axes correspond to an Euler rotation of the local axes at the vertex. Having obtained the global axes, it is then shown how to determine the relative orientation between the global system and other local systems for arbitrary points along the arms.

The Global Coordinate System

The sites were developed to create two orthogonal arms of length L=4000 m. The arms define the plane of the interferometer. The plane is defined by a *global coordinate system*, having its $\hat{x}_G$ and $\hat{y}_G$ axes along the centerlines of the two arms, and the $\hat{z}_G$ axis defined by the cross product: $\hat{z}_G = \hat{x}_G \times \hat{y}_G$. The arms are assigned to the axes $\hat{x}_G, \hat{y}_G$ so as to define a right-handed coordinate system.

The arms may be characterized by their orientations relative to the compass, and the elevation differences between the two ends of the arms. These elevation differences are different for each of the arms and sites. The elevation differences are denoted as $(\Delta z_X, \Delta z_Y)$. These quantities are specified in the LIGO System Specification, Document Number IGO-E950084.

The quantities $(\Delta z_X, \Delta z_Y)$ determine the angular orientation of the global coordinate system relative to a local system at the vertex. This local system is defined by $\hat{z}_L$=the local vertical; $\hat{x}_L$=the projection of the $\hat{x}_G$ interferometer arm onto the local horizontal; $\hat{y}_L$=the projection of the $\hat{y}_G$ interferometer arm onto the local horizontal.
The transformation between global and local coordinate systems at the vertex is specified by Euler angles. A general rotation is defined by the Euler angles $\{\lambda, \mu, \psi\}$, which correspond first to a rotation about $\hat{z}$ by ψ, then about $\hat{x}'$ by μ, and finally about $\hat{z}''$ by λ:

$$R[\lambda, \mu, \psi] = \begin{bmatrix} Cos[\lambda]Cos[\psi] - Sin[\lambda]Cos\mu Sin[\psi] & Sin[\lambda]Cos[\psi] + Cos[\lambda]Cos[\mu]Sin[\psi] & Sin[\mu]Sin[\psi] \\ -Sin[\lambda]Cos[\mu]Cos[\psi] - Cos[\lambda]Sin[\psi] & Cos[\lambda]Cos[\mu]Cos[\psi] - Sin[\lambda]Sin[\psi] & Sin[\mu]Cos[\psi] \\ Sin[\lambda]Sin[\mu] & -Cos[\lambda]Sin[\mu] & Cos[\mu] \end{bmatrix}$$

For LIGO two Euler angles are equal and opposite: $\psi = -\lambda$, for which R reduces to:

$$R[\lambda, \mu, -\lambda] = \begin{bmatrix} Cos^2[\lambda] + Sin^2[\lambda]Cos[\mu] & Sin[\lambda]Cos[\lambda] - Cos[\lambda]Sin[\lambda]Cos[\mu] & -Sin[\lambda]Sin[\mu] \\ Sin[\lambda]Cos[\lambda] - Cos[\mu]Cos[\lambda]Sin[\psi] & Cos^2[\lambda]Cos[\mu] + Sin^2[\lambda] & Cos[\lambda]Sin[\mu] \\ Sin[\lambda]Sin[\mu] & -Cos[\lambda]Sin[\mu] & Cos[\mu] \end{bmatrix}$$

The quantities $\varepsilon_{X,Y} = \Delta z_{X,Y} - \frac{L^2}{2R}$ are a measure of the amount of deviation between the local vertical and the global **Z** axis. Using these quantities, the Euler angles are given (to first order in the ε) by:

$$\lambda = \frac{\pi}{2} + \text{atan}\left[\frac{\varepsilon_Y}{\varepsilon_X}\right] ; \mu = \frac{\sqrt{\varepsilon_X^2 + \varepsilon_Y^2}}{L}$$

The local axes <u>at the vertex</u> are, by definition of the coordinate system, $\hat{z}_L$=(0,0,1)=local vertical; $\hat{x}_L$=(1,0,0)=local horizontal along one arm; $\hat{y}_L$=(0,1,0)=local horizontal along the other arm. The global axes are given by:

$$\hat{x}_G = R[\lambda, \mu, -\lambda] \cdot \hat{x}_L$$
$$\hat{y}_G = R[\lambda, \mu, -\lambda] \cdot \hat{y}_L$$
$$\hat{z}_G = R[\lambda, \mu, -\lambda] \cdot \hat{z}_L$$

<u>Local Coordinate Systems at Mid and End Stations</u>

The <u>local coordinate system</u> at an arbitrary point <u>***S***</u> an arm is related to the vectors $\{\hat{r}, \hat{\phi}, \hat{\theta}\}$ of the spherical coordinate system. Because of how the local axes are defined relative to the global axes, the definition of the local axes with respect to the two interferometer arms is different. The difference arises because it is desired to have the $\hat{y}_G$ and $\hat{y}_L$ be nearly aligned. This corresponds to a different mapping between the local coordinate vectors and spherical coordinate vectors. For the $\hat{x}_G$ arm, $\{\hat{x}_L, \hat{y}_L\} \leftrightarrow \{\hat{\theta}, \hat{\phi}\}$, whereas for the $\hat{y}_G$ arm, $\{\hat{x}_L, \hat{y}_L\} \leftrightarrow \{\widehat{-\phi}, \hat{\theta}\}$.

<u>Local axes along the $\hat{x}_G$ arm.</u>

The spherical coordinate system to which all axes are tied is defined as having the vertex at the pole, for which $\{r, \phi, \theta\} = \{R, 0, 0\}$. Along the $\hat{x}_G$ arm, the spherical angular variables, $\{\phi, \theta\}$, are related to the Euler angles as follows: $\{\lambda, \mu, \psi\} = \left\{\frac{\pi}{2} + \phi, \theta, -\frac{\pi}{2}\right\}$.

Thus, for the NW arm at Hanford and SW arm at Livingston, $\{r', \phi_x, \theta_x\}$ are obtained by solving for them at point <u>***S***$_X$</u>:

$$S_X \equiv (0, 0, R) + s\ \hat{x}_G = (r'Cos[\phi_x]Sin[\theta_x], r'Sin[\phi_x]Sin[\theta_x], r'Cos[\theta_x]) \quad .$$

Local axes along the $\hat{y}_G$ arm.

Along the $\hat{y}_G$ arm, the spherical angular variables, $\{\phi, \theta\}$, are related to the Euler angles as follows: $\{\lambda, \mu, \psi\} = \left\{\phi - \frac{\pi}{2}, -\theta, 0\right\}$.

For the SW arm at Hanford and SE arm at Livingston, $\{r', \phi_y, \theta_y\}$ are obtained by solving:

$$S_Y \equiv (0, 0, R) + s\ \hat{y}_G = (r'Cos[\phi_y]Sin[\theta_y], r'Sin[\phi_y]Sin[\theta_y], r'Cos[\theta_y]) \quad .$$

For the midstations, s=2018 m, and for the end stations, s=4000 m. Given $\{\lambda, \mu, \psi\}$ for either arm, the local axes at point $\underline{\boldsymbol{S}}$ are obtained by:

$$\hat{x}_L = R[\lambda, \mu, \psi] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{y}_L = R[\lambda, \mu, \psi] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{z}_L = R[\lambda, \mu, \psi] \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Direction Cosines Between Global and Local Coordinate Axes

Once the local and global axes are obtained for the various points at both sites, a useful way to represent them is by means of the direction cosines which the local axes make with the global axes. These are given below as sets of 3 x 3 matrices. Although for completeness all cosines are provided, only those of order of magnitude 10^{-4} are required in practice.

HANFORD SITE COORDINATE SYSTEMS

Vertex (L=0 meters along arms)

$$\begin{bmatrix} \hat{x}_G \cdot \hat{x}_L & \hat{x}_G \cdot \hat{y}_L & \hat{x}_G \cdot \hat{z}_L \\ \hat{y}_G \cdot \hat{x}_L & \hat{y}_G \cdot \hat{y}_L & \hat{y}_G \cdot \hat{z}_L \\ \hat{z}_G \cdot \hat{x}_L & \hat{z}_G \cdot \hat{y}_L & \hat{z}_G \cdot \hat{z}_L \end{bmatrix} = \begin{bmatrix} 1 - 1.91\times10^{-7} & -2.65\times10^{-9} & -6.19\times10^{-4} \\ -2.65\times10^{-9} & 1 - 3.67\times10^{-11} & -8.57\times10^{-6} \\ 6.19\times10^{-4} & 8.57\times10^{-6} & 1 - 1.91\times10^{-7} \end{bmatrix}$$

Midstation, NW Arm (L=2018 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-4.57\times10^{-8} & 0 & -3.02\times10^{-4} \\ -2.59\times10^{-9} & 1-3.67\times10^{-11} & -8.57\times10^{-6} \\ 3.02\times10^{-4} & 8.57\times10^{-6} & 1-4.57\times10^{-8} \end{bmatrix}$$

Midstation, SW Arm (L=2018 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-1.91\times10^{-7} & 1.90\times10^{-7} & -6.19\times10^{-4} \\ 0 & 1-4.74\times10^{-8} & 3.08\times10^{-4} \\ 6.19\times10^{-4} & -3.08\times10^{-4} & 1-2.39\times10^{-7} \end{bmatrix}$$

End Station, NW Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-3.67\times10^{-11} & 0 & 8.57\times10^{-6} \\ 0 & 1-3.67\times10^{-11} & -8.57\times10^{-6} \\ -8.57\times10^{-6} & 8.57\times10^{-6} & 1-7.34\times10^{-11} \end{bmatrix}$$

End Station, SW Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-1.91\times10^{-7} & 3.83\times10^{-7} & -6.19\times10^{-4} \\ 0 & 1-1.91\times10^{-7} & 6.19\times10^{-4} \\ 6.19\times10^{-4} & -6.19\times10^{-4} & 1-3.83\times10^{-7} \end{bmatrix}$$

LIVINGSTON SITE COORDINATE SYSTEMS

Vertex (L=0 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-4.92\times10^{-8} & -9.70\times10^{-8} & -3.14\times10^{-4} \\ -9.70\times10^{-8} & 1-1.91\times10^{-7} & -6.19\times10^{-4} \\ 3.14\times10^{-4} & 6.19\times10^{-4} & 1-2.40\times10^{-7} \end{bmatrix}$$

End Station, SW Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-4.92\times10^{-8} & 0 & 3.14\times10^{-4} \\ 1.94\times10^{-7} & 1-1.91\times10^{-7} & -6.19\times10^{-4} \\ -3.14\times10^{-4} & 6.19\times10^{-4} & 1-2.40\times10^{-7} \end{bmatrix}$$

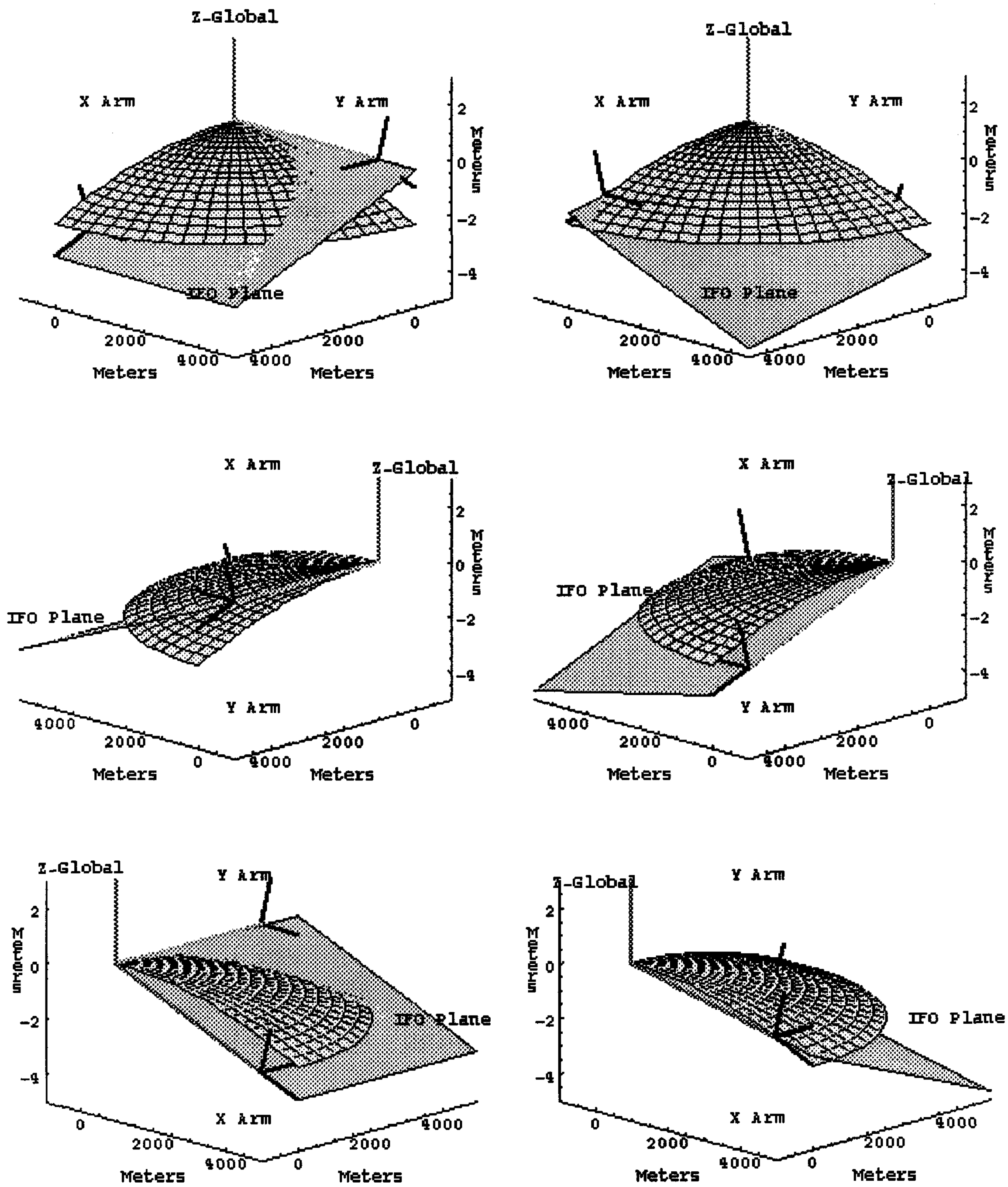
End Station, SE Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G \cdot \hat{x}_L & \hat{x}_G \cdot \hat{y}_L & \hat{x}_G \cdot \hat{z}_L \\ \hat{y}_G \cdot \hat{x}_L & \hat{y}_G \cdot \hat{y}_L & \hat{y}_G \cdot \hat{z}_L \\ \hat{z}_G \cdot \hat{x}_L & \hat{z}_G \cdot \hat{y}_L & \hat{z}_G \cdot \hat{z}_L \end{bmatrix} = \begin{bmatrix} 1 - 4.92\times10^{-8} & 2.69\times10^{-9} & -3.14\times10^{-4} \\ 0 & 1 - 3.67\times10^{-8} & 8.57\times10^{-6} \\ 3.14\times10^{-4} & -8.57\times10^{-6} & 1 - 4.92\times10^{-8} \end{bmatrix}$$

Graphical Representations

Figures 1a-f present graphical representations of the orientations of the interferometer planes at the two sites relative to a surface of constant elevation (referred to the vertex) from various points of view. A spherical earth is assumed here (deviations from geoid or ellipsoid do not affect results at the level of precision required).

Figure 1: Representation of the IFO plane inclinations at the two LIGO sites.



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The quantities $(\Delta z_X, \Delta z_Y)$ determine the angular orientation of the global coordinate system relative to a local system at the vertex. This local system is defined by $\hat{z}_L$=the local vertical; $\hat{x}_L$=the projection of the $\hat{x}_G$ interferometer arm onto the local horizontal; $\hat{y}_L$=the projection of the $\hat{y}_G$ interferometer arm onto the local horizontal.
The transformation between global and local coordinate systems at the vertex is specified by Euler angles. A general rotation is defined by the Euler angles $\{\lambda, \mu, \psi\}$, which correspond first to a rotation about $\hat{z}$ by ψ, then about $\hat{x}'$ by μ, and finally about $\hat{z}''$ by λ:

$$R[\lambda, \mu, \psi] = \begin{bmatrix} Cos[\lambda]Cos[\psi] - Sin[\lambda]Cos\mu Sin[\psi] & Sin[\lambda]Cos[\psi] + Cos[\lambda]Cos[\mu]Sin[\psi] & Sin[\mu]Sin[\psi] \\ -Sin[\lambda]Cos[\mu]Cos[\psi] - Cos[\lambda]Sin[\psi] & Cos[\lambda]Cos[\mu]Cos[\psi] - Sin[\lambda]Sin[\psi] & Sin[\mu]Cos[\psi] \\ Sin[\lambda]Sin[\mu] & -Cos[\lambda]Sin[\mu] & Cos[\mu] \end{bmatrix}$$

For LIGO two Euler angles are equal and opposite: $\psi = -\lambda$, for which R reduces to:

$$R[\lambda, \mu, -\lambda] = \begin{bmatrix} Cos^2[\lambda] + Sin^2[\lambda]Cos[\mu] & Sin[\lambda]Cos[\lambda] - Cos[\lambda]Sin[\lambda]Cos[\mu] & -Sin[\lambda]Sin[\mu] \\ Sin[\lambda]Cos[\lambda] - Cos[\mu]Cos[\lambda]Sin[\psi] & Cos^2[\lambda]Cos[\mu] + Sin^2[\lambda] & Cos[\lambda]Sin[\mu] \\ Sin[\lambda]Sin[\mu] & -Cos[\lambda]Sin[\mu] & Cos[\mu] \end{bmatrix}$$

The quantities $\varepsilon_{X,Y} = \Delta z_{X,Y} - \frac{L^2}{2R}$ are a measure of the amount of deviation between the local vertical and the global Z axis. Using these quantities, the Euler angles are given (to first order in the ε) by:

$$\lambda = \frac{\pi}{2} + \mathrm{atan}\left[\frac{\varepsilon_Y}{\varepsilon_X}\right] ; \mu = \frac{\sqrt{\varepsilon_X^2 + \varepsilon_Y^2}}{L}$$

The local axes <u>at the vertex</u> are, by definition of the coordinate system, $\hat{z}_L$=(0,0,1)=local vertical; $\hat{x}_L$=(1,0,0)=local horizontal along one arm; $\hat{y}_L$=(0,1,0)=local horizontal along the other arm. The global axes are given by:

$$\hat{x}_G = R[\lambda, \mu, -\lambda] \cdot \hat{x}_L$$
$$\hat{y}_G = R[\lambda, \mu, -\lambda] \cdot \hat{y}_L$$
$$\hat{z}_G = R[\lambda, \mu, -\lambda] \cdot \hat{z}_L$$

Local Coordinate Systems at Mid and End Stations

The <u>local coordinate system</u> at an arbitrary point <u>***S***</u> an arm is related to the vectors $\{\hat{r}, \hat{\phi}, \hat{\theta}\}$ of the spherical coordinate system. Because of how the local axes are defined relative to the global axes, the definition of the local axes with respect to the two interferometer arms is different. The difference arises because it is desired to have the $\hat{y}_G$ and $\hat{y}_L$ be nearly aligned. This corresponds to a different mapping between the local coordinate vectors and spherical coordinate vectors. For the $\hat{x}_G$ arm, $\{\hat{x}_L, \hat{y}_L\} \leftrightarrow \{\hat{\theta}, \hat{\phi}\}$, whereas for the $\hat{y}_G$ arm, $\{\hat{x}_L, \hat{y}_L\} \leftrightarrow \{\widehat{-\phi}, \hat{\theta}\}$.

<u>Local axes along the $\hat{x}_G$ arm.</u>

The spherical coordinate system to which all axes are tied is defined as having the vertex at the pole, for which $\{r, \phi, \theta\} = \{R, 0, 0\}$. Along the $\hat{x}_G$ arm, the spherical angular variables, $\{\phi, \theta\}$, are related to the Euler angles as follows: $\{\lambda, \mu, \psi\} = \left\{\frac{\pi}{2} + \phi, \theta, -\frac{\pi}{2}\right\}$.

Thus, for the NW arm at Hanford and SW arm at Livingston, $\{r', \phi_x, \theta_x\}$ are obtained by solving for them at point <u>***S***</u>$_X$:

$$S_X \equiv (0, 0, R) + s\,\hat{x}_G = (r' Cos[\phi_x]Sin[\theta_x], r' Sin[\phi_x]Sin[\theta_x], r' Cos[\theta_x]) \quad .$$

Local axes along the $\hat{y}_G$ arm.

Along the $\hat{y}_G$ arm, the spherical angular variables, $\{\phi, \theta\}$, are related to the Euler angles as follows: $\{\lambda, \mu, \psi\} = \left\{\phi - \frac{\pi}{2}, -\theta, 0\right\}$.

For the SW arm at Hanford and SE arm at Livingston, $\{r', \phi_y, \theta_y\}$ are obtained by solving:

$$S_Y \equiv (0, 0, R) + s\ \hat{y}_G = (r'Cos[\phi_y]Sin[\theta_y], r'Sin[\phi_y]Sin[\theta_y], r'Cos[\theta_y])\quad .$$

For the midstations, s=2018 m, and for the end stations, s=4000 m. Given $\{\lambda, \mu, \psi\}$ for either arm, the local axes at point ***S*** are obtained by:

$$\hat{x}_L = R[\lambda, \mu, \psi] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{y}_L = R[\lambda, \mu, \psi] \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{z}_L = R[\lambda, \mu, \psi] \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Direction Cosines Between Global and Local Coordinate Axes

Once the local and global axes are obtained for the various points at both sites, a useful way to represent them is by means of the direction cosines which the local axes make with the global axes. These are given below as sets of 3 x 3 matrices. Although for completeness all cosines are provided, only those of order of magnitude 10^{-4} are required in practice.

HANFORD SITE COORDINATE SYSTEMS

Vertex (L=0 meters along arms)

$$\begin{bmatrix} \hat{x}_G \cdot \hat{x}_L & \hat{x}_G \cdot \hat{y}_L & \hat{x}_G \cdot \hat{z}_L \\ \hat{y}_G \cdot \hat{x}_L & \hat{y}_G \cdot \hat{y}_L & \hat{y}_G \cdot \hat{z}_L \\ \hat{z}_G \cdot \hat{x}_L & \hat{z}_G \cdot \hat{y}_L & \hat{z}_G \cdot \hat{z}_L \end{bmatrix} = \begin{bmatrix} 1 - 1.91\times10^{-7} & -2.65\times10^{-9} & -6.19\times10^{-4} \\ -2.65\times10^{-9} & 1 - 3.67\times10^{-11} & -8.57\times10^{-6} \\ 6.19\times10^{-4} & 8.57\times10^{-6} & 1 - 1.91\times10^{-7} \end{bmatrix}$$

Midstation, NW Arm (L=2018 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-4.57\times10^{-8} & 0 & -3.02\times10^{-4} \\ -2.59\times10^{-9} & 1-3.67\times10^{-11} & -8.57\times10^{-6} \\ 3.02\times10^{-4} & 8.57\times10^{-6} & 1-4.57\times10^{-8} \end{bmatrix}$$

Midstation, SW Arm (L=2018 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-1.91\times10^{-7} & 1.90\times10^{-7} & -6.19\times10^{-4} \\ 0 & 1-4.74\times10^{-8} & 3.08\times10^{-4} \\ 6.19\times10^{-4} & -3.08\times10^{-4} & 1-2.39\times10^{-7} \end{bmatrix}$$

End Station, NW Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-3.67\times10^{-11} & 0 & 8.57\times10^{-6} \\ 0 & 1-3.67\times10^{-11} & -8.57\times10^{-6} \\ -8.57\times10^{-6} & 8.57\times10^{-6} & 1-7.34\times10^{-11} \end{bmatrix}$$

End Station, SW Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-1.91\times10^{-7} & 3.83\times10^{-7} & -6.19\times10^{-4} \\ 0 & 1-1.91\times10^{-7} & 6.19\times10^{-4} \\ 6.19\times10^{-4} & -6.19\times10^{-4} & 1-3.83\times10^{-7} \end{bmatrix}$$

LIVINGSTON SITE COORDINATE SYSTEMS

Vertex (L=0 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-4.92\times10^{-8} & -9.70\times10^{-8} & -3.14\times10^{-4} \\ -9.70\times10^{-8} & 1-1.91\times10^{-7} & -6.19\times10^{-4} \\ 3.14\times10^{-4} & 6.19\times10^{-4} & 1-2.40\times10^{-7} \end{bmatrix}$$

End Station, SW Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G\cdot\hat{x}_L & \hat{x}_G\cdot\hat{y}_L & \hat{x}_G\cdot\hat{z}_L \\ \hat{y}_G\cdot\hat{x}_L & \hat{y}_G\cdot\hat{y}_L & \hat{y}_G\cdot\hat{z}_L \\ \hat{z}_G\cdot\hat{x}_L & \hat{z}_G\cdot\hat{y}_L & \hat{z}_G\cdot\hat{z}_L \end{bmatrix} = \begin{bmatrix} 1-4.92\times10^{-8} & 0 & 3.14\times10^{-4} \\ 1.94\times10^{-7} & 1-1.91\times10^{-7} & -6.19\times10^{-4} \\ -3.14\times10^{-4} & 6.19\times10^{-4} & 1-2.40\times10^{-7} \end{bmatrix}$$

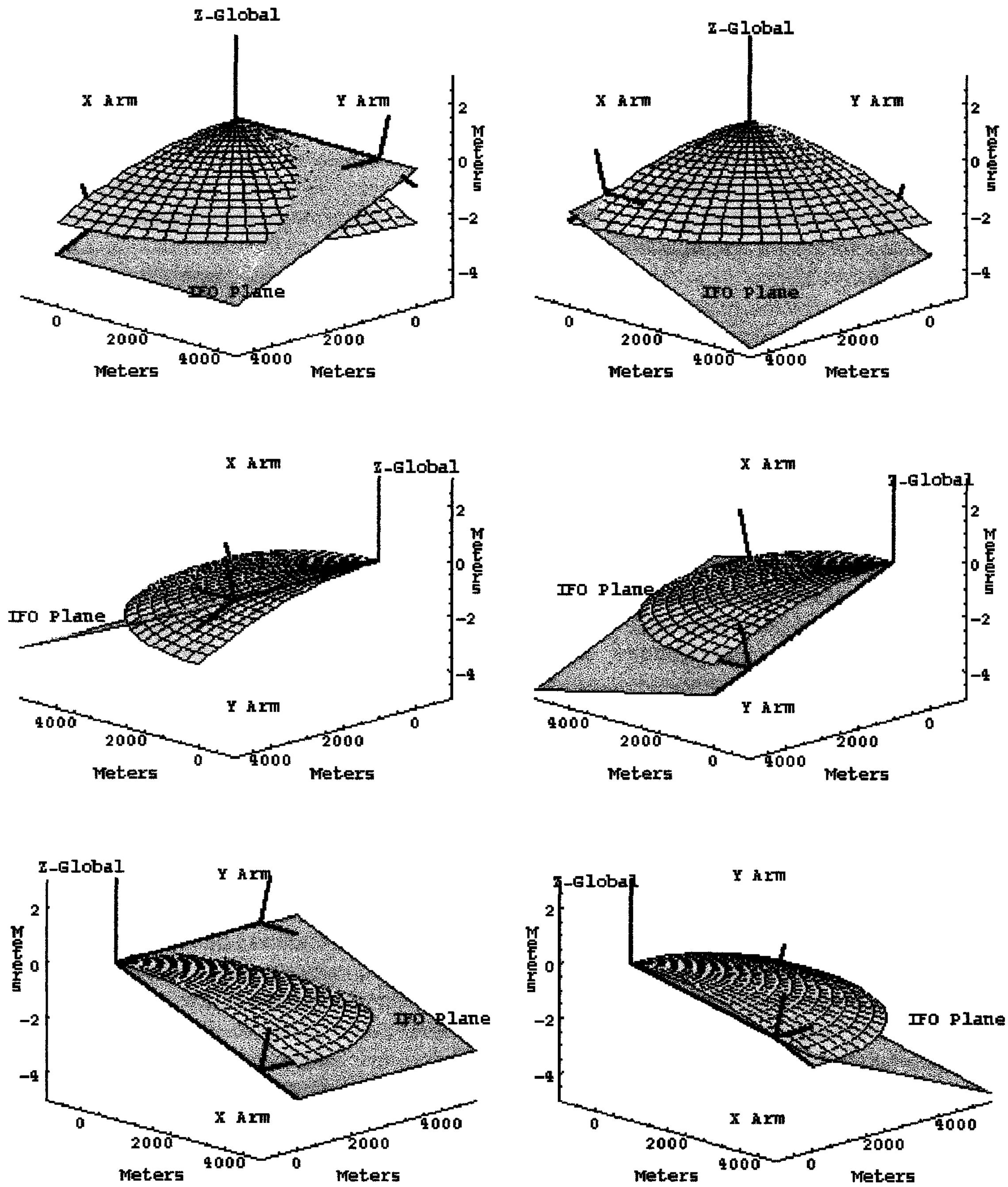
End Station, SE Arm (L=4000 meters along arms)

$$\begin{bmatrix} \hat{x}_G \cdot \hat{x}_L & \hat{x}_G \cdot \hat{y}_L & \hat{x}_G \cdot \hat{z}_L \\ \hat{y}_G \cdot \hat{x}_L & \hat{y}_G \cdot \hat{y}_L & \hat{y}_G \cdot \hat{z}_L \\ \hat{z}_G \cdot \hat{x}_L & \hat{z}_G \cdot \hat{y}_L & \hat{z}_G \cdot \hat{z}_L \end{bmatrix} = \begin{bmatrix} 1 - 4.92\times10^{-8} & 2.69\times10^{-9} & -3.14\times10^{-4} \\ 0 & 1 - 3.67\times10^{-8} & 8.57\times10^{-6} \\ 3.14\times10^{-4} & -8.57\times10^{-6} & 1 - 4.92\times10^{-8} \end{bmatrix}$$

Graphical Representations

Figures 1a-f present graphical representations of the orientations of the interferometer planes at the two sites relative to a surface of constant elevation (referred to the vertex) from various points of view. A spherical earth is assumed here (deviations from geoid or ellipsoid do not affect results at the level of precision required).

Figure 1: Representation of the IFO plane inclinations at the two LIGO sites.



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Derivation of Global and Local Coordinate Axes for the LIGO Sties

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Purpose

Memorandum LIGO-L950128 gives the operational definition of the global and local coordinates to be used at the two LIGO sites. Therein, the arm directions and geographical coordinates of the site vertices are specified. This note documents how the local coordinate systems are defined, relative to the global coordinates. For this analysis, the earth was assumed to be spherical, with R=6,378,164 meters and the LIGO arm lengths were taken to be 4000 meters, with midstations located 2018 meters from the vertex.

Using the measured, as leveled, elevation differences between the vertex and the end of each arm, the Euler angles representing the relative orientations of a set of global axes, tied to the interferometer plane, and a set of local axes tied to the "plumb" and local horizontal directions are obtained. The global axes correspond to an Euler rotation of the local axes at the vertex. Having obtained the global axes, it is then shown how to determine the relative orientation between the global system and other local systems for arbitrary points along the arms.

The Global Coordinate System

The sites were developed to create two orthogonal arms of length L=4000 m. The arms define the plane of the interferometer. The plane is defined by a *global coordinate system*, having its $\hat{x}_G$ and $\hat{y}_G$ axes along the centerlines of the two arms, and the $\hat{z}_G$ axis defined by the cross product: $\hat{z}_G = \hat{x}_G \times \hat{y}_G$. The arms are assigned to the axes $\hat{x}_G, \hat{y}_G$ so as to define a right-handed coordinate system.

The arms may be characterized by their orientations relative to the compass, and the elevation differences between the two ends of the arms. These elevation differences are different for each of the arms and sites. The elevation differences are denoted as $(\Delta z_X, \Delta z_Y)$. These quantities are specified in the LIGO System Specification, Document Number TBD.

The quantities $(\Delta z_X, \Delta z_Y)$ determine the angular orientation of the global coordinate system relative to a local system at the vertex. This local system is defined by $\hat{z}_L$=the local vertical; $\hat{x}_L$=the projection of the $\hat{x}_G$ interferometer arm onto the local horizontal; $\hat{y}_L$=the projection of the $\hat{y}_G$ interferometer arm onto the local horizontal.
The transformation between global and local coordinate systems at the vertex is specified by Euler angles. A general rotation is defined by the Euler angles $\{\lambda, \mu, \psi\}$, which correspond first to a rotation about $\hat{z}$ by ψ, then about $\hat{x}'$ by μ, and finally about $\hat{z}''$ by λ:

$$R[\lambda,\mu,\psi] = \begin{bmatrix} Cos[\lambda]Cos[\psi] - Sin[\lambda]Cos\mu Sin[\psi] & Sin[\lambda]Cos[\psi] + Cos[\lambda]Cos[\mu]Sin[\psi] & Sin[\mu]Sin[\psi] \\ -Sin[\lambda]Cos[\mu]Cos[\psi] - Cos[\lambda]Sin[\psi] & Cos[\lambda]Cos[\mu]Cos[\psi] - Sin[\lambda]Sin[\psi] & Sin[\mu]Cos[\psi] \\ Sin[\lambda]Sin[\mu] & -Cos[\lambda]Sin[\mu] & Cos[\mu] \end{bmatrix}$$

For LIGO two Euler angles are equal and opposite: $\psi = -\lambda$, for which R reduces to:

$$R[\lambda,\mu,-\lambda] = \begin{bmatrix} Cos^2[\lambda]+Sin^2[\lambda]Cos[\mu] & Sin[\lambda]Cos[\lambda]-Cos[\lambda]Sin[\lambda]Cos[\mu] & -Sin[\lambda]Sin[\mu] \\ Sin[\lambda]Cos[\lambda]-Cos[\mu]Cos[\lambda]Sin[\psi] & Cos^2[\lambda]Cos[\mu]+Sin^2[\lambda] & Cos[\lambda]Sin[\mu] \\ Sin[\lambda]Sin[\mu] & -Cos[\lambda]Sin[\mu] & Cos[\mu] \end{bmatrix}$$

The quantities $\varepsilon_{X,Y} = \Delta z_{X,Y} - \frac{L^2}{2R}$ are a measure of the amount of deviation between the local vertical and the global Z axis. Using these quantities, the Euler angles are given (to first order in the ε) by:

$$\lambda = \frac{\pi}{2} + \mathrm{atan}\left[\frac{\varepsilon_Y}{\varepsilon_X}\right]; \mu = \frac{\sqrt{\varepsilon_X^2+\varepsilon_Y^2}}{L}$$

The local axes <u>at the vertex</u> are, by definition of the coordinate system, $\hat{z}_L$=(0,0,1)=local vertical; $\hat{x}_L$=(1,0,0)=local horizontal along one arm; $\hat{y}_L$=(0,1,0)=local horizontal along the other arm. The global axes are given by:

$$\hat{x}_G = R[\lambda,\mu,-\lambda]\cdot\hat{x}_L$$
$$\hat{y}_G = R[\lambda,\mu,-\lambda]\cdot\hat{y}_L$$
$$\hat{z}_G = R[\lambda,\mu,-\lambda]\cdot\hat{z}_L$$

<u>Local Coordinate Systems at Mid and End Stations</u>

The <u>local coordinate system</u> at an arbitrary point <u>**S**</u> an arm is related to the vectors $\{\hat{r},\hat{\phi},\hat{\theta}\}$ of the spherical coordinate system. Because of how the local axes are defined relative to the global axes, the definition of the local axes with respect to the two interferometer arms is different. The difference arises because it is desired to have the $\hat{y}_G$ and $\hat{y}_L$ be nearly aligned. This corresponds to a different mapping between the local coordinate vectors and spherical coordinate vectors. For the $\hat{x}_G$ arm, $\{\hat{x}_L,\hat{y}_L\} \leftrightarrow \{\hat{\theta},\hat{\phi}\}$, whereas for the $\hat{y}_G$ arm, $\{\hat{x}_L,\hat{y}_L\} \leftrightarrow \{-\hat{\phi},\hat{\theta}\}$.

<u>Local axes along the $\hat{x}_G$ arm.</u>

The spherical coordinate system to which all axes are tied is defined as having the vertex at the pole, for which $\{r,\phi,\theta\} = \{R,0,0\}$. Along the $\hat{x}_G$ arm, the spherical angular variables, $\{\phi,\theta\}$, are related to the Euler angles as follows: $\{\lambda,\mu,\psi\} = \{\frac{\pi}{2}+\phi,\theta,-\frac{\pi}{2}\}$.

Thus, for the NW arm at Hanford and SW arm at Livingston, $\{r',\phi_x,\theta_x\}$ are obtained by solving for them at point <u>S_X</u>:

$$S_X \equiv (0,0,R) + s\,\hat{x}_G = (r'Cos[\phi_x]Sin[\theta_x], r'Sin[\phi_x]Sin[\theta_x], r'Cos[\theta_x]).$$