LIGO Laser Interferometer Gravitational-Wave Observatory

Introduction to Fabry-Perot Cavities and 40m Power-Recycling Interferometer

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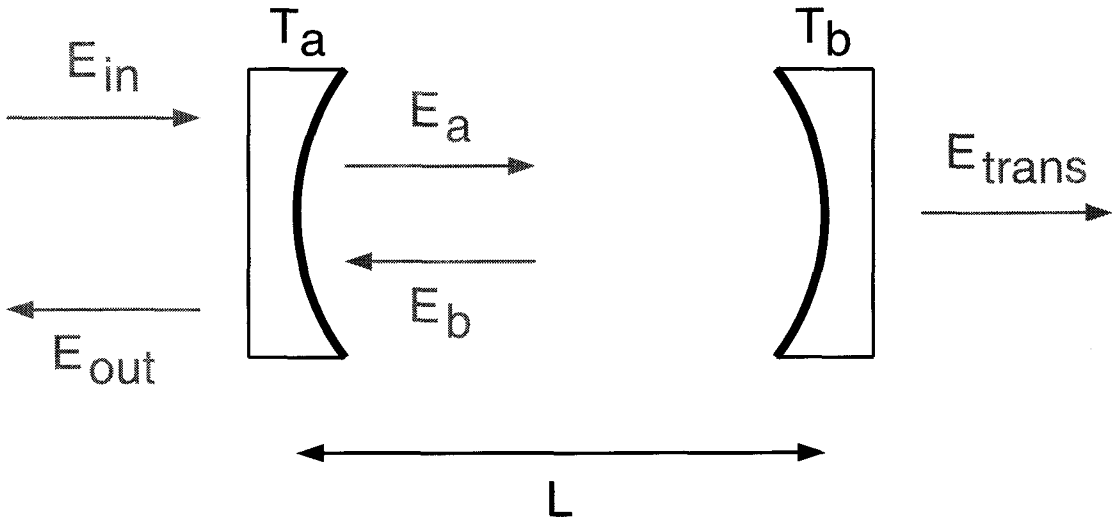
Overview

- The physics of Fabry Perot cavities
  - ›› setting the scene for recycling
- Recycling at the 40m: Optical Design and Choice of Parameters
  - ›› coupled cavities
  - ››choice of optical parameters
- Displacement Calibration and Shot Noise

The Physics of Fabry Perot Cavities

- definitions of standard parameters
- cavity DC response
- phase modulation
- reflection locking

Fields in Fabry Perot Cavities

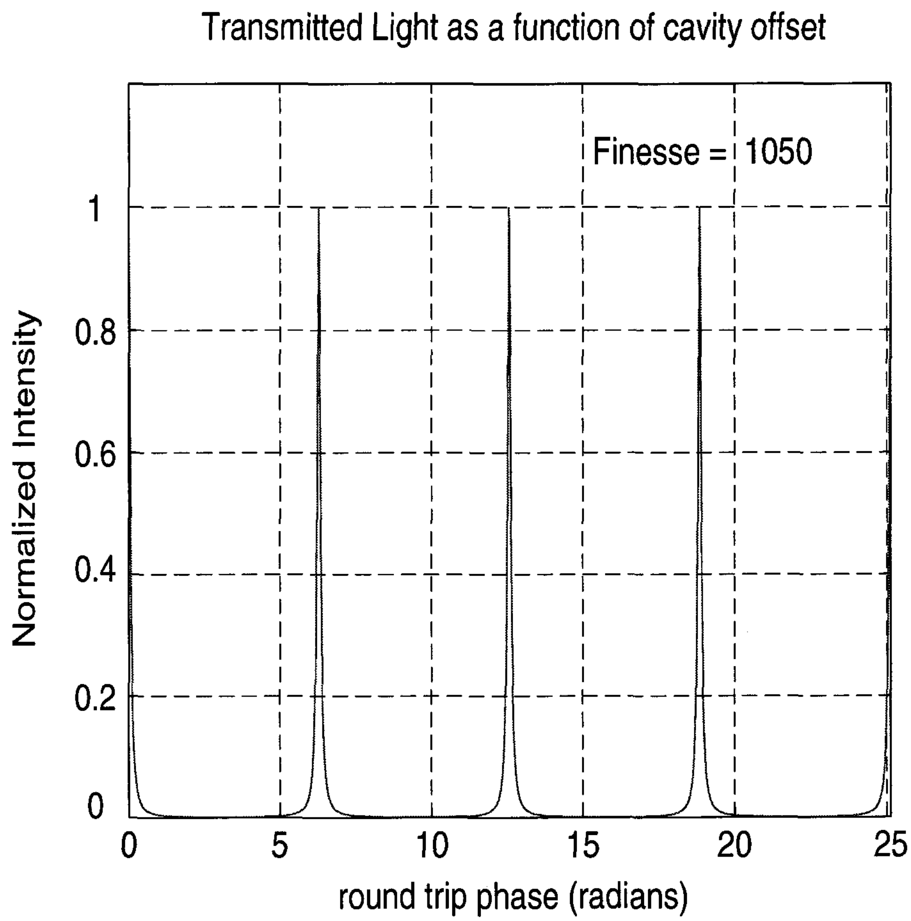


The transmitted light power

$$\frac{P_{trans}}{P_{in}} = \frac{T_a T_b}{(1 - r_a r_b)^2 + 4 r_a r_b \sin^2(\phi/2)}$$

Light in the cavity experiences a round trip phase of ϕ. On RESONANCE

$$\phi = 2n\pi \text{ or equivalently } f = \frac{c}{2L}n$$



The transmission peaks are spaced in frequency by

$$\frac{c}{2L} = \text{Free Spectral Range of the Cavity}$$

The cavity finesse is defined to be

$$\text{Finesse} = \frac{\text{Separation of Adjacent Transmission Peaks}}{\text{Full width of peak at half maximum}} = \frac{\pi\sqrt{r_a r_b}}{(1 - r_a r_b)}$$

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Another useful quantity is the Co-efficient of Finesse

$$F = \frac{4r_a r_b}{(1 - r_a r_b)^2} = \frac{4(\text{Finesse})^2}{\pi^2}$$

Some typical numbers:

Arm cavity	FSR (MHz)	Finesse	δf (Hz)
40m recombined	3.9	12,770	305
40m recycling	3.9	1050	3700
LIGO	0.04	206	180

Arm cavity	Ta	Tb	L
40m recombined	280 ppm	12 ppm	100 ppm
40m recycled	5750 ppm	12 ppm	100 ppm
LIGO	0.03	10 ppm	30 ppm

Cavity Reflectivity

$$\frac{E_{out}}{E_{in}} = \frac{r_a - r_b(1 - L_a)e^{i\phi}}{1 - r_a r_b e^{i\phi}}$$

-> the cavity looks like a mirror of variable reflectivity. At resonance

$$r_{cavity} = \left(\frac{E_{out}}{E_{in}}\right)_{\phi = 0} = \frac{r_a - r_b(1 - L_a)}{1 - r_a r_b}$$

For small deviations from resonance

$$C_o = \left|\frac{\partial}{\partial\phi}\left(\frac{E_{out}}{E_{in}}\right)\Big|_{\phi = 0}\right| = \frac{T_a r_b}{(1 - r_a r_b)^2}$$

This is the DC cavity response. Typical numbers:

Arm cavity	r_{cavity}	C_o
40m recombined	-0.138	4626
40m recycled	-0.929	645
LIGO arm cavity	-0.995	130

Coupling of Cavity

$$\left(\frac{E_{out}}{E_{in}}\right)_{\phi = 0} = \frac{r_a - r_b(1 - L_a)}{1 - r_a r_b}$$

Optimal Coupling occurs when $E_{out} = 0$. For small transmission and losses this occurs when

$$T_a = T_b + L_a + L_b$$

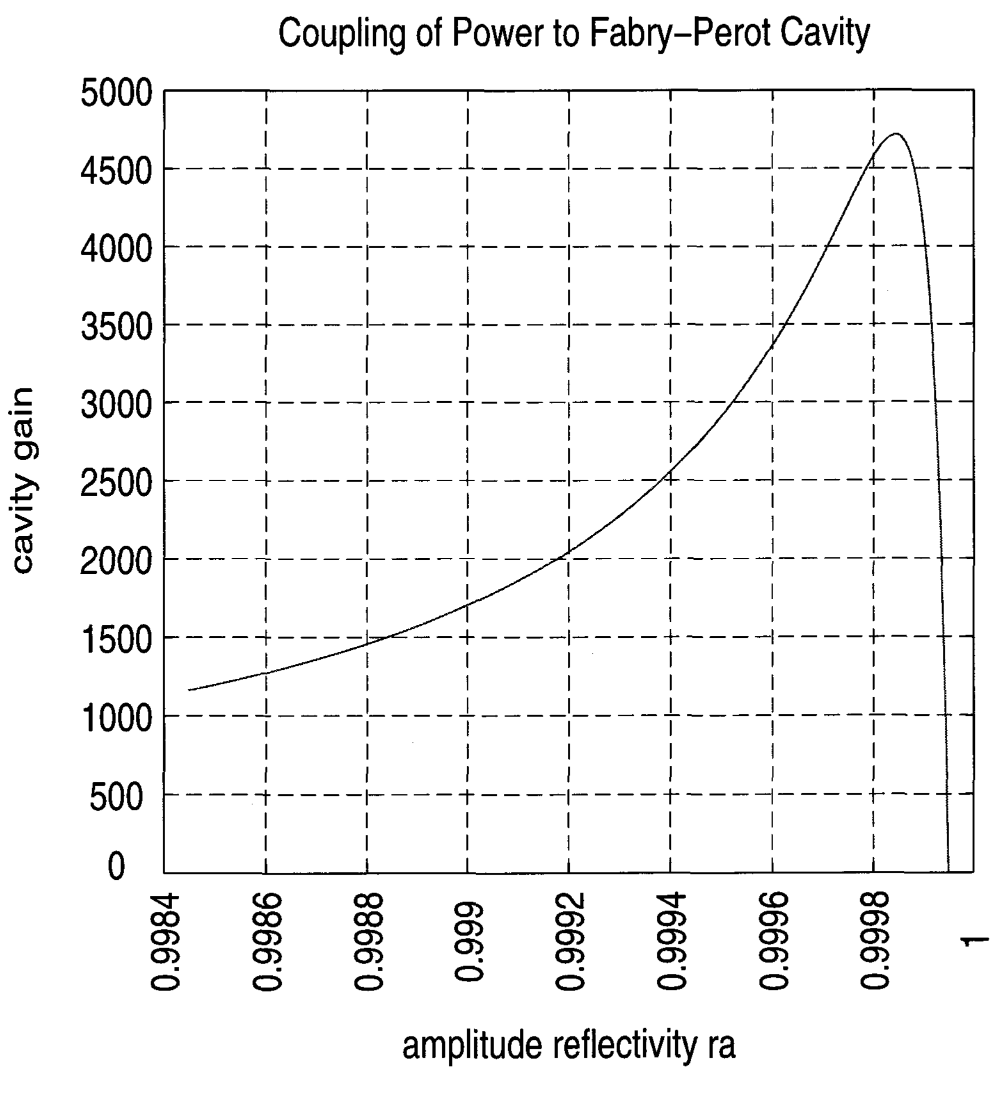
If $T_a < T_b + L_a + L_b$ then E_{out} is +ve i.e. in phase with incident light - cavity is undercoupled.

If $T_a > T_b + L_a + L_b$ then E_{out} is -ve i.e. out of phase with incident light - cavity is overcoupled.

Cavity Gain

On resonance the optical gain of the cavity is:

$$G = \left|\frac{E_a}{E_{in}}\right|^2 = \frac{T_a}{(1 - r_a r_b)^2}$$



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Optimal coupling => maximum cavity gain

$$G_{opt} \approx \frac{1}{T_b + L_a + L_b}$$

cavity response for small changes in ϕ

$$C_o = G r_b$$

∴ for maximum cavity response want to be (close to) optimally coupled

Cavity Visibility

Cavity visibility

$$V = 1 - \frac{P_{min}}{P_{max}}$$

where P_{min} and P_{max} is the in lock and out of lock power respectively reflected by the arm.

Note that since in general only a fraction M of the input laser light, P_o, is modematched to the cavity

$$P_{min} = R_{cavity} M P_o + (1 - M) R_a P_o$$

$$P_{max} = R_a P_o$$

∴ for low modulation depth (see later)

$$V = M\left(1 - \frac{R_{cavity}}{R_a}\right)$$

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For 40m, M ~ 0.9,

Arm cavity	R_{cavity}	Ta	V
40m recombined	0.019	280 ppm	0.88
40m recycled	0.86	5750 ppm	0.12

Phase Modulation and Signal for Locking Fabry Perot Cavity

Simple Example

Phase Modulation Sidebands

Phase Modulation Parameters

Bessel Functions

Sideband Initial Phases

Sideband Propagation

Power in the Sidebands

Intensity on the Photodiode

Photodiode Responsivity

Photodiode Quantum Efficiency

Photodiode Output

Demodulation

Signal

Signal's Maximum

Optimal Modulation

Simple Example

Frequencies: ω_0 , $\omega_0 + \Omega$

Field incident on F-P cavity

$$E = E_0 \; e^{i\omega_0 t} \; (1 + a \; e^{i\Omega t})$$

Field after the reflection

$$E = E_0 \; e^{i\omega_0 t} \; (\; \rho(x) + a \; e^{i\Omega t} \;)$$

Intensity at the photodiode

$$|E|^2 = E_0^2 \; (\; 2a \; \mathrm{Im}\{\rho(x)\} \; \sin\Omega t \ldots)$$

Demodulation picks up 1Ω-component:

$$V \sim \mathrm{Im}\{\rho(x)\}.$$

Can lock on the sideband: less signal, wrong sign

Phase Modulation Sidebands

Frequencies: $\omega_0 + \Omega$, ω_0, $\omega_0 - \Omega$

Field after Pockel Cell

$$E = E_0 \, e^{i\omega_0 t} \, e^{i\Gamma \sin \Omega t}.$$

Expansion

$$1 + i\Gamma \sin \Omega t \;\; = \;\; 1 \; + \; \frac{\Gamma}{2} \, e^{i\Omega t} \;\; - \;\; \frac{\Gamma}{2} \, e^{-i\Omega t}$$

Same result

$$V \sim \mathrm{Im}\{\rho(x)\}.$$

Phase Modulation Parameters

Modulation frequency

$$\Omega = 2\pi f_{mod}$$

Modulation wavelength

$$\lambda_{mod} = \frac{c}{f_{mod}}.$$

40m and LIGO Parameters

	f_{mod}(MHz)	λ_{mod}(m)
Recombined 40m	12.3	24.4
Recycled 40m	32.7	9.18
LIGO	24.5	12.2

Bessel Functions

Fourier expansion

$$e^{i\Gamma \sin \Omega t} = \sum_{n=-\infty}^{\infty} J_n(\Gamma) e^{in\Omega t}$$

where $J_n(\Gamma)$ are Bessel functions

Γ - Modulation Index

There is infinite number of sidebands

Frequencies

$$\omega_n = \omega_0 + n\,\Omega$$

The amplitudes

$$E_n = J_n(\Gamma)\,E$$

Sideband Initial Phases

Bessel functions with negative index flip sign

$$J_{-n}(\Gamma) = (-1)^n \; J_n(\Gamma)$$

General rule:

Upper sidebands are all positive

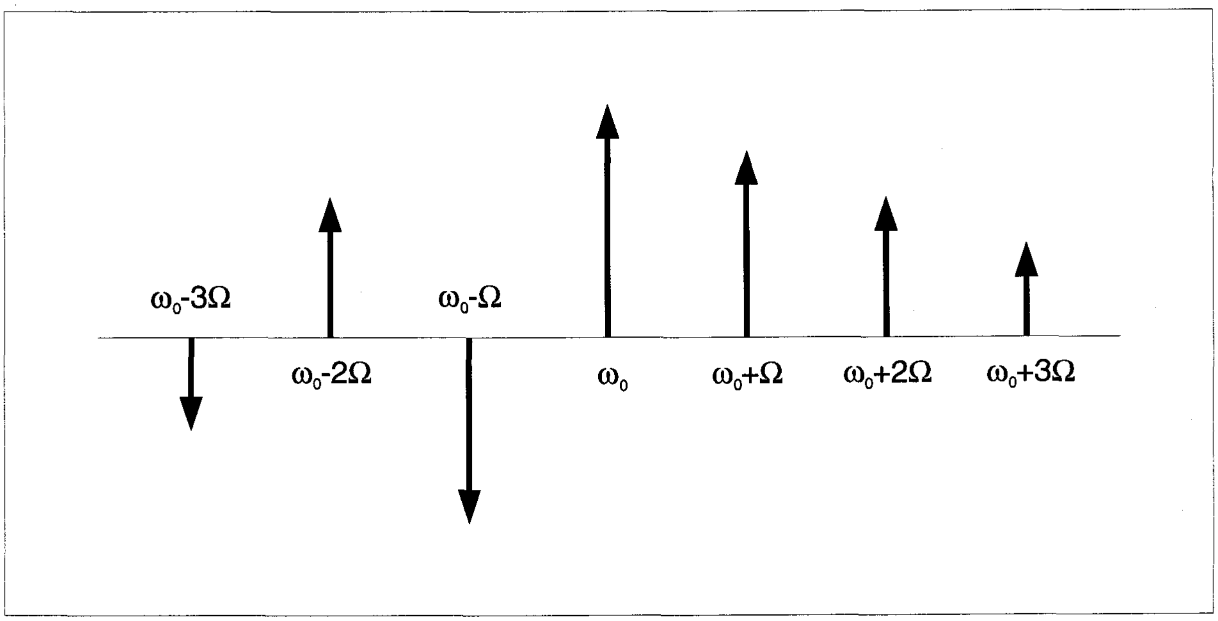
Lower even number sidebands are positive

Lower odd number sidebands are negative

Example:

$$\begin{aligned} E_{-1} &= -E_{+1}, \\ E_{-2} &= +E_{+2} \end{aligned}$$

Figure 1: Sideband Initial Phases



Sideband Propagation

Picture 1

Sidebands have different frequencies.

For propagation in positive z-direction

$$E_0 = E\ J_0\ e^{i\omega_0(t-\frac{z}{c})},$$
$$E_n = E\ J_n\ e^{i\omega_n(t-\frac{z}{c})}$$

Picture 2

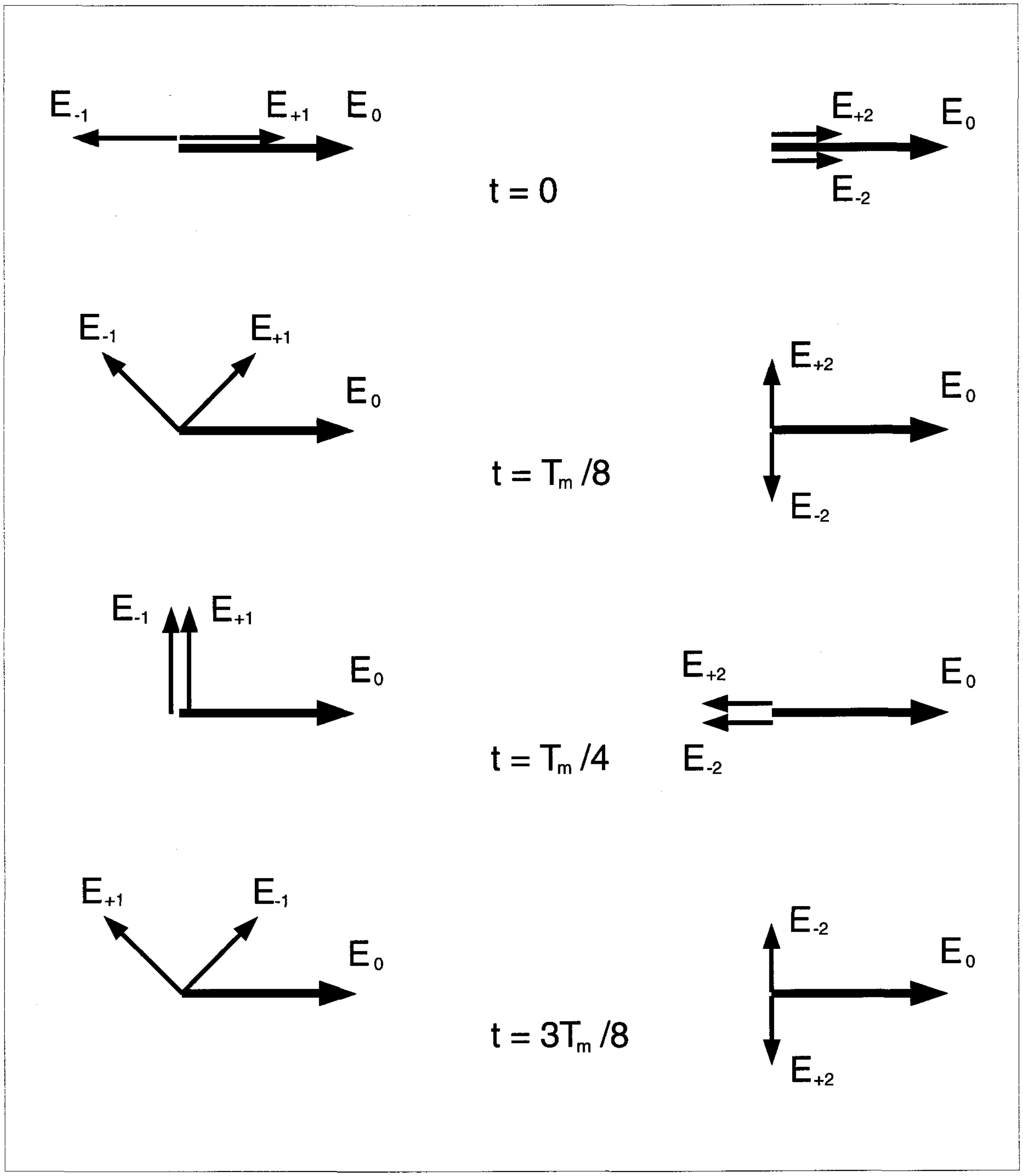
Sidebands have the same frequency as the carrier, but their phases ***slowly*** change with time

$$E_n = E\ J_n\ e^{i\omega_0(t-\frac{z}{c})} e^{iU_n}$$

$U_n(z,t)$ is the phase of the nth-sideband relative to carrier

$$U_n = n\ \Omega\ (t - \frac{z}{c}).$$

Figure 1: Phasor Diagram for 1st and 2nd Order Sidebands*



Power in the Sidebands

Modulation index, Γ, can be found from measurements of sideband power.
The power in nth-sideband

$$P_n = P\, J_n^2(\Gamma)$$

The measurements are done using optical spectrum analyzer.
Resolving sidebands is necessary for accurate measurements of contrast defect.

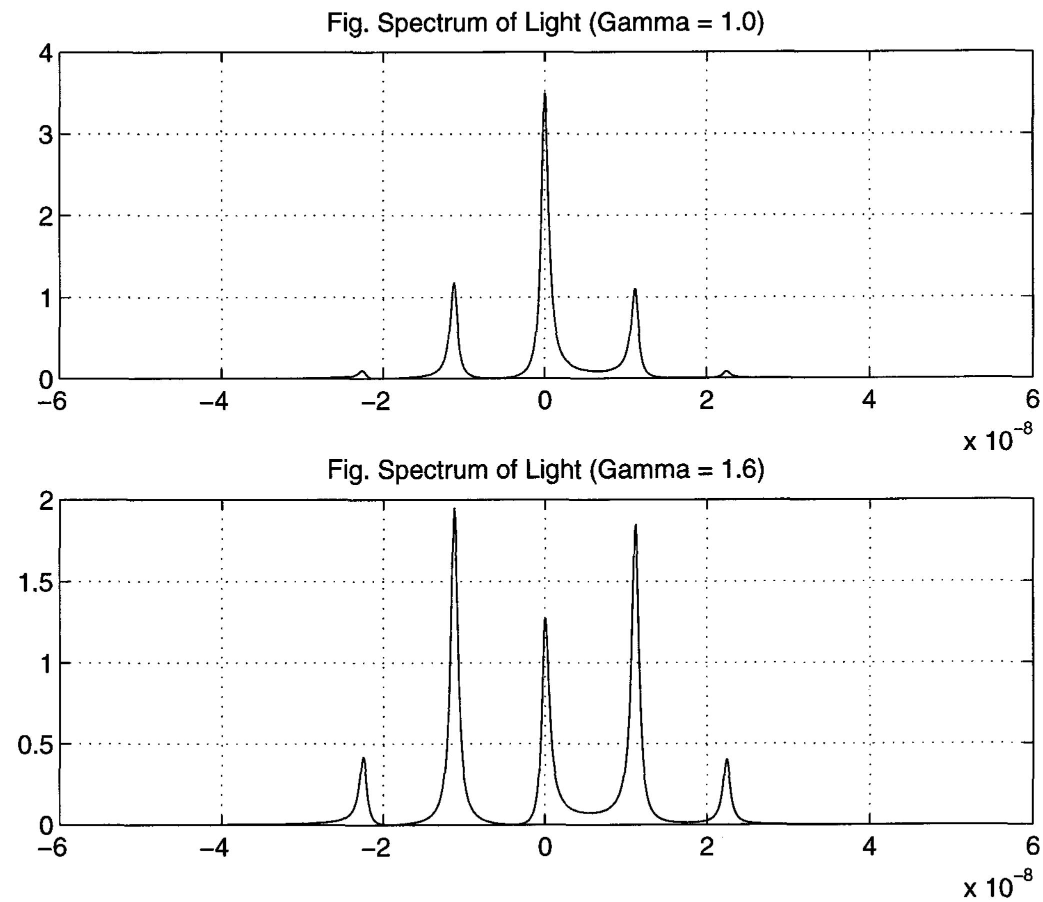


Fig. Spectrum of Light (Gamma = 1.0)
0
1
2
3
4
−6
−4
−2
0
2
4
6
x 10^{-8}
Fig. Spectrum of Light (Gamma = 1.6)
0
0.5
1
1.5
2
−6
−4
−2
0
2
4
6
x 10^{-8}

Intensity at the Photodiode

If cavity is on resonance intensity is constant in the first order.
If cavity deviates from resonance the intensity is modulated

$$P = A_0 + 2A_1 \cos \Omega t + 2B_1 \sin \Omega t.$$

Either A_1 or B_1 is proportional to deviation of cavity from resonance.
The coefficients are

$$\begin{aligned} A_0 &= |E_0|^2 + |E_{+1}|^2 + |E_{-1}|^2, \\ A_1 &= \mathrm{Re}\left\{E_0^*(E_{+1} + E_{-1})\right\}, \\ B_1 &= \mathrm{Im}\left\{E_0^*(E_{+1} - E_{-1})\right\}. \end{aligned}$$

Photodiode Responsivity

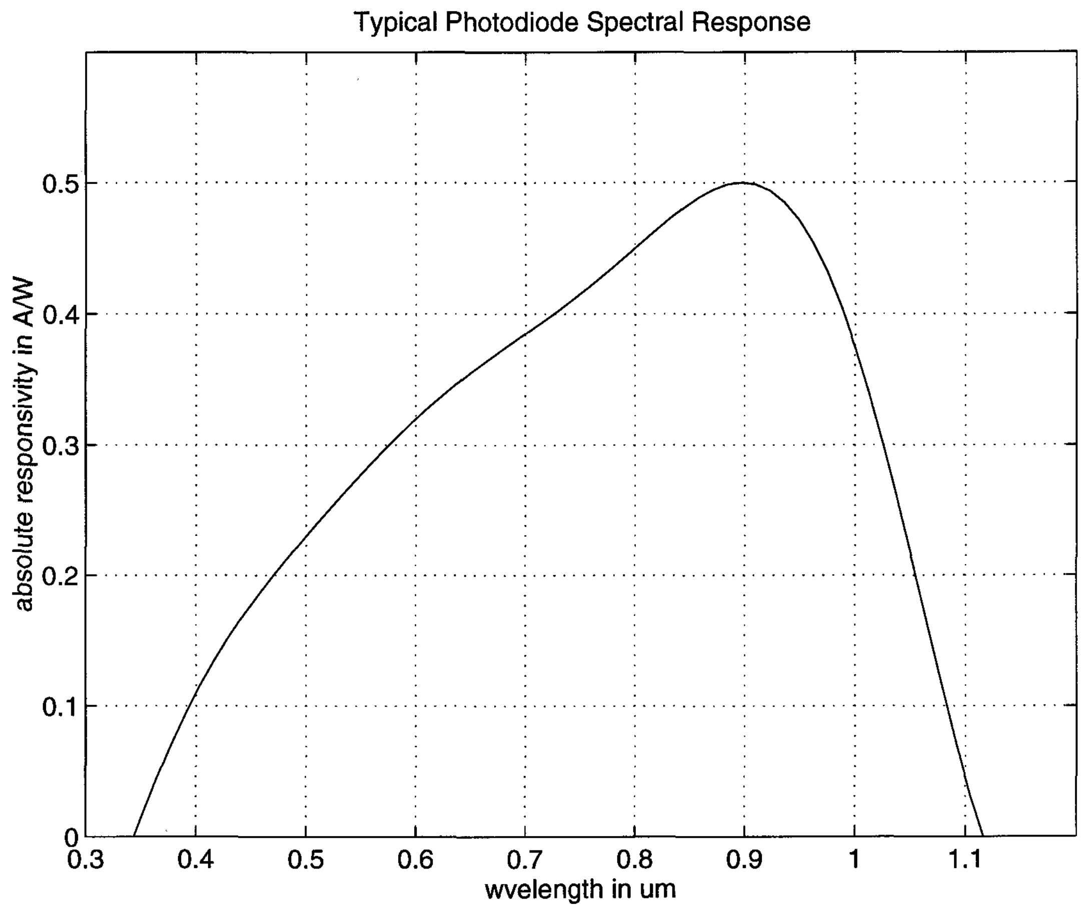
At the photodiode the intensity of light is converted to the photocurrent

$$I(t) = \sigma P(t)$$

σ is the responsivity the photodiode

For the green light $\lambda = 514.5$ nm the responsivity is

$$\sigma = 0.25 \ \ \mathrm{A/W}$$


Typical Photodiode Spectral Response
absolute responsivity in A/W
0
0.1
0.2
0.3
0.4
0.5
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
wvelength in um

Photodiode Quantum Efficiency

Each photon carries energy $h\nu$.

Intensity of light = total energy carried by photons in 1 sec

$$P = h\nu N_{ph}$$

Photocurrent = total charge carried across the photodiode in 1 sec

$$I = eN_e$$

Quantum efficiency of the photodiode $\kappa < 1$

$$N_e = \kappa N_{ph}$$

Photodiode responsivity

$$\sigma = \frac{\kappa e}{h\nu}$$

The choice above corresponds to $\kappa = 0.6$

Photodiode Output

Photodiode has resonant circuit

$$\tilde{Z}(\omega) = \frac{R}{1 + i\omega RC(1 - \frac{\omega_r^2}{\omega^2})}$$

The resonance frequency $\omega_r = 1/\sqrt{LC}$ is tuned to coinside with the modulation frequency Ω.

$C = 70$ pF (under bias),

L is tuned inductance,

$R \approx 600$ Ohm

Photodiode output

$$V_{pd}(t) = R\ I(t)$$

Demodulation

Demodulation signal $D(t)$ is a square wave.

β adjustable phase

Approximation:

leave only the first term of the Fourier series

$$D(t) = \frac{4}{\pi} \sin(\Omega t + \beta).$$

Coefficient $\frac{4}{\pi}$ is important for comparison with calibration.

Mixer output

$$V = V_{pd}(t)\ D(t)$$

Signal

In stationary approximation

$$V(x) = V_0 \, \frac{\sin 2kx}{1 + F \sin^2 kx}$$

40m recycling arm cavity $F = 4.48 \times 10^5$
The magnitude of the signal is defined by the constant

$$V_0 = \frac{4}{\pi} R \sigma P M J_0 J_1 C_0$$

$$2 \; k \; V_0 = 160 \;\; \mathrm{V/nm}$$

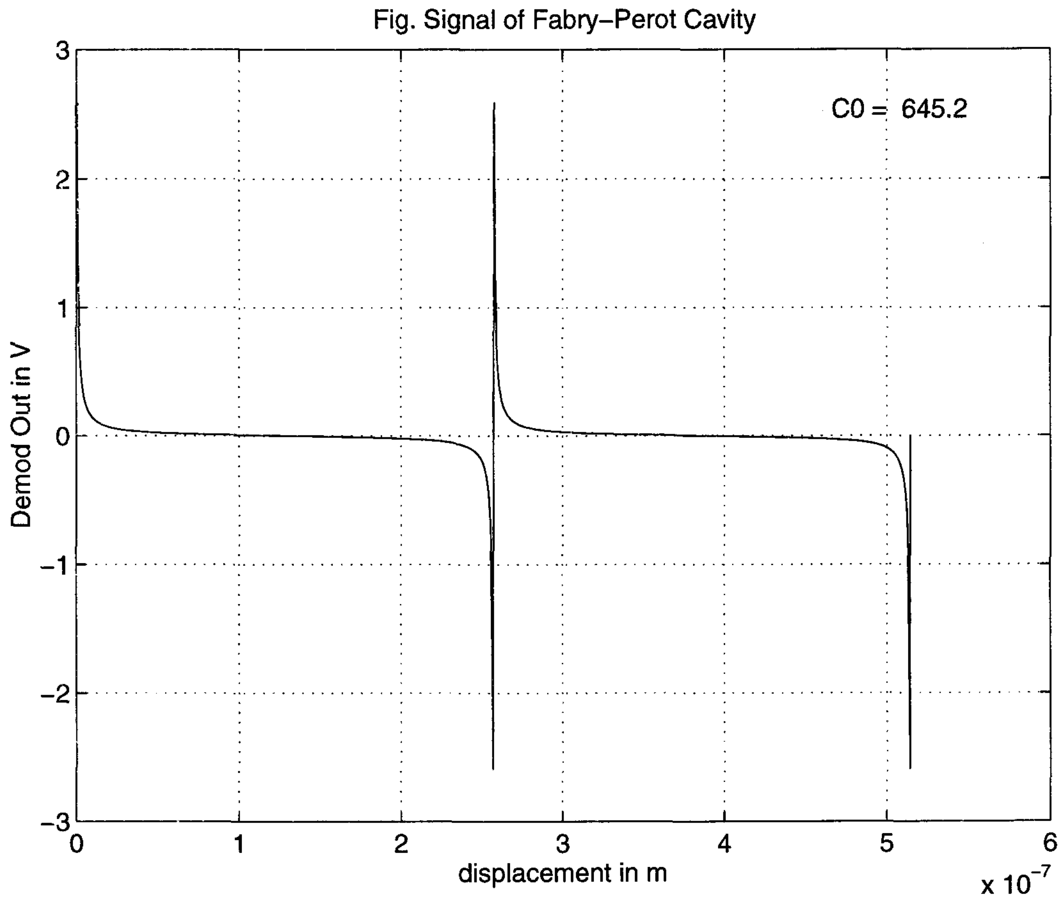


Fig. Signal of Fabry–Perot Cavity
C0 = 645.2
Demod Out in V
displacement in m
x 10^-7
3
2
1
0
−1
−2
−3
0
1
2
3
4
5
6

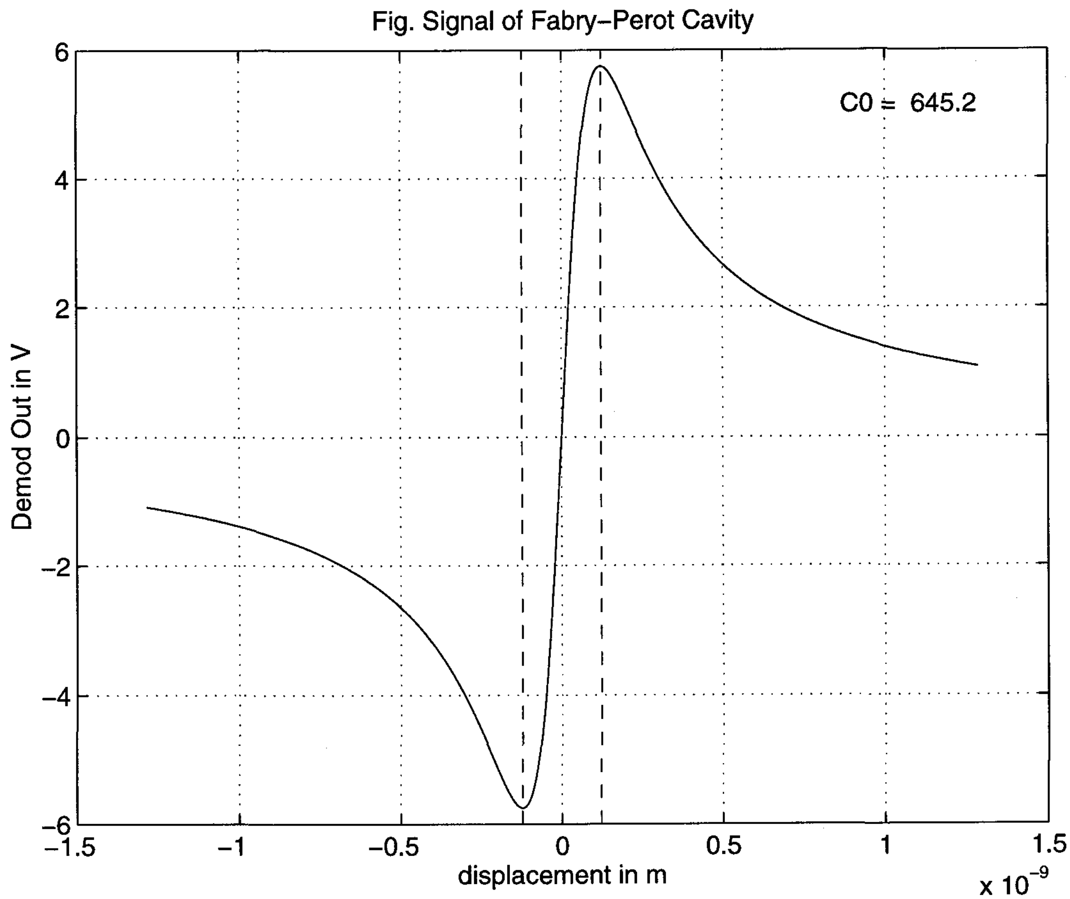


Fig. Signal of Fabry–Perot Cavity
C0 = 645.2
Demod Out in V
displacement in m
x 10^-9
6
4
2
0
−2
−4
−6
−1.5
−1
−0.5
0
0.5
1
1.5

Maximum of Signal

Fine tuning

Signal as a function of modulation index

$$V_0 \sim J_0(\Gamma)\ J_1(\Gamma)$$

Maximum of the function is at

$$\Gamma_1 = 1.08$$

It corresponds to

$$\frac{J_1^2}{J_0^2} = 0.43$$

Beyond this point there is no increase in the signal

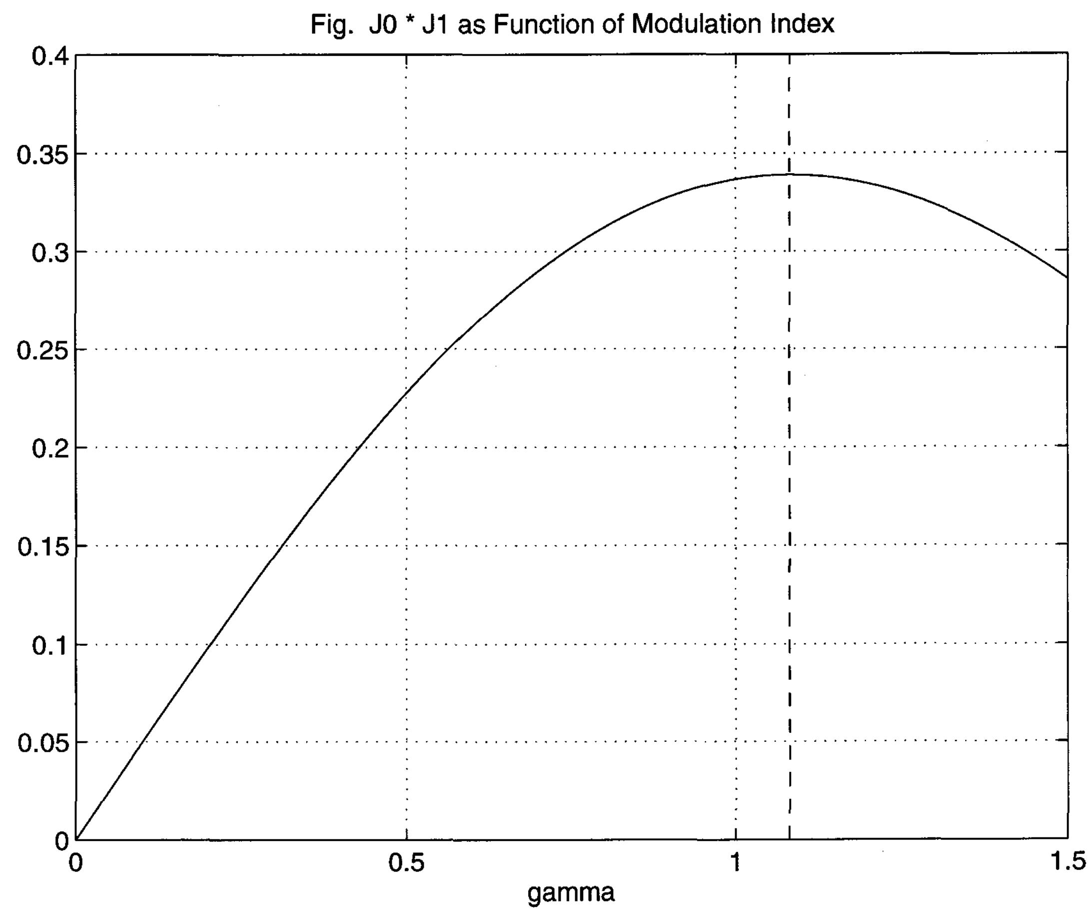


Fig. J0 * J1 as Function of Modulation Index
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
gamma

Optimal Modulation

If we are shot noise limited then the important signal to noise ratio is

$$K(\Gamma) = \frac{J_0 \; J_1}{\sqrt{\rho^2 J_0^2 + 2 J_1^2}}$$

For single Fabry-Perot cavity ρ^2 is a square of static cavity reflectivity

For recycled interferometer ρ^2 is contrast defect

In general the range $\rho^2 < 1$

For any particular ρ^2 maximum can be found numerically. This will define optimal modulation index Γ_{opt} .

Γ_{opt} as a function of ρ^2 is found numerically

$$\Gamma_{opt} = \Gamma_{opt}(\rho^2)$$

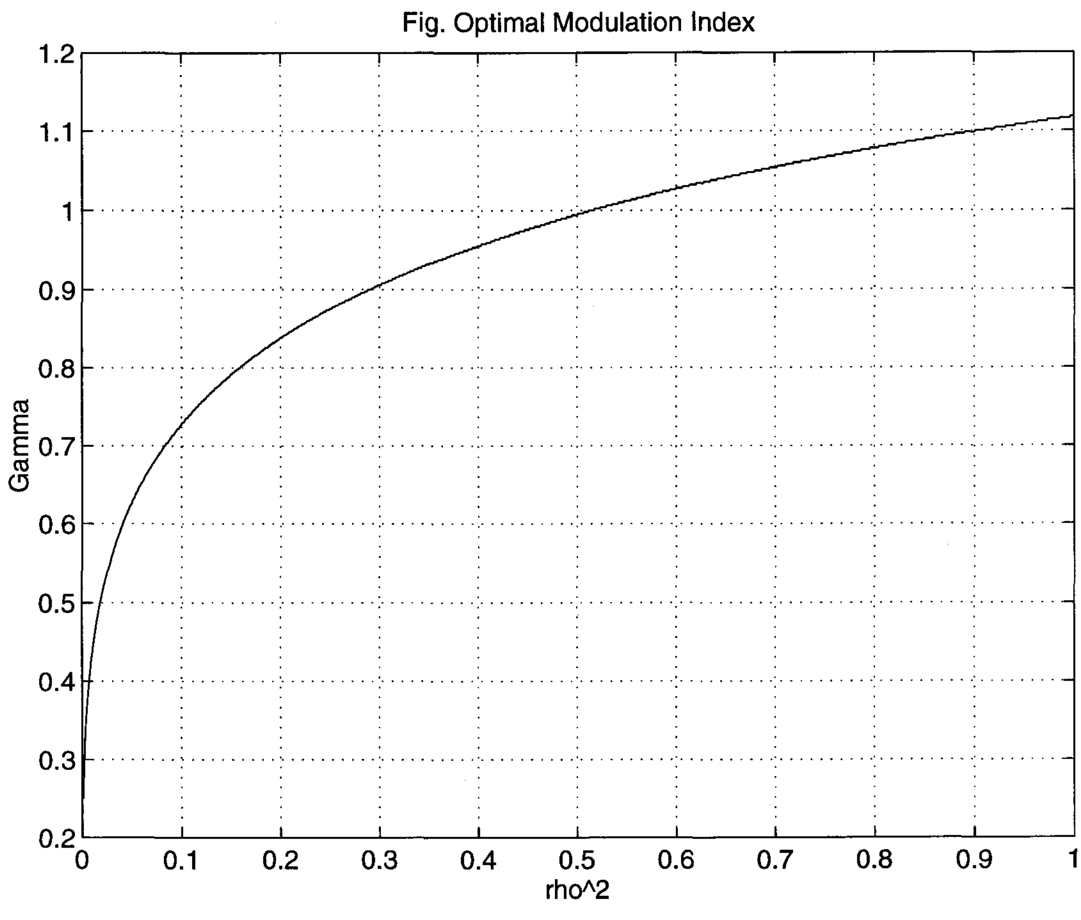


Fig. Optimal Modulation Index
Gamma
rho^2
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Recycling II: Optical Design and Choice of Parameters

- the use of an asymmetric Michelson
- choice of modulation frequency
- choice of mirror reflectivites
- cavity dynamics
    - ringdowns
    - transfer functions

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Frontal / Schnupp Modulation

- Have already seen the use of modulation to control a single Fabry Perot cavity

How do we apply this principle to a system with

several degrees of freedom?

Modulate the light immediately before it enters the interferometer - called frontal or Schnupp modulation.

- Need an asymmetric Michelson.

Why?

- Asymmetric Michelson is dispersive $\therefore$ whilst carrier light appears at only one port, sidebands are transmitted to both ports.

e.g. consider control of beamsplitter for simple Michelson:

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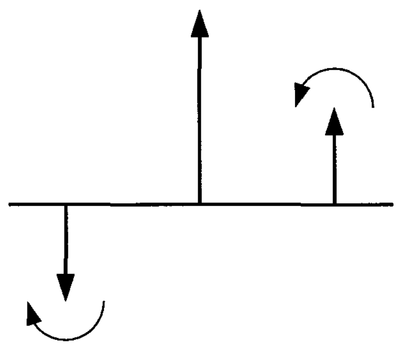
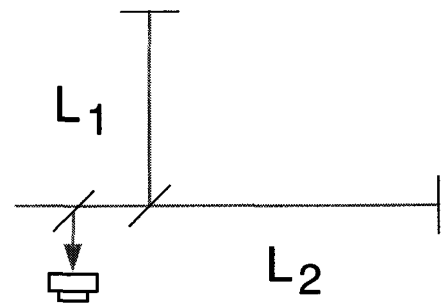
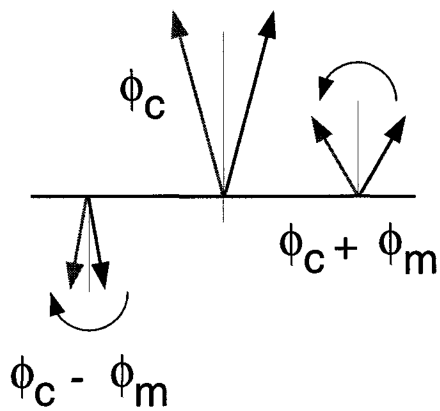


Input light



Light from 1 has phase retarded
Light from 2 has phase advanced



If $\phi_c \neq 0$ then one sideband is larger than the other $\therefore$ locking is equivalent to equalising the sideband heights.

Optimal Asymmetry

The optimal asymmetry is calculated by considering the shot noise limited sensitivity of the GW. It is achieved for

$$\cos\left(\frac{2\pi\delta}{\lambda_{mod}}\right) = r_R T_P$$

For the 40m parameters:

$$\delta_{opt} = 67 \text{ cm}$$

Due to space considerations

$$\delta = 54 \text{ cm}$$

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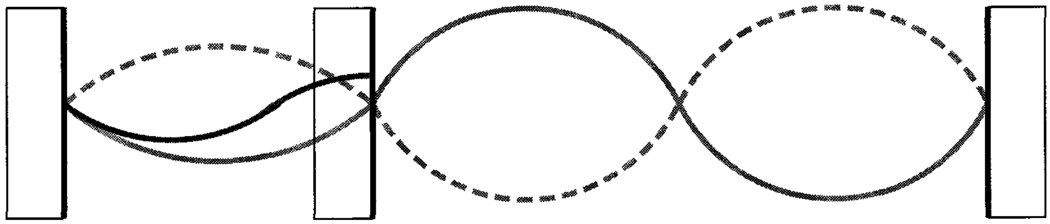
Choice of Modulation Frequency

- Sidebands + carrier are resonant in the recycling cavity, only carrier is resonant in the arms.



- Carrier has phase change of π between light entering and leaving arm $\therefore$ for carrier resonant in the recycling cavity $\phi_c = 2k\pi$
- For sidebands to be resonant in the recycling cavity, we require a π phase change i.e.

$$\phi_s = \phi_c + \phi_{\text{mod}} = 2n\pi + \pi$$

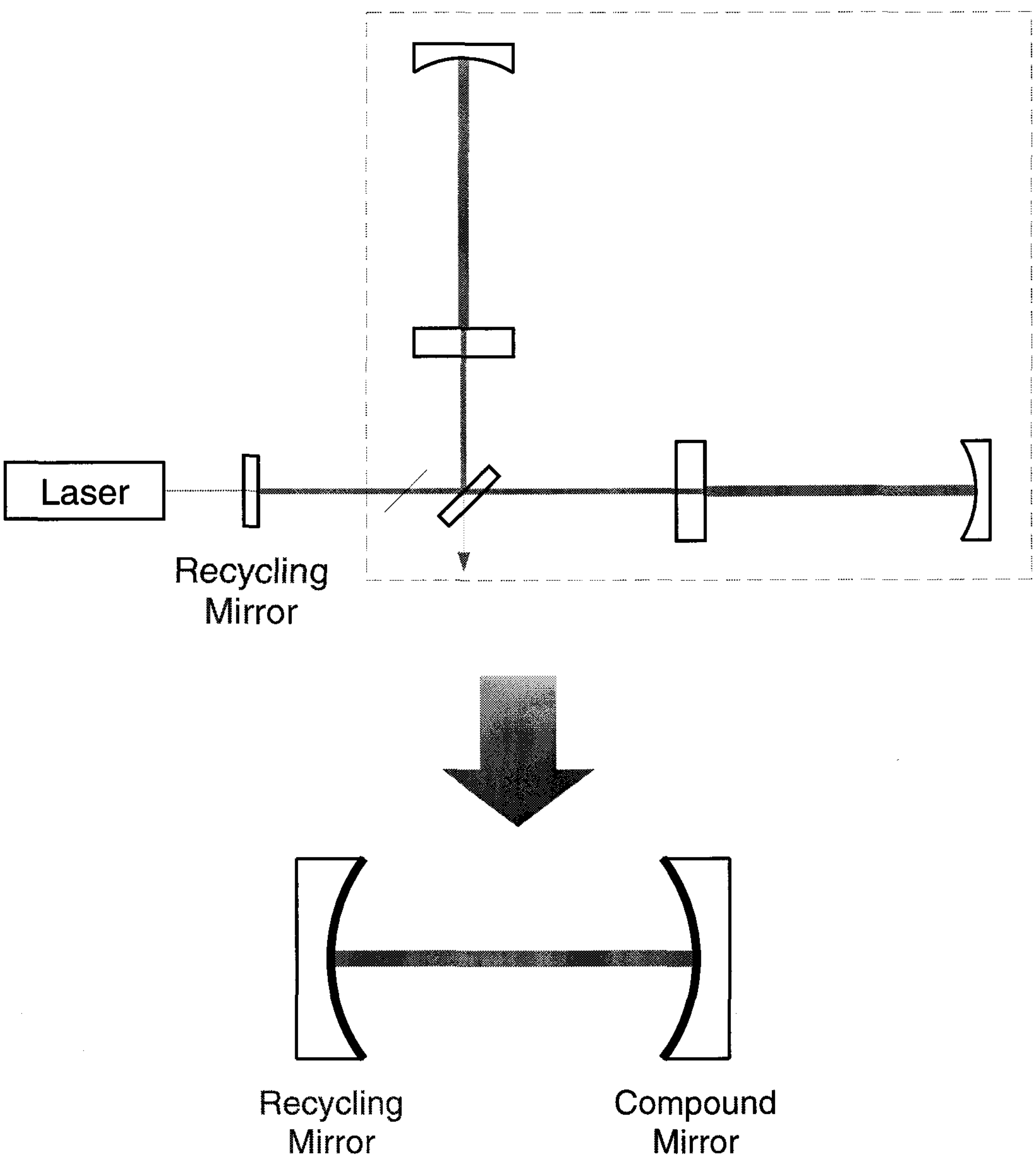
$$\phi_{\text{mod}} = \frac{4\pi l_R}{\lambda_{\text{mod}}} = 2m\pi + \pi$$

$$l_r = \frac{\lambda_{\text{mod}}}{2}\left(m + \frac{1}{2}\right)$$

From space considerations at 40m, $l_r = 2.3$m

=> $f_{\text{mod}} = 32.7$ MHz

Recycling: Choice of Mirror Parameters



Recombined Interferometer as a Compound Mirror

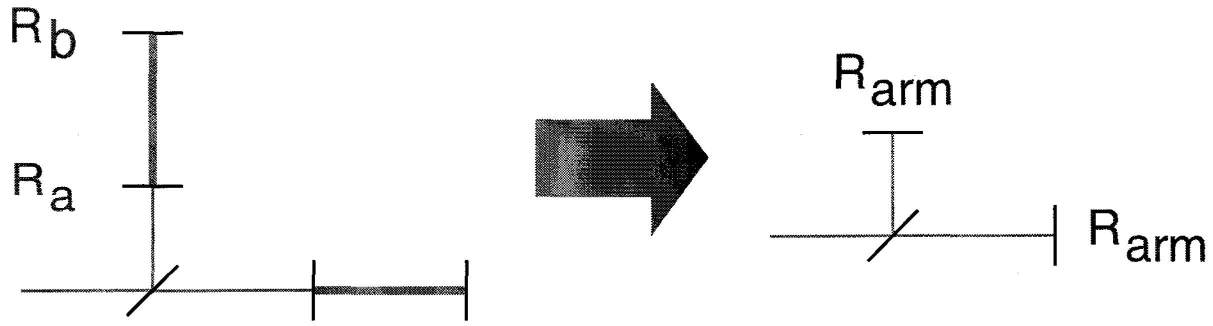
Recall, single FP cavity has a reflectivity

$$R_{\text{cavity}} = \left[\frac{r_a - r_b(1 - L_a)e^{i\phi}}{1 - r_a r_b e^{i\phi}}\right]^2$$

In reality only a fraction, M, of the input power is modematched to the cavity ∴ (1-M) of the light reflects directly off the front mirror. Define

$$R_{\text{arm}} = MR_{\text{cavity}} + (1 - M)R_a$$

Thus we may reduce the recombined topology to



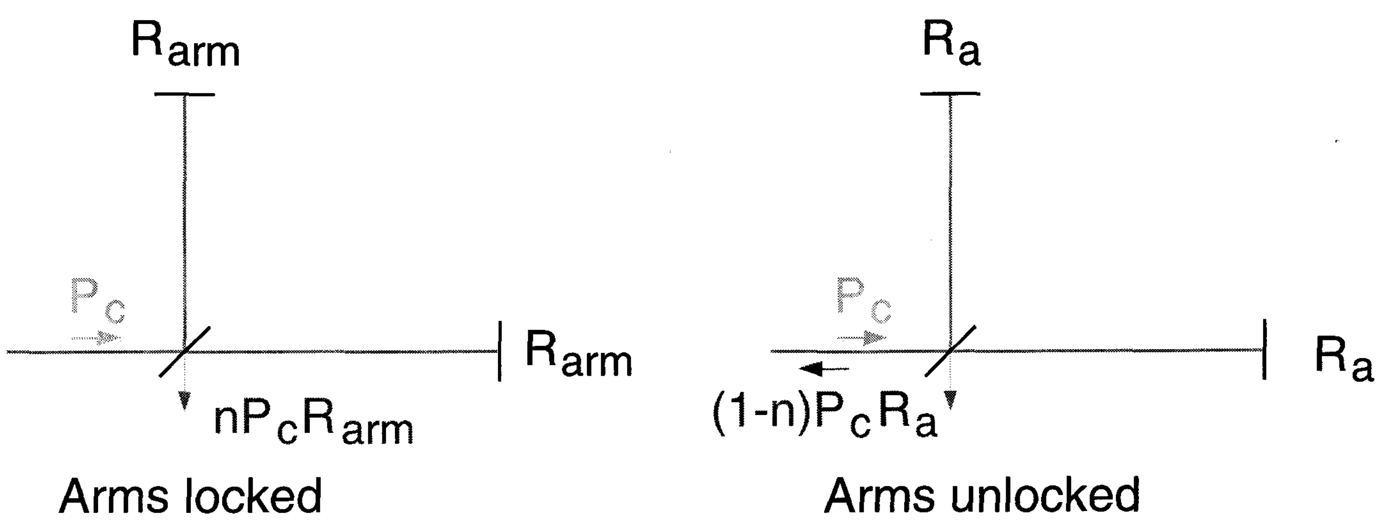
Contrast

The contrast of the interferometer is a measure of how perfectly light interferes at the beamsplitter:

$$C = \frac{P_B - P_D}{P_B + P_D}$$

P_D is minimum carrier power at dark port with both arms in lock. P_B is maximum carrier power at bright port with arms out of lock.

Some fraction, n, of carrier light recombining at the beamsplitter leaks out of the dark port



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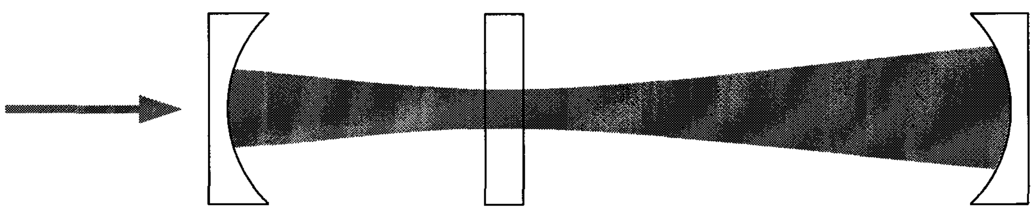
Thus the fraction of power at the dark port

$$n = \frac{R_a(1-C)}{R_{arm}(1+C)}$$

From inspection we can see that the reflectivity of the recombined ifo is $(1\text{-}n)P_cR_{arm}$. What about the recycled interferometer?



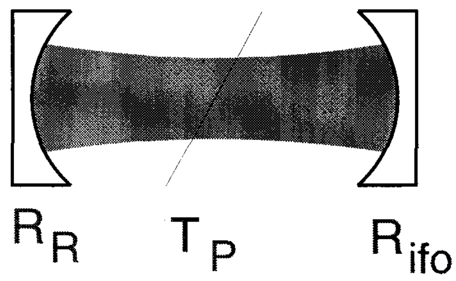
The recycling cavity should provide some mode-cleaning. However the 40m will use a flat recycling mirror -> degenerate recycling cavity.

Non-mode matched sideband light will resonate in the recycling cavity aswell as higher order carrier modes produced by misalignments. (produces no useful signal, only noise). Model this as loss at recycling mirror.

Viewed from recycling mirror

$$R_{\text{ifo}} = (1-n)R_{\text{cavity}}$$

Recycling Factor



The recycling factor

$$N = \left[\frac{t_r t_P}{1 - T_P r_R |r_{ifo}|}\right]^2$$

This is a maximum when the recycling cavity is optimally coupled i.e. when

$$r_R = (1 - L_R) T_P |r_{ifo}|$$

For this case

$$N_{max} = \frac{(1 - L_R)^2 T_P}{T_R}$$

Loss at the Recycling Mirror

To match the recycling mirror in curvature requires R_R = 378.9m. We chose to flat mirror

∴ loss at the recycling mirror is dominated by mode mismatching.

Fraction of power scattered from TEM_{00} mode:

$$L_R = 1 - \frac{1}{1 + \left(\frac{k\omega_R^2}{2R_R}\right)^2}$$

where ω_R is the beam waist at the recycling mirror (2.2 mm)

=> $L_R \sim 0.01$

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Choice of Recycling Factor

- LIGO has N ~ 30
- want 40m to model LIGO as closely as possible
- light storage time in arm cavity is a function of cavity length ∴ don't want this to be too different between LIGO and 40m
- Desire to balance these issues

-> N ~ 5

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40m Optics - Design Parameters

Parameter	Symbol	Recycled
Vertex mirror transmission (ppm)	T_a	5750
End mirror transmission (ppm)	T_b	12
Loss per mirror (ppm)	L_a	100
Corner frequency (Hz)	f_c	1850
Finesse	$\mathcal{F}$	1050
Modematching	M	0.9
Modulation Frequency (MHz)	f_{mod}	32.7
Asymmetry Length (m)	δ	0.54
Contrast Defect	δC	0.03
Pick-off transmission	T_p	0.9915
Recycling Factor	N	5
Recycling mirror transmission	T_R	0.175

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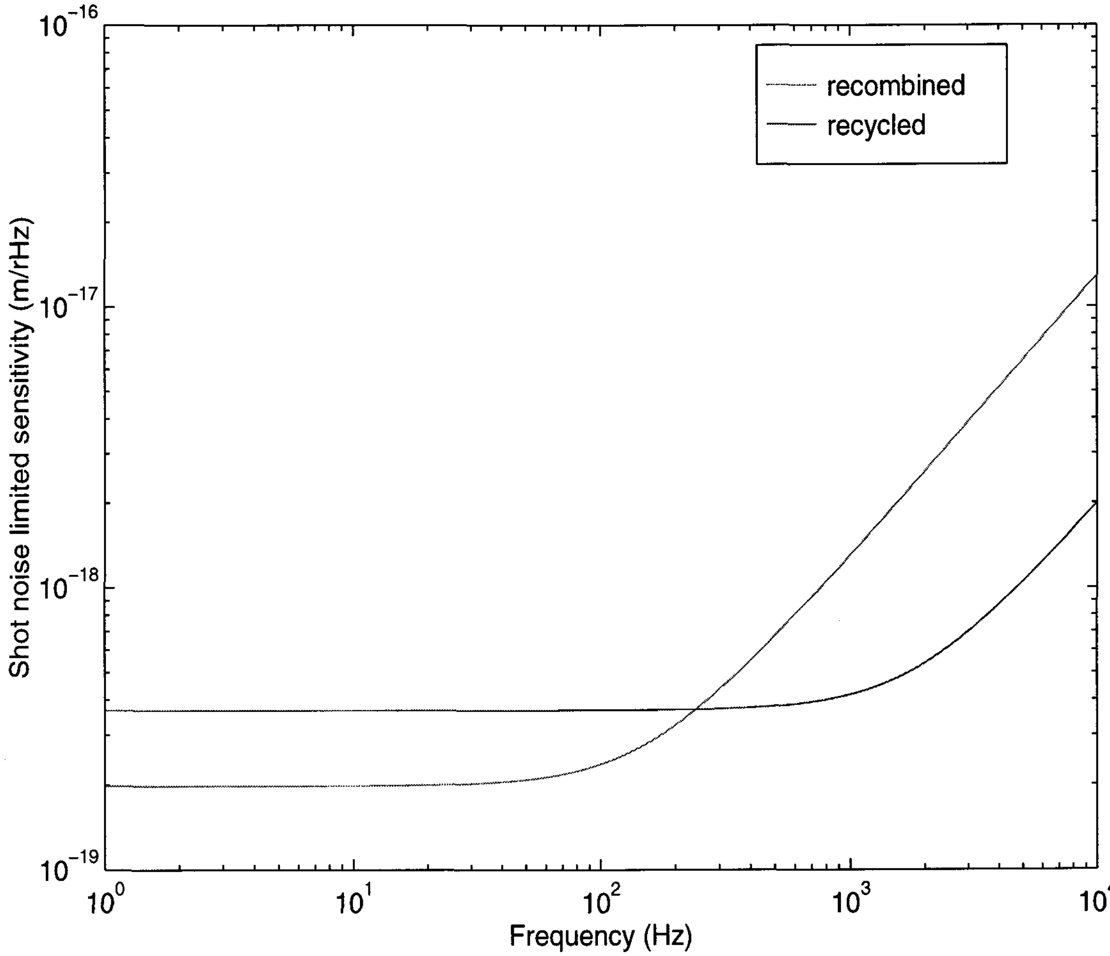
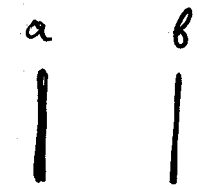
Shot Noise Limited Sensitivity



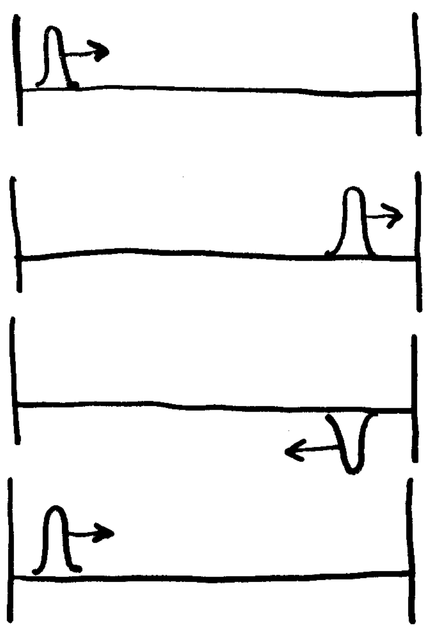
Ringdowns

1) Cavity on resonance. Mirrors are fixed.



Amp. refl. r_a, r_b

2) Short Light Pulse



E_0 – amp.

E_0

$E_0(-r_b)$

$E_0 r_a r_b$

$$E_n = E_0 (r_a r_b)^n,$$ where

n – number of round trips

$\frac{1}{e}$ - amplitude → Effective number of bounces

$$B = \frac{1}{\ln\left(\frac{1}{r_a r_b}\right)} ,$$

For 40m

$$B \approx 320$$

Storage time

$$\tau = B * 2T , \qquad T = \frac{L}{c} = 0.125 \mu sec$$

for 40m $\quad \tau \approx 80 \mu sec$

In general $\tau = \dfrac{2T}{\ln\left(\frac{1}{r_a r_b}\right)}$.

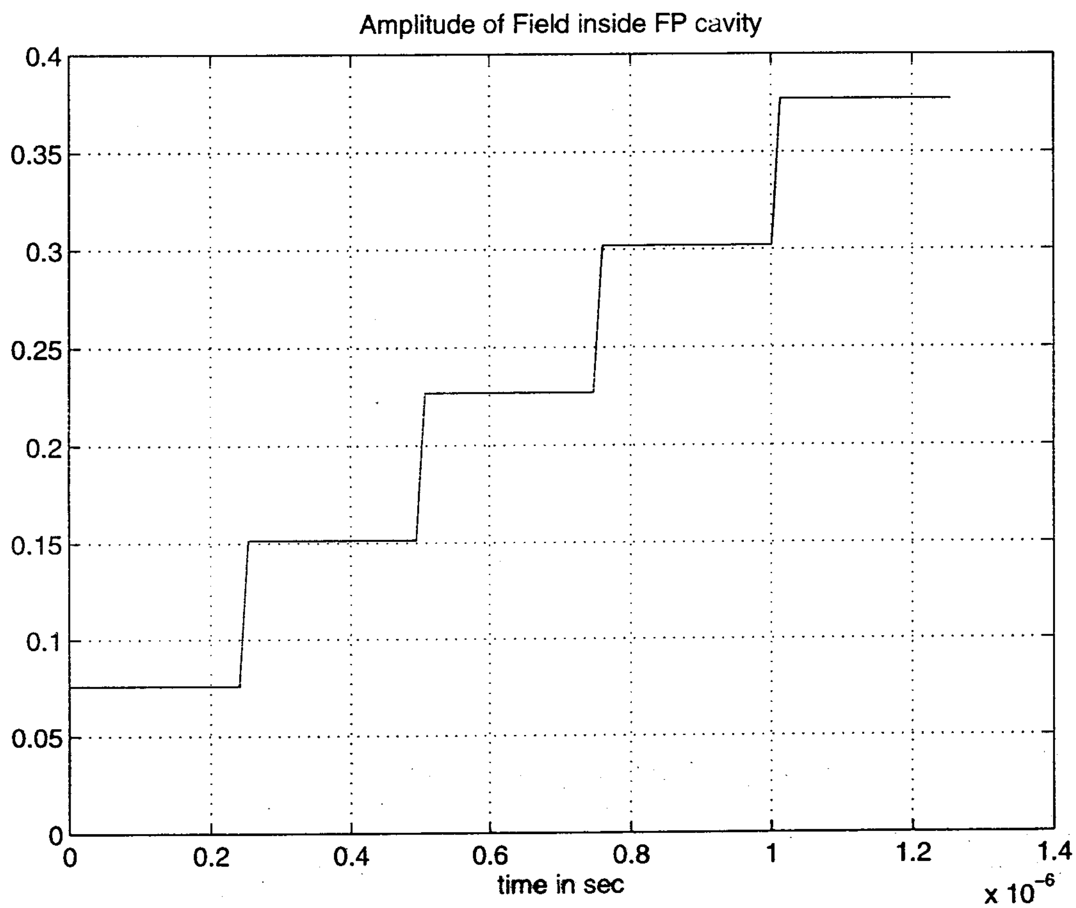
Exponential Decay and Buildup

decay: $$E(t) = E_1 e^{-\frac{t}{\tau}}$$

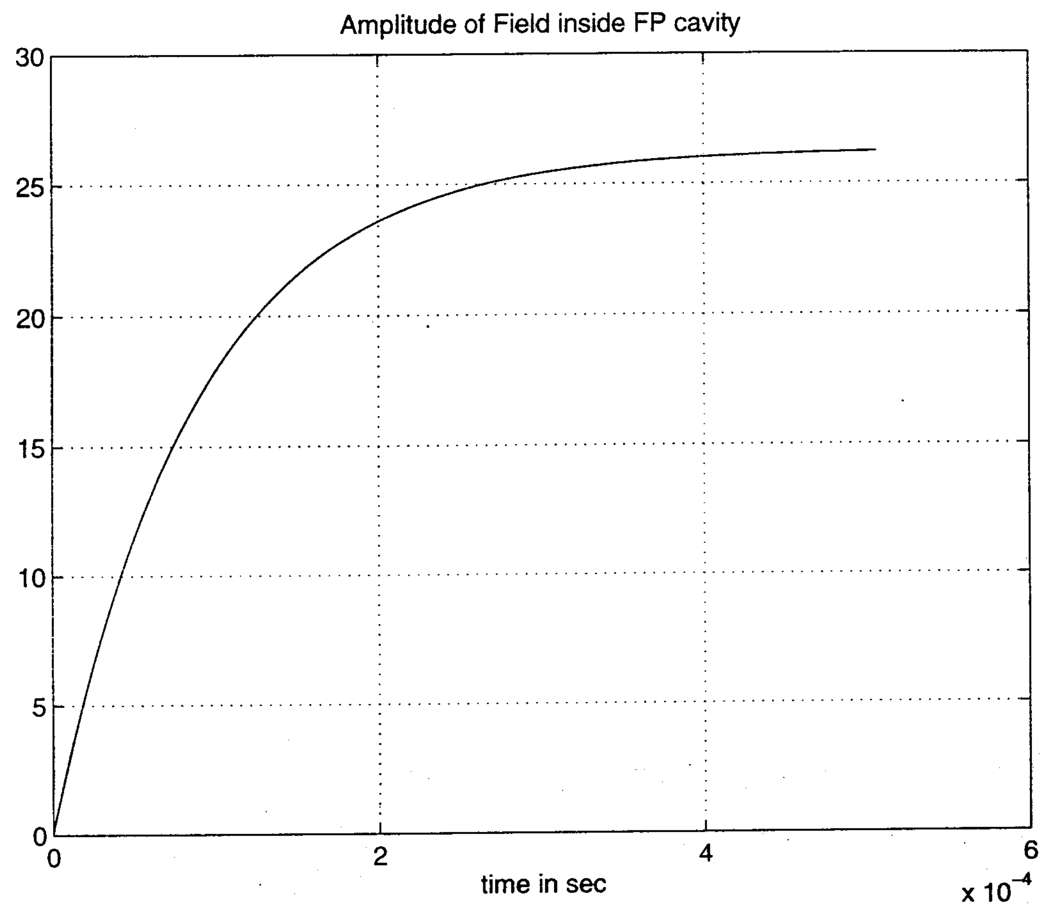
buildup: $$E(t) = E_1 \left(1 - e^{-\frac{t}{\tau}}\right)$$

$$E_1 = E_0 \frac{t_a}{1 - r_a r_b}$$ – equilibrium field

for 40m: $$E_1 \approx 26\, E_0$$



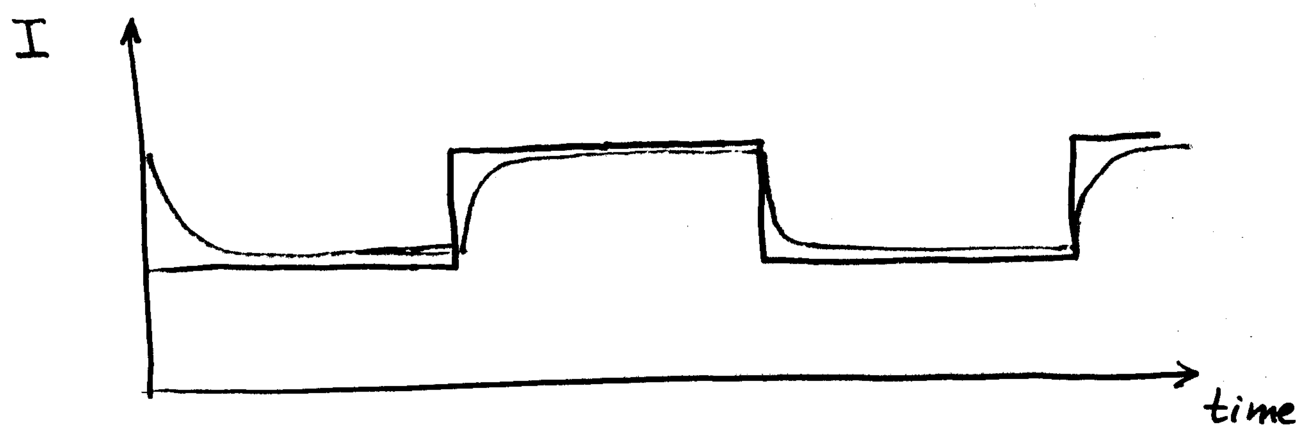
Amplitude of Field inside FP cavity
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
time in sec
x 10^-6



Amplitude of Field inside FP cavity
30
25
20
15
10
5
0
0
2
4
6
time in sec
x 10^-4

Ringdown Measurements

- Intensity of input light is modulated by square wave
- Cavity on resonance
- Measure transmitted light power, $I(t)$



Double Exponential Decay

$$E(t) = E + \Delta E\, e^{-\frac{t}{\tau}} \quad \text{— Amplitude}$$

$$I(t) = E(t)^2 = E^2 + 2E\,\Delta E\, e^{-\frac{t}{\tau}} + \Delta E^2 e^{-2\frac{t}{\tau}}$$

Four Parameter Fit

$$I(t) = a + b e^{ct} + d e^{2ct}$$

a, b, c, d — parameters

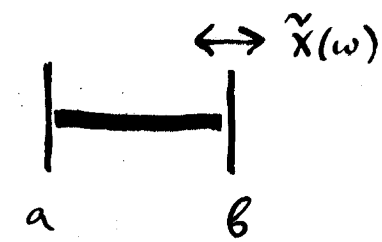
Fabry – Perot Cavity Transfer Functions

- Cavity near resonance
- Linear regime
- Perturbation $\sim \cos \omega t$ (input) = $\tilde{y}(\omega)$
- Demod-out – (output) = $\tilde{V}(\omega)$

Transfer Function $H(\omega)$

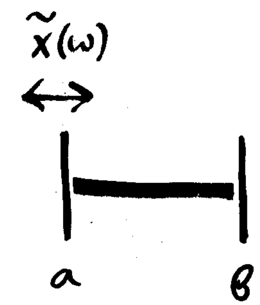
$$\tilde{V}(\omega) = H(\omega)\, \tilde{y}(\omega)$$

Back Mirror Motion



$$H_1(\omega) = \frac{e^{-i\omega T}}{1 - r_a r_b e^{-2i\omega T}}$$

Front Mirror Motion



$$H_2(\omega) = -\frac{1}{1 - r_a r_b e^{-2i\omega T}}$$

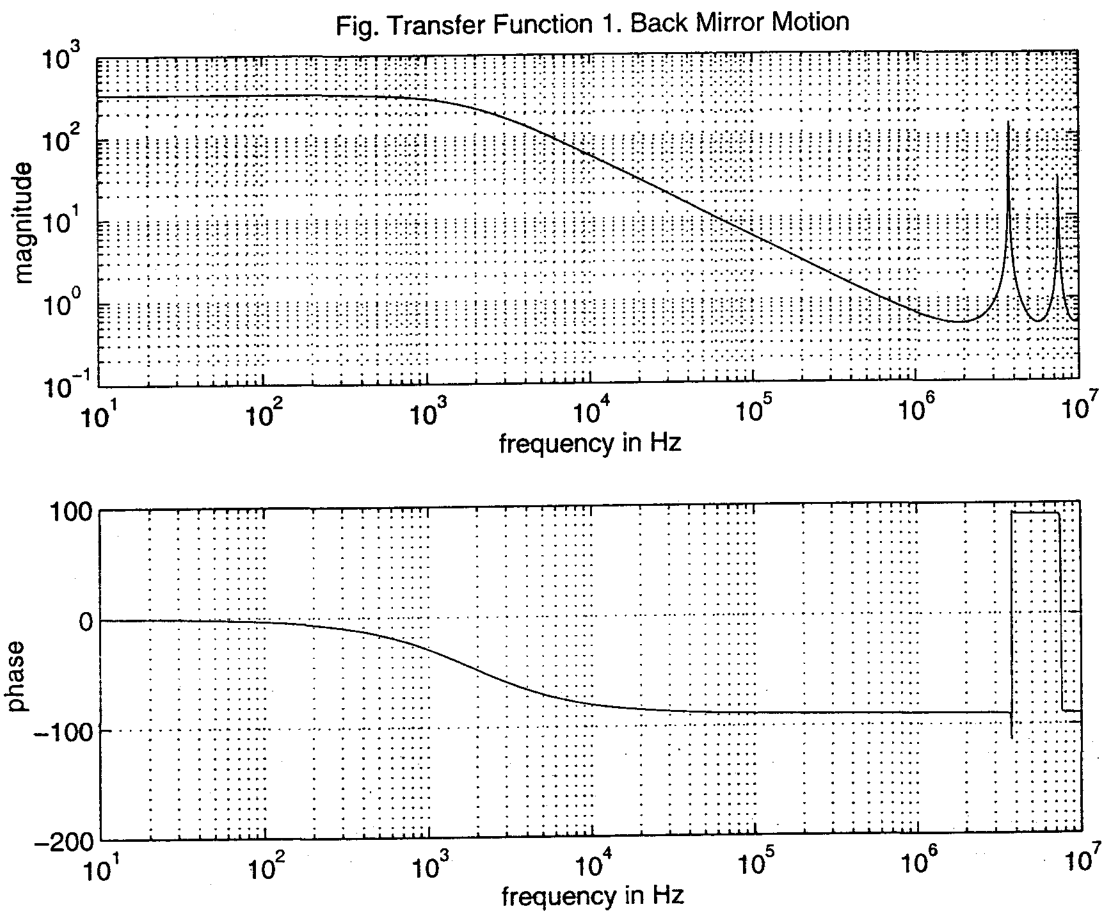
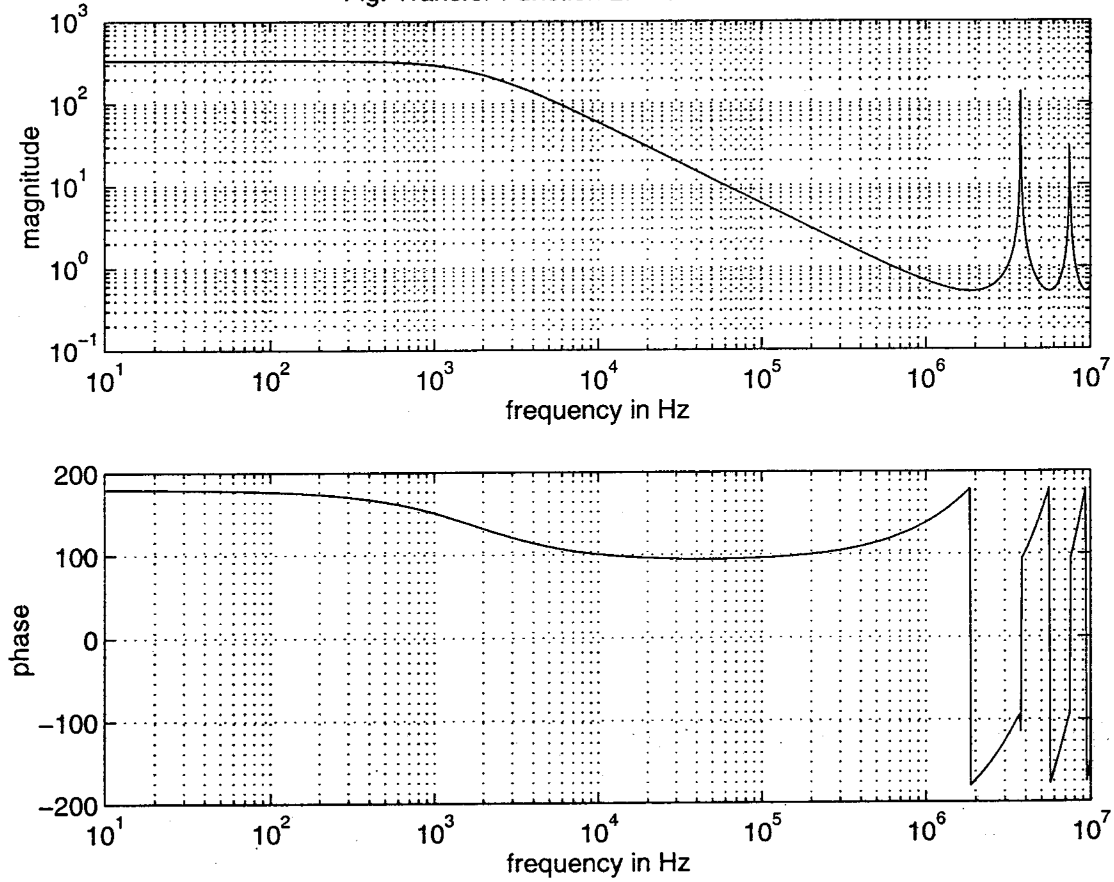


Fig. Transfer Function 1. Back Mirror Motion
magnitude
frequency in Hz
phase
frequency in Hz

Fig. Transfer Function 2. Front Mirror Motion



Both Mirrors Move in Phase

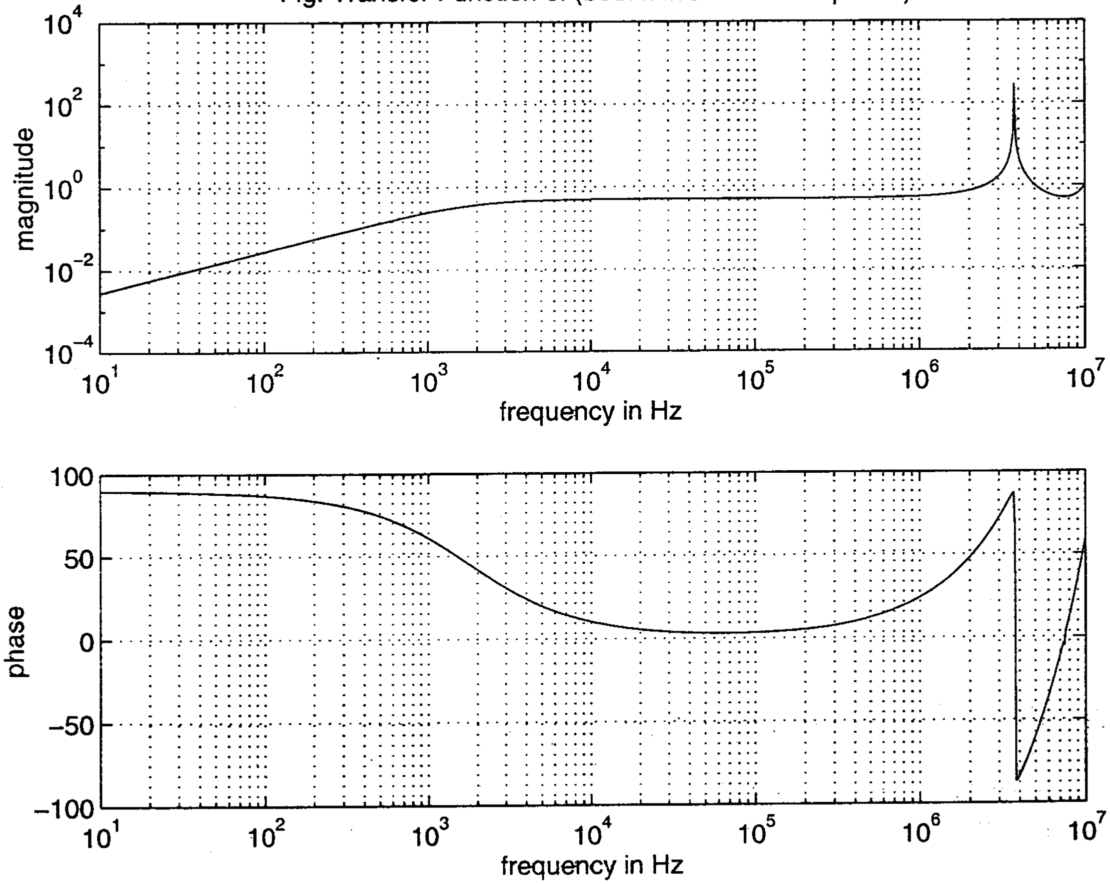
$$H(\omega) = H_1(\omega) + H_2(\omega)$$

$$H(\omega) = \frac{1 - e^{-i\omega T}}{1 - r_a r_b e^{-2i\omega T}}$$

$e^{-i\omega T} = -1$, $f = \frac{\omega}{2\pi} = f_{FSR}$, 3.7 MHz 40m

(37 kHz) LIGO

Fig. Transfer Function 3. (both mirrors move in phase)



Cavity Pole

General feature : $H(\omega) \sim \dfrac{1}{1 - r_a r_e \, e^{-2i\omega T}}$

In the low frequency regime $\omega T \ll 1$
expand

$$e^{-2i\omega T} \approx 1 - 2i\omega T \quad \Rightarrow$$

$$H(\omega) \sim \frac{1}{1 + i\frac{\omega}{\omega_c}}, \text{ where } \quad \omega_c = \frac{1}{2T} \frac{1 - r_a r_e}{r_a r_e}$$

ω_c – cavity pole.

For 40m $\quad f_c = \dfrac{\omega_c}{2\pi} = 1860 \text{ Hz}$

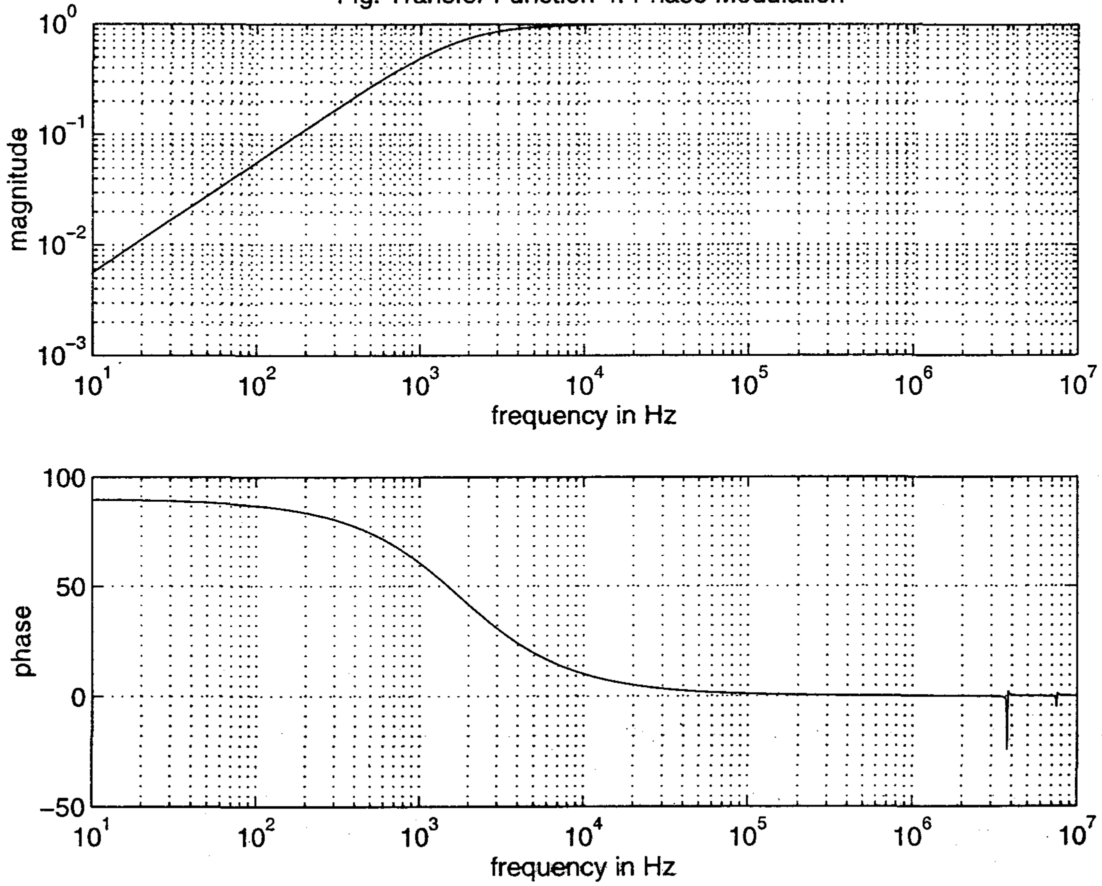
Phase Modulation (Phase Noise)

Input field: $E(t) = E_0 e^{i\delta\phi(t)}$

Transfer Function: $H(\omega) = \dfrac{\tilde{V}(\omega)}{\delta\tilde{\phi}(\omega)}$

$$H(\omega) = \frac{1 - e^{-2i\omega T}}{1 - r_a r_b e^{-2i\omega T}}$$

Fig. Transfer Function 4. Phase Modulation



Intensity Noise

The formalism allows to predict the effect of intensity modulation on demod-out.

In the reflection locking scheme V, demod-out, is not sensitive to intensity modulation in the 1st order approximation.

To see any effect on V, we need non-zero deviation from resonance X_0. Assume $X_0 = \text{const}$ and small enough to be within a fringe.

Then

$$\tilde{V}(\omega) \sim (2kX_0)\{\text{Intensity Noise}\}$$

Intensity Fluctuations

$I(t) = E_o^2(t)$, $\quad E_o(t)$ – Amp. of input field

In frequency domain $\rightarrow$ convolution

$$\tilde{I}(\omega) = \int \tilde{E}_o(\omega-\omega')\,\tilde{E}_o(\omega')\,\frac{d\omega'}{2\pi}$$

There is no transfer function $\tilde{I}(\omega) \rightarrow \tilde{V}(\omega)$.

However,

$$\tilde{V}(\omega) = \underline{(2kx_o)} \int E_o(\omega-\omega')\,E_o(\omega')\,F(\omega')\,\frac{d\omega'}{2\pi}$$

So intensity noise → convolution with the function

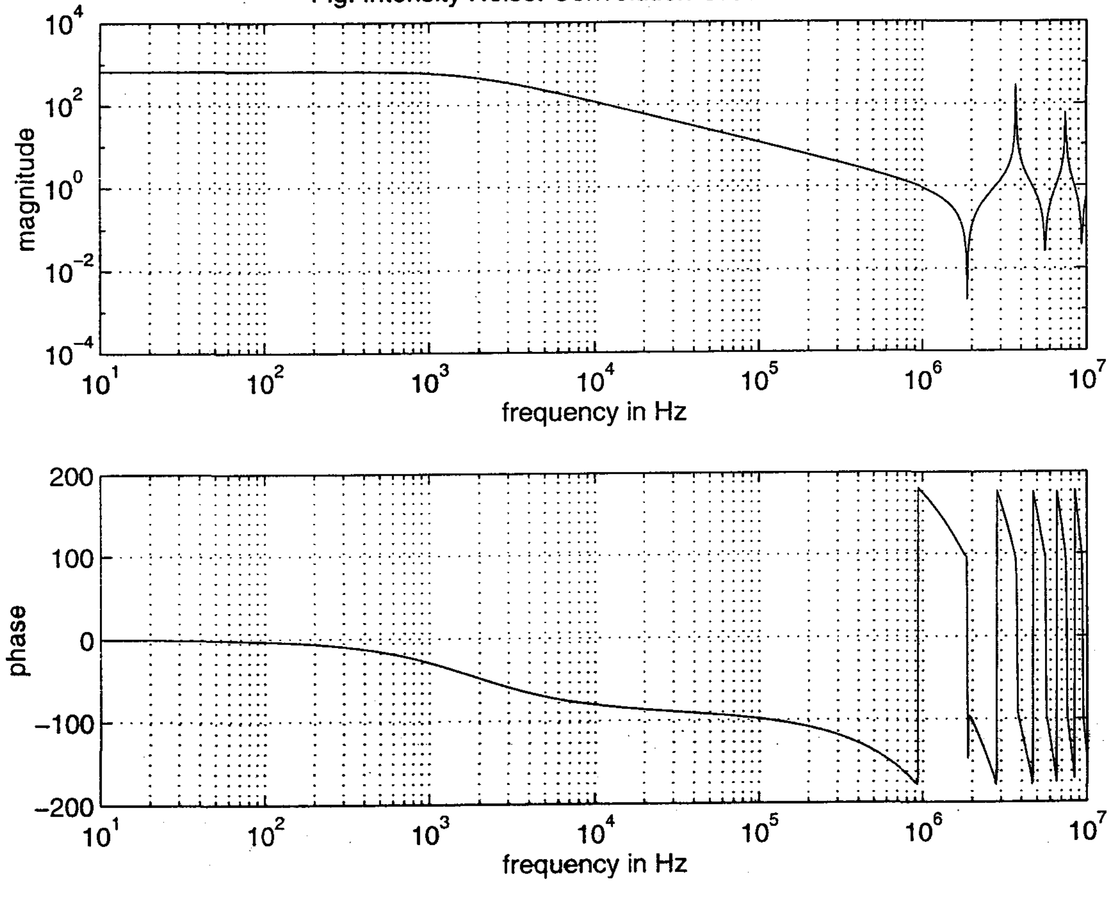
$$\tilde{F}(\omega) = e^{-2i\omega T} \frac{1 + r_a r_b e^{-2i\omega T}}{1 - r_a r_b e^{-2i\omega T}},$$

For low frequencies the weight function $\tilde{F}(\omega)$ is essentially low-pass filter with delay.

For high frequencies $\tilde{F}(\omega)$ has zeros and poles.

Since $\tilde{F}$ enters the convolution all frequencies matter.

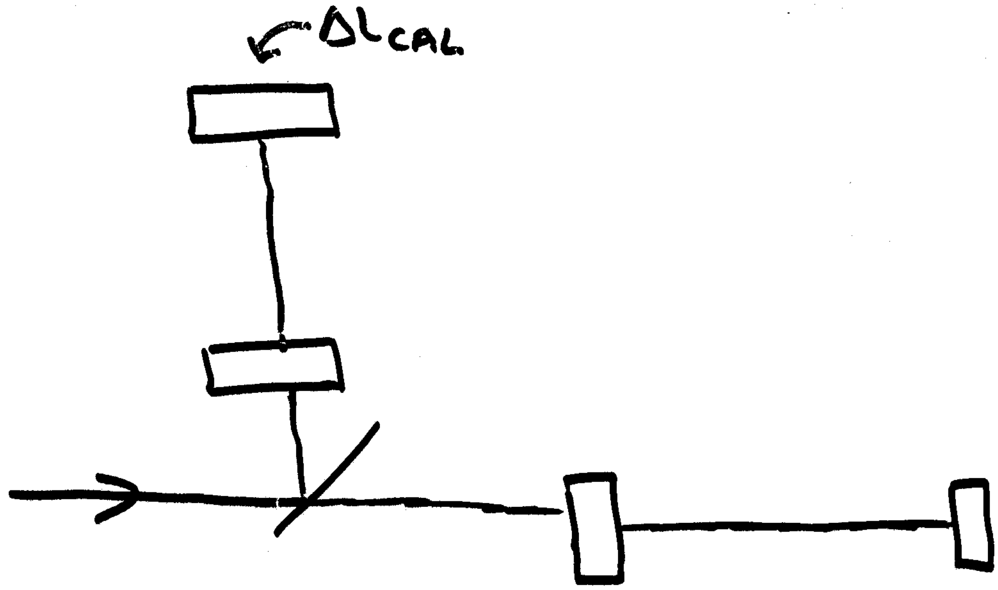
Fig. Intensity Noise. Convolution Green Function



Recycling III: Calibration and Shot Noise

- Displacement calibration
- Shot noise
  - ››Theoretical calculation
  - ››Experimental determination
- Transfer functions of coupled cavities

DISPLACEMENT CALIBRATION

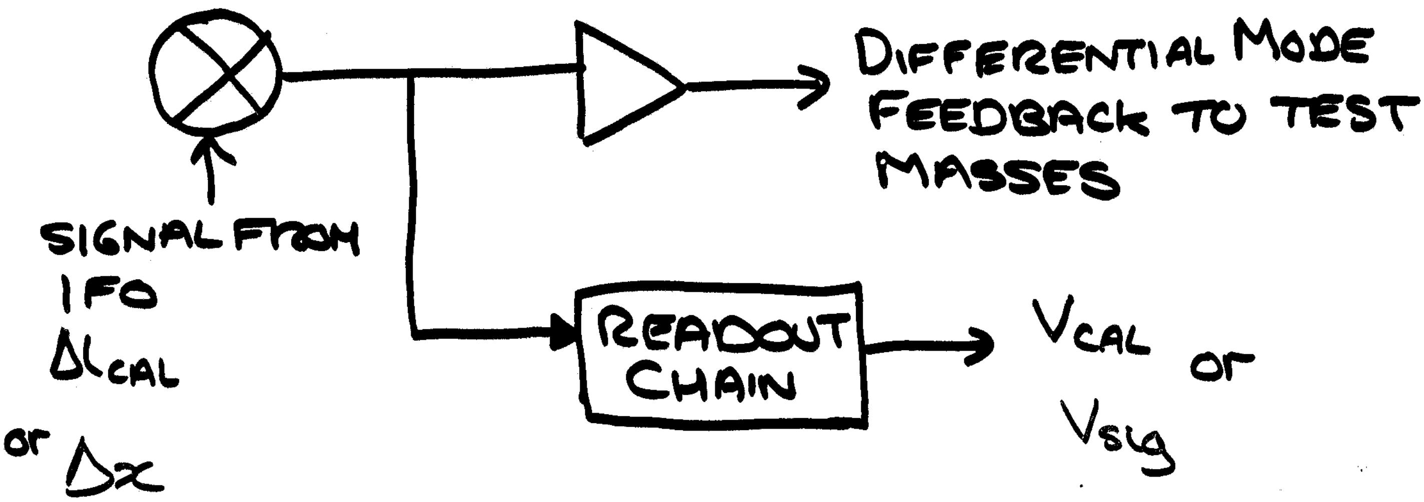


- APPLY SINUSOIDAL DISPLACEMENT Δl_{CAL} VIA FORCE ACTUATOR, BY APPLYING V_{COIL}
- Δl_{CAL} IS LARGER THAN NOISE FLOOR OF IFO BUT NOT SO LARGE AS TO INDUCE NON-LINEARITIES. TYPICALLY $\sim 10^{-16}$ m rms.

$$\Delta l_{CAL} = \beta \frac{V_{COIL}}{\omega^2} \qquad \omega \gg \omega_{PEND}$$

β CONSTANT, DEPENDS ON MASS OF TEST MASS, MAGNET STRENGTH ETC ONLY INFREQUENTLY MEASURED

INTERFEROMETER RESPONSE



Response to induced calibration : V_{CAL} out of readout chain, proportional to drive V_{COIL}

$$V_{CAL} = \underbrace{\alpha(f)}_{} V_{COIL}$$

↑ Depends on frequency of calibration sinusoid

$\alpha(f)$ IS SENSITIVE TO SYSTEM GAINS - MEASURED FREQUENTLY

$\alpha(f)$ IS THE SWEPT SINE FREQUENCY RESPONSE

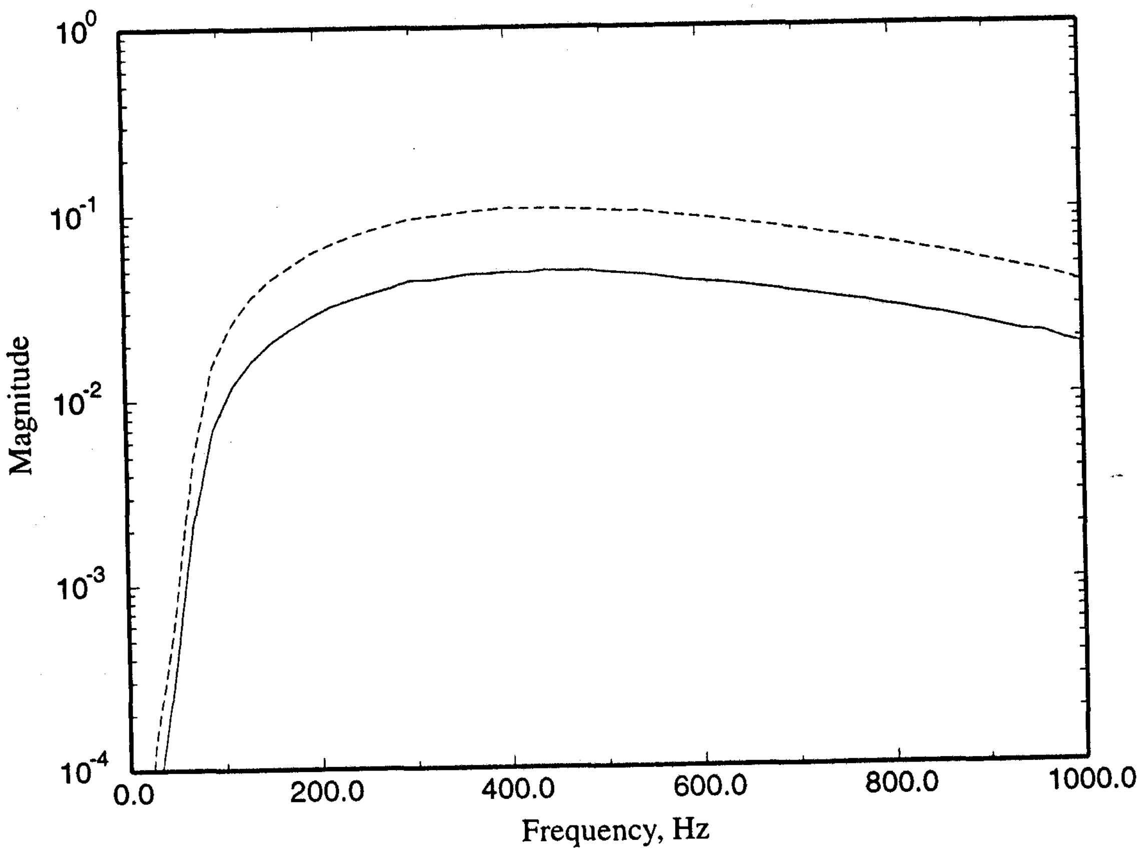


Figure 3-3: Transfer function from driving the east end test mass (solid) or both end masses differentially (dotted) to the interferometer output.

determine the voltage necessary to push the test mass $\lambda/2$ at the drive frequency.[1] Because the test mass is suspended as a pendulum, this calibration rolls off as f^{-2}. The voltage to displacement calibration used here is[2]

$$x(f) = \left(3.14\times10^{-8}\ \frac{\mathrm{m\ Hz}^2}{\mathrm{V}}\right)\frac{V_{\mathrm{source}}}{f^2} \qquad \text{3-2}$$

1. An alternate method used in the Fabry-Perot configuration was to measure the displacement induced by a given drive voltage using an auxiliary Michelson interferometer. Both methods gave comparable results.
2. The uncertainty in this calibration is approximately 20%.

CALIBRATION OF Δx

IN NORMAL OPERATION V_{COIL} IS REMOVED. DISPLACEMENT Δx PRODUCES READOUT CHAIN OUTPUT V_{SIG}.

$$\frac{\Delta x}{\Delta l_{CAL}} = \frac{V_{SIG}}{V_{CAL}}$$

$$\Delta x = \beta \frac{V_{COIL}}{\omega^2} \frac{V_{SIG}}{V_{CAL}}$$

$$\boxed{\Delta x = \frac{\beta V_{SIG}}{\alpha(f)\,\omega^2}}$$

How To Measure β?

Full Fringe Method

Apply large enough sinusoidal $V_{coil,fringe}$ to drive mass through adjacent "fringes" which are separated by $\lambda/2$.

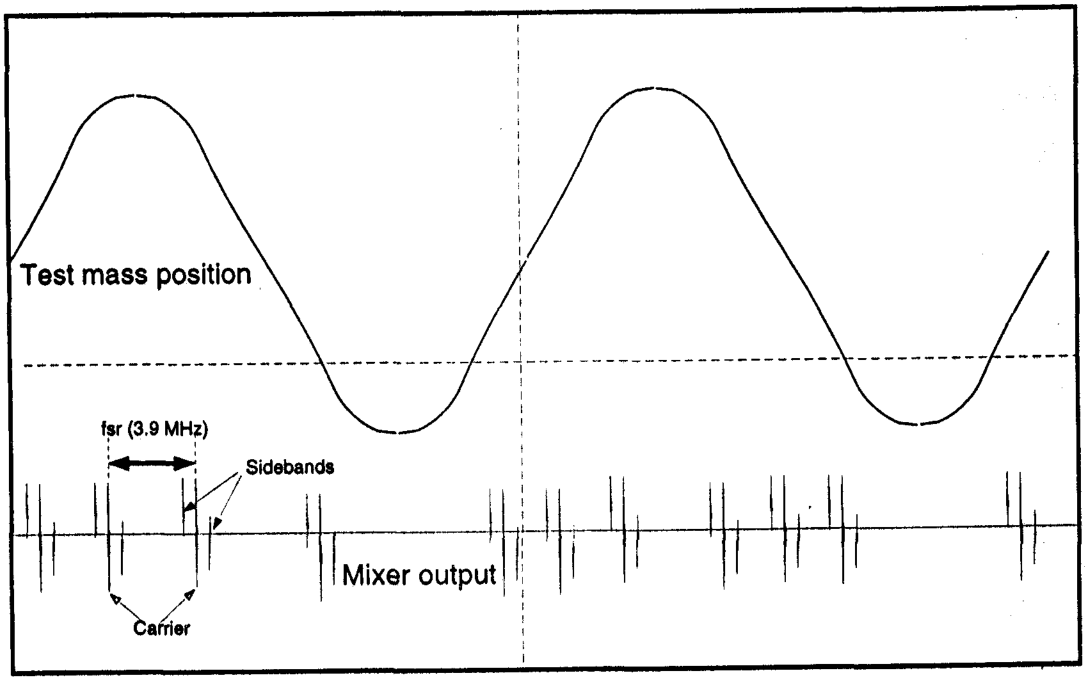
Frequency of drive is low $f_{fringe} \sim 10Hz$

∴ usual calibration circuitry must be bypassed

∴ need to separately measure the attenuation factor h_{calib} of calibration circuitry

$$\beta = \frac{\Delta L \omega^2}{V_{coil}}$$

$$\beta = \frac{(2\pi f_{fringe})^2}{V_{coil,fringe}} \frac{\lambda}{2} h_{calib}$$


Test mass position
fsr (3.9 MHz)
Sidebands
Mixer output
Carrier

NUMERICAL VALUES

	$h_{CAL,B}$	β	k (1kHz)
AUG 96	-17.6dB	5.36×10^{-5} mHz^2/V	25502 $\frac{V/\sqrt{Hz}}{mHz^2}$
INCORRECT	-9.4dB		9930

WHERE k IS MULTIPLIER FOR SWEPT SINE MEASUREMENT WHICH IS ENTERED INTO SPEC. ANALYSER

$$k(1kHz\ SPAN) = \frac{\sqrt{1.87Hz}}{\beta}$$

↑ 1.87 Hz BANDWIDTH

REASON FOR INCORRECT CALIBRATION

- ASSUMED A 50Ω TERMINATOR SUPPLIED A PATH TO GROUND FOR CALIBRATION CURRENT
- THIS WAS REMOVED WHEN THE ELECTRONICS WERE MODIFIED FROM NON-RECOMBINED → RECOMBINED

∴ NEED TO APPLY A CORRECTION FACTOR OF -8.5dB TO ALL SPECTRA FROM MAY 95 – AUGUST 96

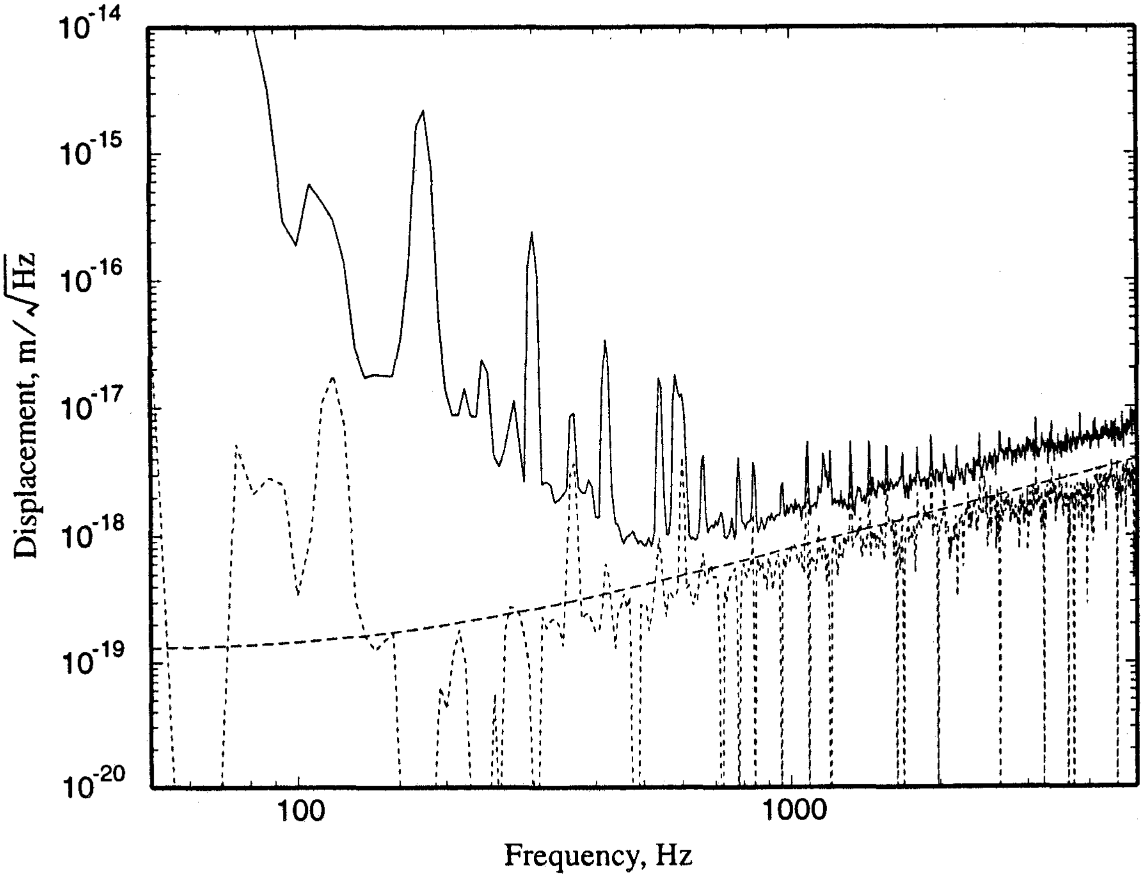


Figure 3-4: Calculated shot noise contribution to interferometer displacement spectrum (dashed), with empirical measurement of shot noise contribution (dotted) and interferometer displacement spectrum taken shortly before on January 10, 1996 (solid).

the shot noise contribution to the gravitational wave signal (discussed below) and to the interferometer displacement spectrum taken at the time these measurements were done. The empirical measurement of the shot noise contribution agrees with the calculation to within the uncertainties of the parameters in the calculation and in the calibration. Note that the interferometer was not limited by shot noise at any frequency. This was confirmed by attenuating the light leaving the antisymmetric port by 37.5% and adding incandescent

Photodiode's Output

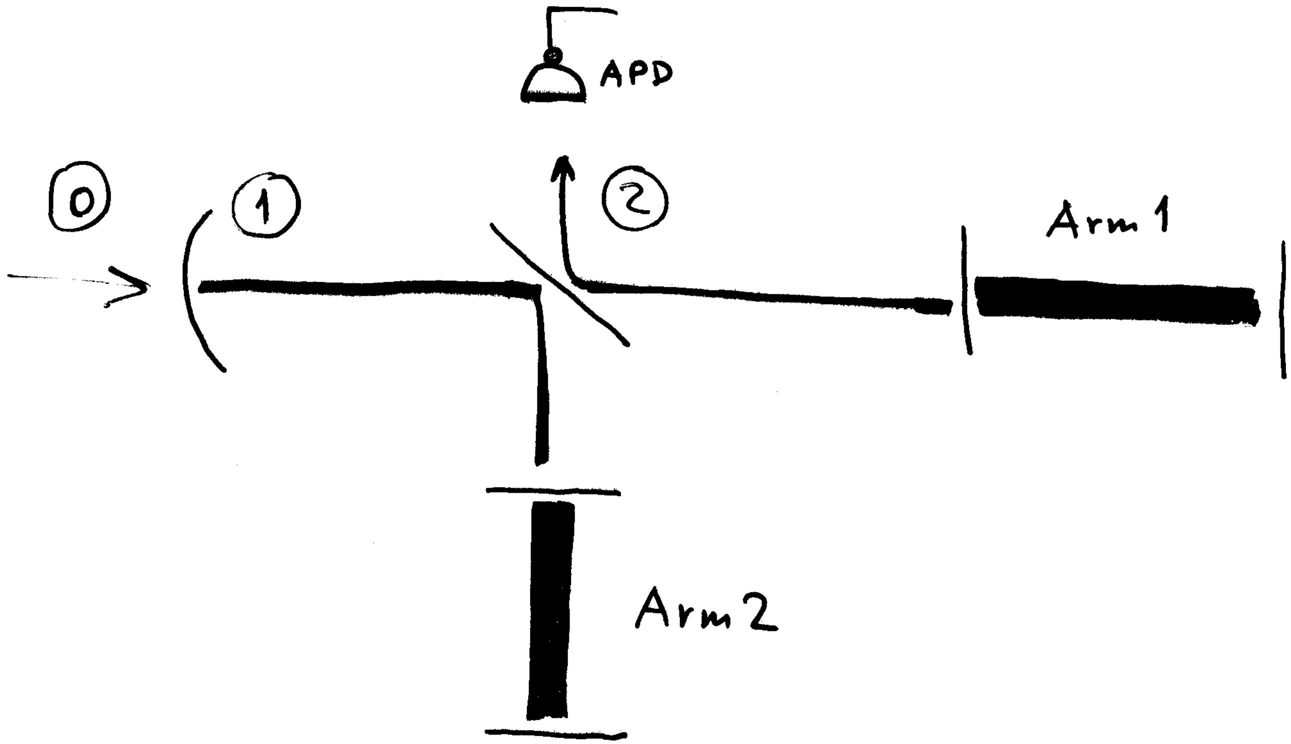
photocurrent : $$I(t) = I_0 + I_1 \sin\Omega t + I_2 \cos 2\Omega t$$

signal : $$V(t) \sim I_1(t)$$

noise (shot noise) : $$S_y^{(0)} \sim I_0$$

$$S_y^{(2)} \sim I_2$$

(0) $E_{in} = \sqrt{PM}$



(1) Fields

$$E'_0 = E_{in} J_0 \sqrt{N_0}$$

$$E'_{\pm 1} = \pm E_{in} J_1 \sqrt{N_1}$$

(2)

$$E_0 = 0$$

$$E_{\pm 1} = E'_{\pm 1} \sin\alpha$$

$$\alpha = k_m \delta l$$

Carrier at APD: $E_0 \neq 0$

$$E_0 = \delta E_0 + i \frac{1}{2} E_0' H(t) .$$

1) $\delta E_0 = E_0' \sqrt{\delta C}$ due to contrast defect

2) $H(t) = \int C(t-t_1)\, \Phi(t_1)\, dt_1$

↑ DM Phase

$$\frac{1}{k} \Phi(t) = X_1(t) - X_2(t)$$

$$\tilde{C}(\omega) = \frac{C_0}{1 + i \frac{\omega}{\omega_c}} \; ; \quad C_0 = 645 \quad (40m)$$

Photocurrent Components $(I = \sigma P)$

$$I_0 = \sigma \left(\delta E_0^2 + 2 E_1^2 \right) ,$$

$$I_1 = 4 \sigma E_0' E_1 H(t) ,$$

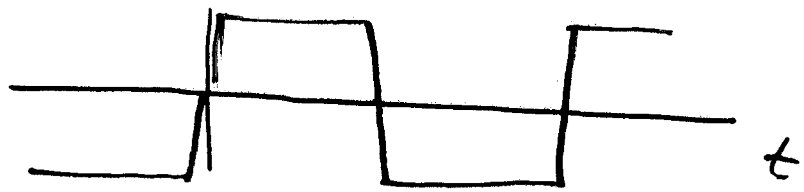
$$I_2 = \sigma \left(2 E_1^2 - 4 \, \delta E_0 \cdot E_2 \right) .$$

$\sigma = 0.25$ A/W (photodiode's responsivity)

Mixer Output

$$V_{mix}(t) = V(t)\, D(t)$$

$D(t) =$



$$V(t) = \frac{4}{\pi} R\, I_1(t)$$

$$V = \upsilon\, \Phi \quad ; \qquad \upsilon = \frac{4}{\pi} R\, \sigma\, E_0'\, E_1\, C_0$$

or $V = k \upsilon X$; X – DM displacement.

Signal to Noise Ratio

$$SNR = \frac{\sqrt{S_v(\omega)}}{\sqrt{S_y(\omega)}}$$

Sensitivity $\rightarrow$ $SNR = 1$

$$[V] = \sqrt{S_y(\omega)} \quad , \quad (V/\sqrt{Hz}).$$

$$[\Phi_\omega] = \frac{\sqrt{S_y(\omega)}}{v\,|C(\omega)|} \quad , \quad (rad/\sqrt{Hz}).$$

$$[X_\omega] = \frac{\sqrt{S_y(\omega)}}{kv\,|C(\omega)|} \quad , \quad (m/\sqrt{Hz}).$$

25-Mar-97

SHOT NOISE CALCULATIONS

FOR RECYCLED 40m

****

Input Parameters

TR = 0.194 , transmissivity of recycling mirror

TP = 0.99 , transmissivity of pick-off

Ta = 0.00575 , transmissivity of input mirror

La = 0.0001 , losses of input mirror

M = 0.8 , modematching coefficient

dC = 0.04 , contrast defect

****

Results of the Calculation

fc = 1866.27 , cavity corner frequency (Hz)

cp0 = -0.102339 , carrier coupling to rec. cav.

cp1 = -0.122677 , 1st SB coupling to rec. cav.

N0 = 5.96665 , carrier recycling factor

N1 = 6.16858 , 1st SB recycling factor

N2 = 0.0703348 , 2nd SB recycling factor

I0 = 7.80432 , DC-photocurrent (mA)

I2 = 3.93739 , RF-photocurrent (mA)

S = 9.13722e-16 , shot noise power (V^2/Hz)

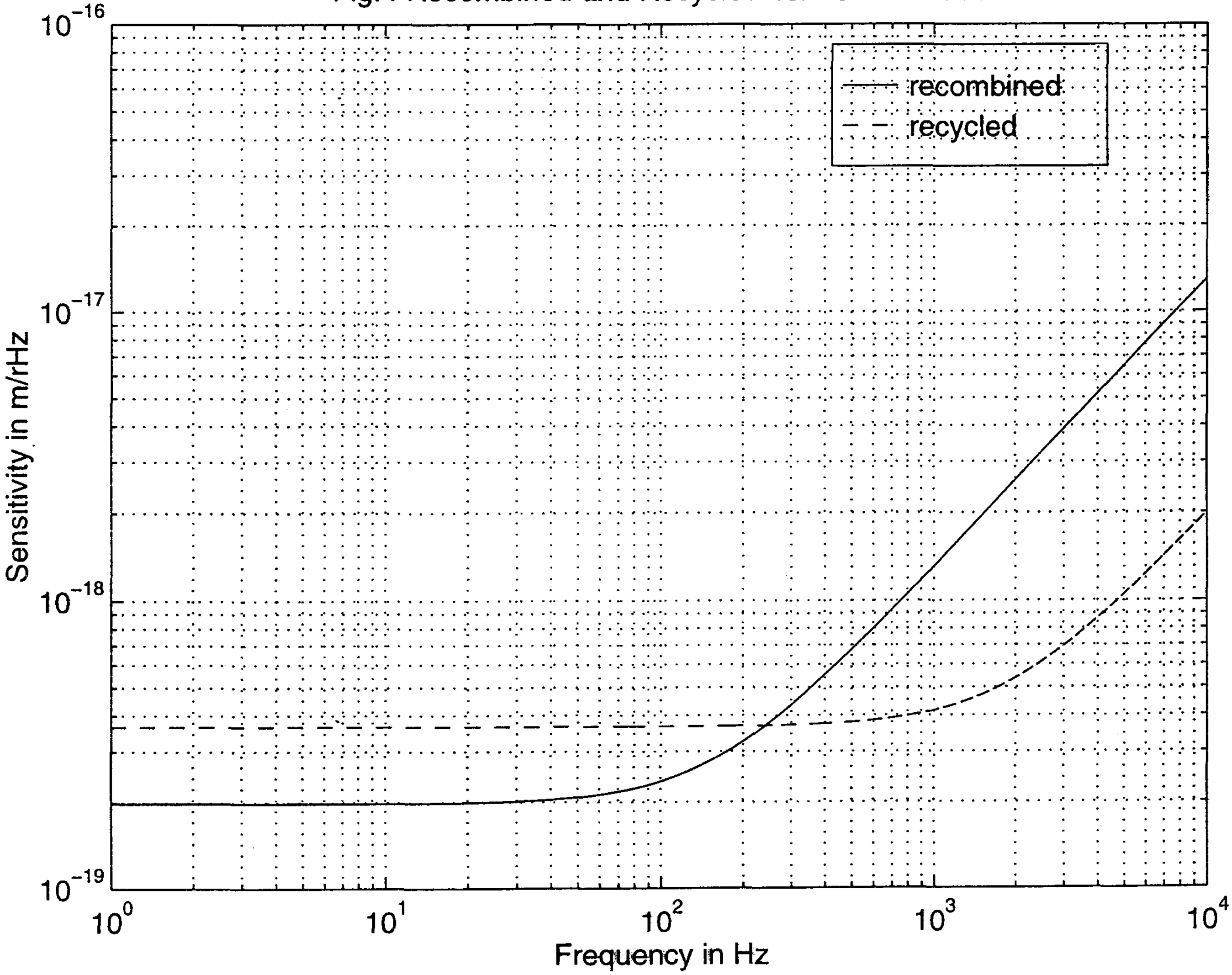
ns = 0.252257 , non-stationarity

vm = 1.64818e+11 , calibration factor (V/m)

Phi0 = 4.47948e-12 , phase DC-sensitivity (rad/rHz)

X0 = 3.66803e-19 , displacement DC-sensitivity (m/rHz)

Fig.4 Recombined and Recycled 40m Sensitivities



Contrast and Contrast Defect

Contrast $$C = \frac{P_B - P_D}{P_B + P_D}$$

Contrast defect $$dC = 1 - C$$

Poor contrast $\longrightarrow$ contrast defect

dC - measure of how well two beam interfere at APD at DC-level.

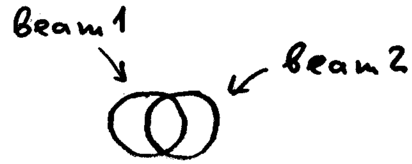
Reasons for poor contrast

1) External (also geometrical)
(match two Beams)

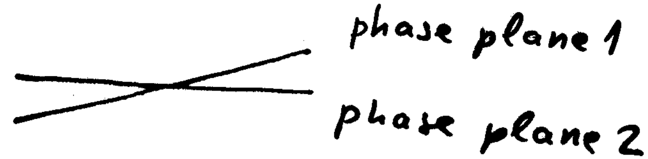
2) Internal.
(If the beams overlap perfectly (?)
they still may not interfere perfectly.)

Mechanisms:

1a) Spatial mismatch



1b) Angular mismatch



2a) Assume 1a + 1b. Beams are different.
power mismatch $A_1 \neq A_2$

2b) Beam sizes are different.
waist positions are different
asymmetry

2c) polarizations are different

2d) modematching differences etc.

Gaussian Beams

$$\vec{E} = \hat{n}\, e^{i(\omega t - kz)} \left[\frac{w_0}{w(z)} e^{-\frac{r^2}{w^2}} \right] e^{i\frac{kr^2}{2R}}\, e^{i\eta(z)}$$

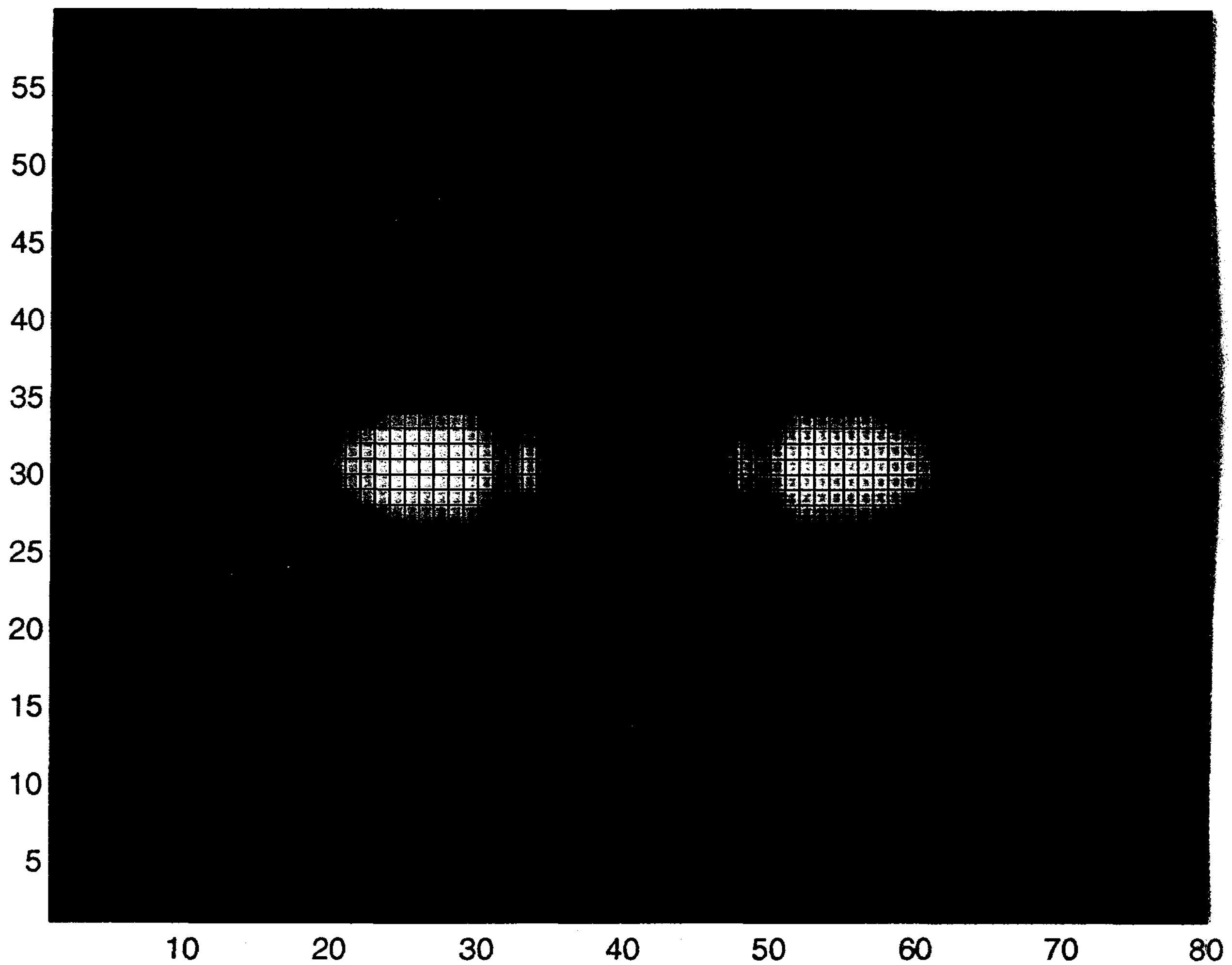
- $\hat{n}$ — polarization
- $e^{i(\omega t - kz)}$ — usual thing
- $\left[\frac{w_0}{w(z)} e^{-\frac{r^2}{w^2}} \right]$ — gaussian profile
- $e^{i\frac{kr^2}{2R}}$ — curvature
- $e^{i\eta(z)}$ — Guyon phase

Interference

$$E(x,y) = E_1(x,y) - E_2(x,y)$$

Contrast defect $\qquad dC = \frac{1}{P_0} \int |E|^2 \, dx\, dy$

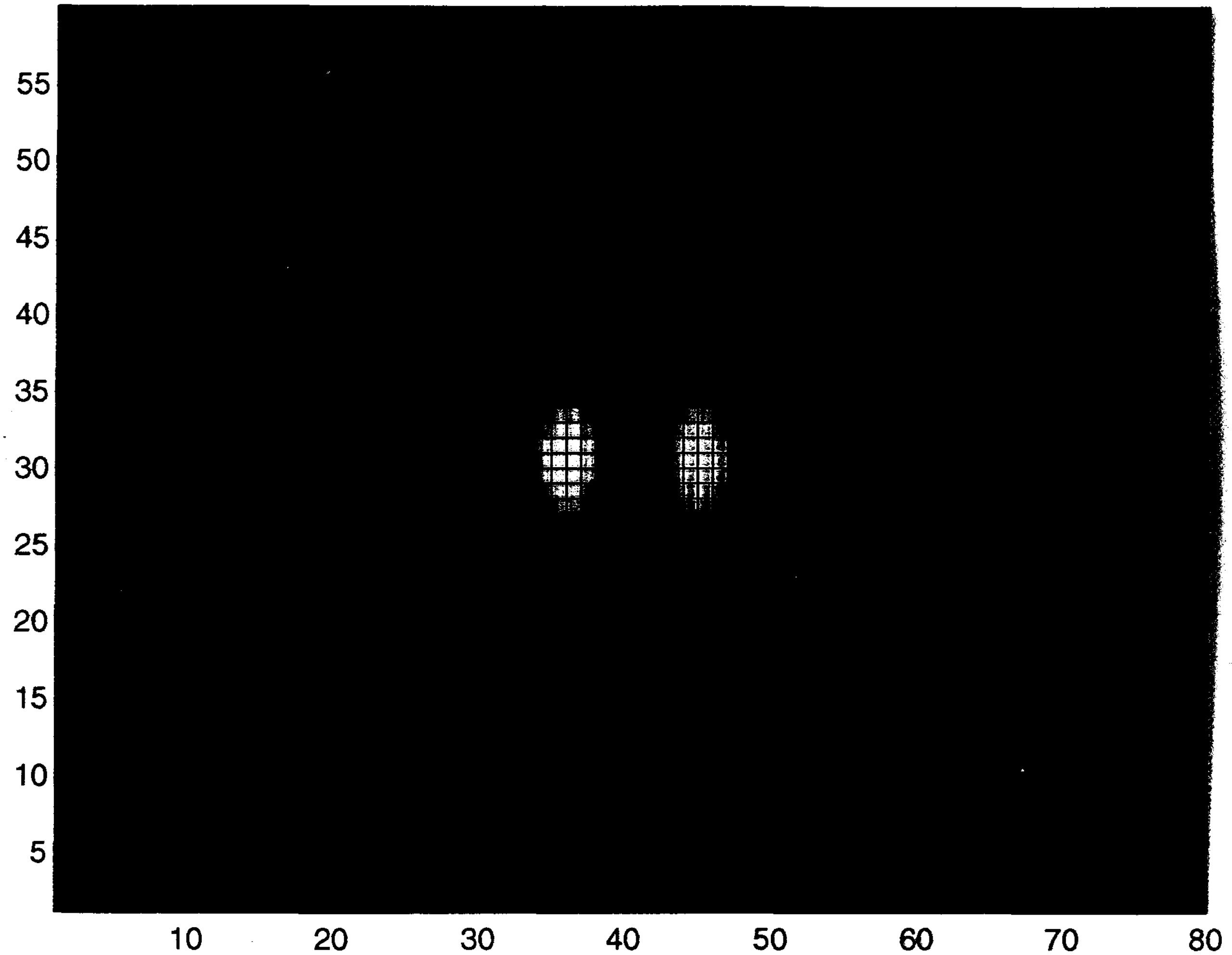
Intensity on Antisymmetric Photodiode. dC = 0.86



Separation = 2 W_0

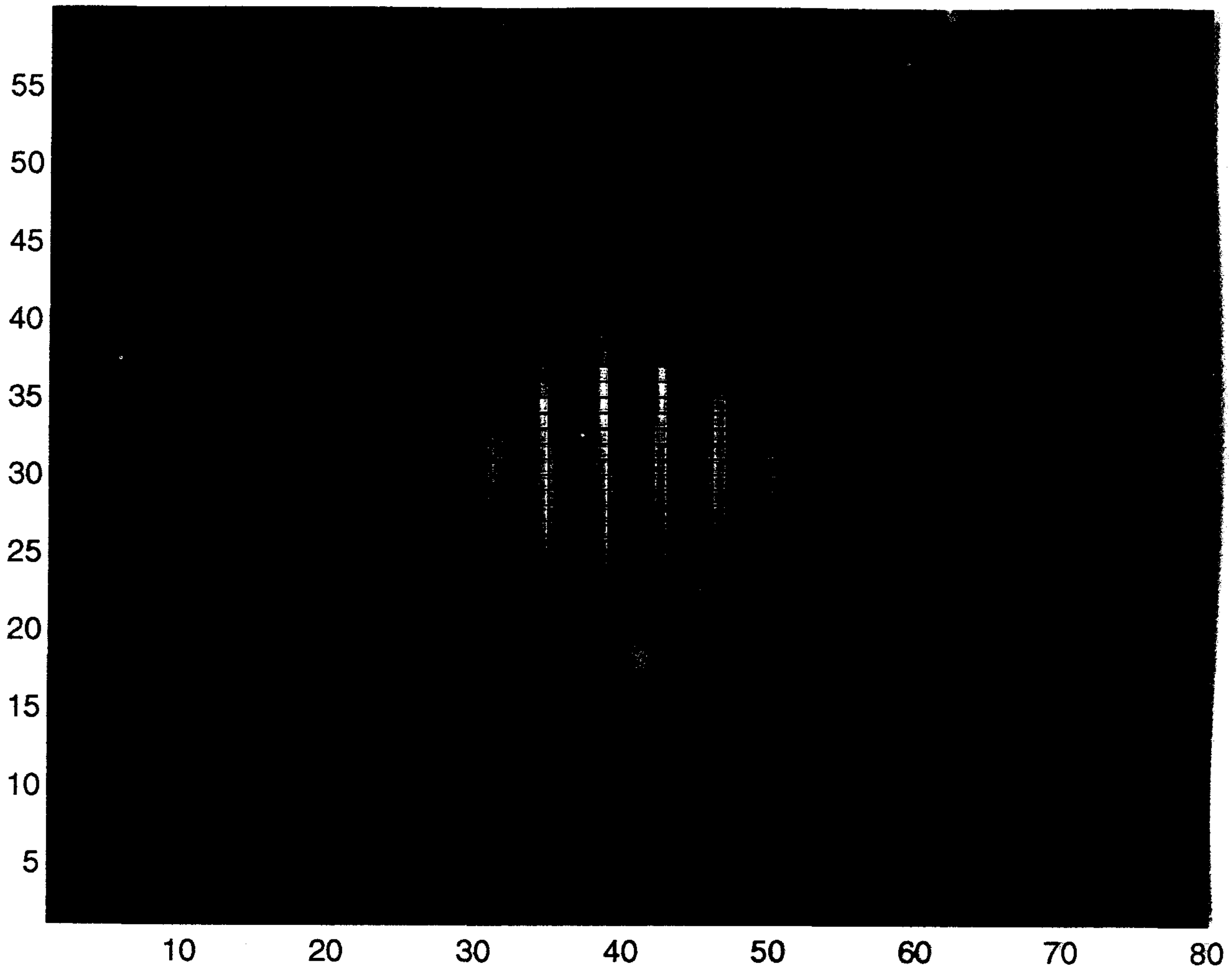
W_0 = 2.2 mm

Intensity on Antisymmetric Photodiode. dC = 0.8631

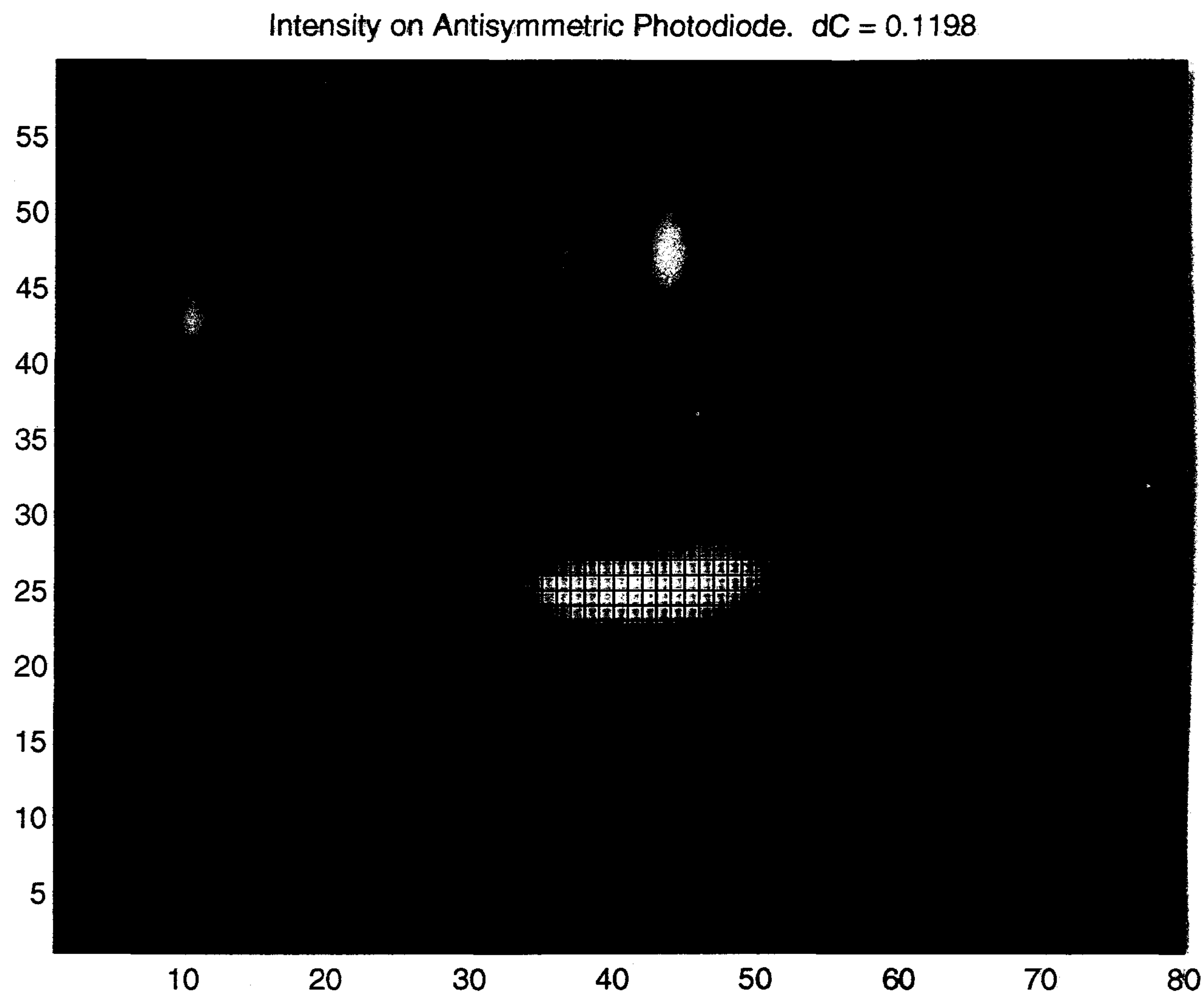


Separation = w_0

Intensity on Antisymmetric Photodiode. dC = 0.8631



$\Theta = 0.1^\circ$



Asymmetry = 0.54 m,

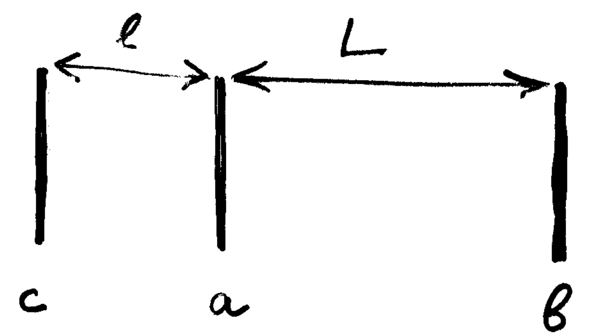
X-separation = 0.02 w_0,

Y-separation = 0.2 w_0,

Ta (Arm1) = 5500 e-6,

Ta (Arm2) = 6000 e-6,

Coupled Cavities



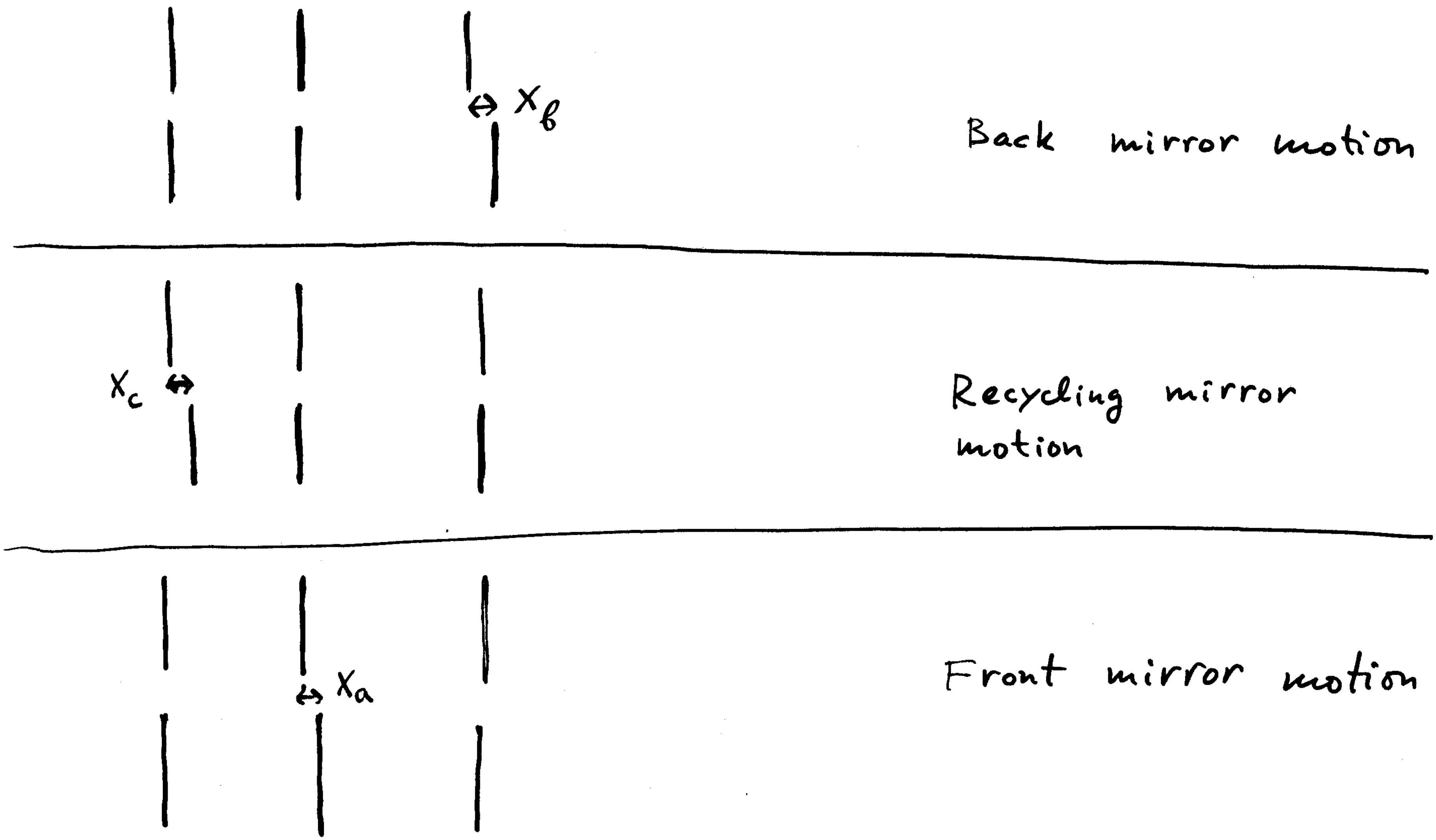
lengths : $\ell = 2.3\ m$, $L = 38.2\ m$ } 40m

Time scales :

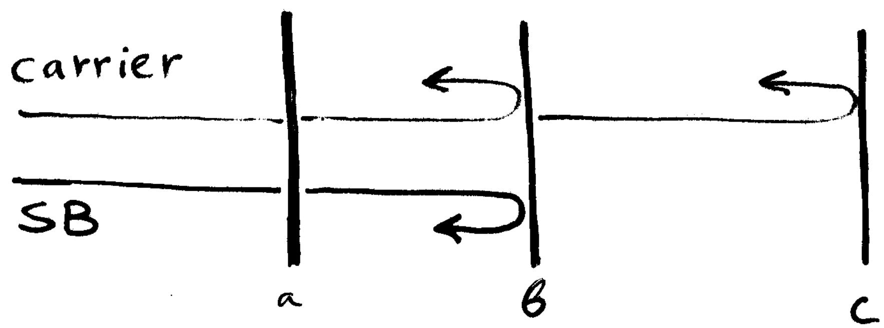
$$T = \frac{L}{c}$$

$$\tau = \frac{\ell}{c}$$

Degrees of Freedom



Resonance Conditions

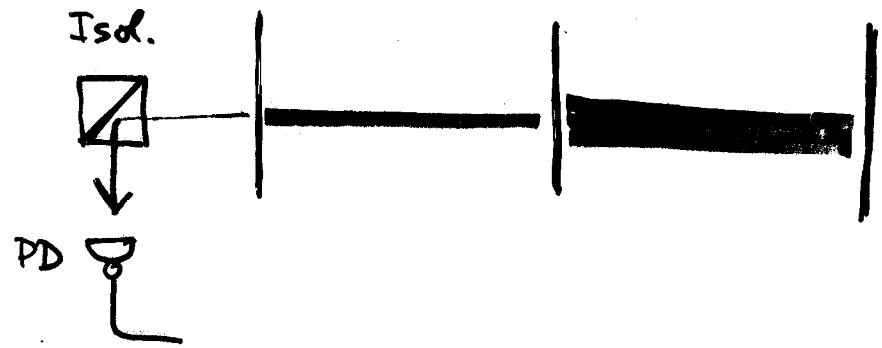


*) Carrier resonates everywhere

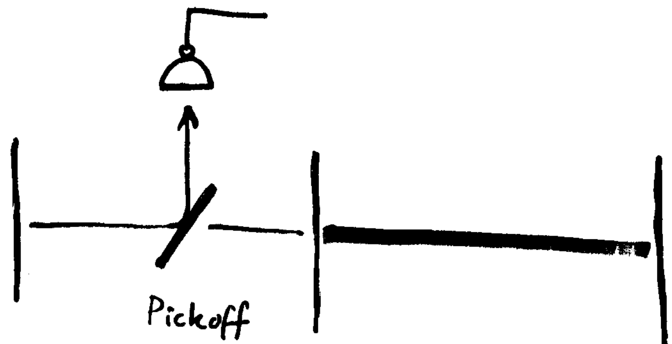
*) SB's resonate in recycling cavity

Photodiode Positions

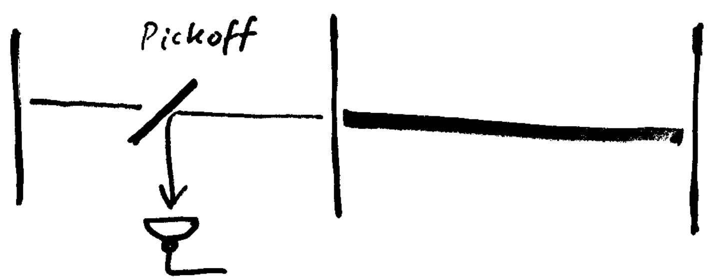
1) Isolator



2) Pick-offs



*



Transfer Function

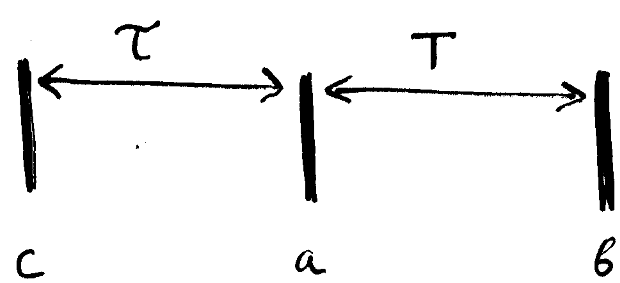
End mirror motion : $x_\ell(t)$

Demod - out : $V(t)$

Transfer Function : $$H(\omega) = \frac{\tilde{V}(\omega)}{2k\,\tilde{x}_\ell(\omega)}$$

Result : $$H(\omega) = \frac{1}{\det[\Delta(\omega)]}$$

$$\Delta(\omega) = \begin{pmatrix} 1 - r_a r_b e^{-2i\omega T} & t_a r_c e^{-2i\omega \tau} \\ t_a r_b e^{-2i\omega T} & 1 + r_a r_c e^{-2i\omega \tau} \end{pmatrix}$$



t's – Amp. transm.

r's – Amp. refl.

$$\det \Delta(\omega) = 1 - r_a r_b e^{-2i\omega T} + r_a r_c e^{-2i\omega\tau} - r_b r_c \left(r_a^2 + t_a^2\right) e^{-2i\omega(T+\tau)}$$

Compare with single cavity

$$A(\omega) = 1 - r_a r_b e^{-2i\omega T}$$

Low frequency expansion

$$e^{-2i\omega T} \approx 1 - 2i\omega T$$

$$e^{-2i\omega \tau} \approx 1 - 2i\omega \tau$$

$$\det \Delta(\omega) = \text{const} \left(1 + i\frac{\omega}{\omega_c}\right)$$

$$\frac{1}{2\pi}\omega_c = 140 \text{ Hz}.$$

compare with single cavity $\frac{\omega_c}{2\pi} = 1860$ Hz

Fig. Coupled Cavity [redacted] TF