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Non-Linear Response of Test Mass to External Forces and Arbitrary Motion of Suspension Point

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Abstract

Non-linear equations for planar motion of a suspended test mass are obtained. We included an arbitrary motion of the suspension point. The dynamics is described in a constrained and unconstrained formalism. In the constraint formalism a general expression for the wire tension is found. We show that it results in non-linearities and couplings between horizontal and vertical degrees of freedom.

Keywords

non-linear dynamics, constraint, test mass, suspension, seismic motion, pendulum, wire tension.

1 Introduction

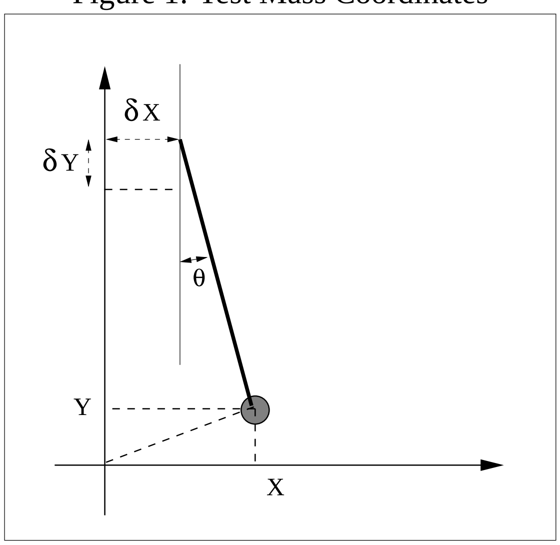
The test masses in LIGO are rigid bodies suspended from wires. They present a dynamical system with many degrees of freedom and constraints. In addition, the test mass is under a constant action of a number of forces. Any approach to study the dynamics of such system must separate the rotational degrees of freedom of the test mass from the motion of its center of mass. This paper focuses on the motion of the center of mass. Linear equations of motion of the center of mass of the suspended test mass were studied in [1]. We obtain a non-linear equation of motion for the center of mass. In most cases of interest the mass moves with a very small amplitude. Thus a complete non-linear dynamics can be substituted by its linearized version. The linear equations obtained this way are only approximations. One can attempt to obtain the linear equations without knowing the underlying non-linear equations. However, finding the corrections to these linear equations, such as cross-couplings, is not easy if one does not have non-linear equations.

The present analysis is the first step in deriving equations for the full dynamics of the test mass. The analysis is relevant for a suspension design and study of seismic noise propagation. The study of the full dynamics of the test mass is on the way, [2].

2 Coordinate System

Fig.1 shows the suspended test mass and the coordinate system. The origin for the coordinate system is chosen at the equilibrium point for the center of mass. That is when the suspension point is not moving and no external forces are applied to the mass the center of mass resides at the origin.

Figure 1: Test Mass Coordinates



The suspension point of the wire moves due to ambient seismic noise. Let $\delta x(t)$ and $\delta y(t)$ be displacements of the suspension point due to the seismic noise. The coordinates of the suspension point in the coordinate system described above are

$$x_{sp}(t) = \delta x(t), \quad (1)$$

$$y_{sp}(t) = L + \delta y(t). \quad (2)$$

Thus the motion of the suspension point is given and does not depend on the test mass motion. As a consequence we know the velocity $(\dot{x}_{sp}, \dot{y}_{sp})$, of the suspension point and its acceleration, $(\ddot{x}_{sp}, \ddot{y}_{sp})$.

3 Unconstraint Dynamics

The position of the test mass is described by two coordinates x and y. These are not independent coordinates. To formulate unconstraint dynamics we need an independent coordinate. Such coordinate can be θ, the angle of deflection of the wire from vertical. The coordinates x and y are expressed in terms of θ as follows

$$x = x_{sp} + l\sin\theta, \quad (3)$$
$$y = y_{sp} - l\cos\theta. \quad (4)$$

The kinetic energy is constructed from velocity of the test mass, which is given by

$$v^2 = v_{sp}^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}(\dot{x}_{sp}\cos\theta + \dot{y}_{sp}\sin\theta). \quad (5)$$

Potential energy is

$$U = mgy.$$

The Lagrangian for the motion of the test mass is found through the standard formula

$$\mathcal{L} = \frac{mv^2}{2} - U \quad (6)$$
$$= \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}(\dot{x}_{sp}\cos\theta + \dot{y}_{sp}\sin\theta) + mgl\cos\theta, \quad (7)$$

We omitted the terms that can be written as a total derivative and thus are irrelevant for the Lagrangian dynamics, see [3].

To obtain the equation of motion for the test mass write Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0,$$

and substitute the Lagrangian into it. Thus we obtain the equation of motion

$$\ddot{\theta} + \omega_0^2\sin\theta = -\frac{1}{l}(\ddot{x}_{sp}\cos\theta + \ddot{y}_{sp}\sin\theta).$$

This is an equation of motion of a non-linear oscillator driven by an external force. The force appears due to the motion of the suspension point.

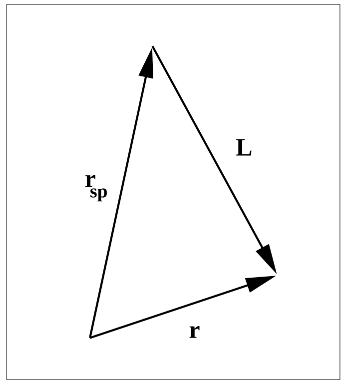
The approach described above has the advantage of dealing with one independent degree of freedom, θ. However, within this formalism it is difficult to include external forces. This is because the external forces are naturally defined in the coordinates parallel and perpendicular to the beam, that is along x and y axes. Transforming these forces into a frame of reference associated with θ is complicated. An alternative is to formulate the dynamics in x, y-coordinates from the beginning. This waywe can easily incorporate the external forces. However, we encounter a different problem. The x and y coordinates of the mass are not independent. There is a constraint. The constraint occurs due to the fixed length of the wire. In the following sections we write the equations of motion in x, y-coordinates and solve the constraint.

4 Constraint Dynamics

4.1 Constraint due to Fixed Wire Length

The length of the wire is constant during the motion on the test mass. This is the constraint of the dynamics. Let the wire be represented by a vector **L**, see Fig.2.

Figure 2: Constraint in Vector Form



Thus the constraint can be written in vector form as

$$\mathbf{r} - \mathbf{r}_{sp} - \mathbf{L} = 0.$$

In terms of components the constraint reads

$$[x - x_{sp}(t)]^2 + [y - y_{sp}(t)]^2 = L^2.$$

We also need a unit vector, $\hat{\mathbf{L}}$, pointed along the wire from the suspension point to the center of mass

$$\hat{\mathbf{L}} = \hat{\mathbf{x}}\ \sin\theta - \hat{\mathbf{y}}\ \cos\theta.$$

4.2 Forces Acting on Test Mass

There are several forces acting on the test mass. These are the wire tension,the gravitational force of Earth, the force due to the coil-magnet actuator and etc. It is not necessary to specify all these forces. The forces have the same effect on the motion of the center of mass. Thus they all can be combined, except the wire tension. The wire tension requires a special treatment because it depends on other forces and also on the motion of mass.

We briefly comment on few forces. The gravitational force is due to the local gravitational acceleration, g. We choose the vertical axis y to be collinear with the vector g. Then the components of the gravitational force are

$$(\mathbf{F}_{gr})_x = 0, \quad (8)$$
$$(\mathbf{F}_{gr})_y = -mg. \quad (9)$$

The actuator force, $\mathbf{F}_{act}$, is due to the coil-magnet system. It should be pointed in the horizontal direction. However, if the coils are unbalanced or magnets are misaligned the actuator force may develop a small component in the vertical direction.

There is a non-zero friction in both x and y directions. It is mostly due to motion of the test mass relative to the suspension point

$$(\mathbf{F}_{fr})_x = -\gamma_x(\dot{x} - \dot{x}_{sp}), \quad (10)$$

$$(\mathbf{F}_{fr})_y = -\gamma_y(\dot{y} - \dot{y}_{sp}). \quad (11)$$

The coefficients of damping in x-direction and y-direction can be very different. The order of magnitude for the damping coefficients is $\gamma/\omega_0 \sim 10^{-3}$.

Let $\mathbf{F}$ be a superposition of all the forces except the wire tension. The total force acting on the mass is $\mathbf{F} + \mathbf{T}$.

4.3 Equation of Motion for Test Mass

The motion of the center of mass of the test mass is described by

$$\begin{cases} m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{T}, \\ \mathbf{r} = \mathbf{r}_{sp} + \mathbf{L}. \end{cases}$$

The wire tension $\mathbf{T}$ is a complicated function of all other forces and the motion of the suspension point. The equations of motion in terms of components are

$$\begin{cases} m\ddot{x} = F_x + T_x, \\ y = y_{sp} + [L^2 - (x - x_{sp})^2]^{1/2}. \end{cases}$$

Note that the vertical coordinate of the test mass does not have its own dynamics. The dynamics of the vertical degree of freedom is defined by the constraint.

4.4 Non-linearities and Wire Tension

All the non-linearities in the dynamics come from the wire tension. In this section we show in details how to calculate the wire tension. This is a typical example of how to solve a constraint in Mechanics.

The wire tension is a vector pointed along the wire

$$\mathbf{T} = -T\,\hat{\mathbf{L}},$$

where $\hat{\mathbf{L}}$ is the unit vector along the wire.

Differentiate the constraint twice to find acceleration of the test mass

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{sp} + \ddot{\mathbf{L}}.$$

Since the length of the vector $\mathbf{L}$ does not change

$$\ddot{\mathbf{L}} = -\frac{(\dot{\mathbf{r}} - \dot{\mathbf{r}}_{sp})^2}{L}\hat{\mathbf{L}} + \mathbf{t},$$

where $\mathbf{t}$ is tangential acceleration, i.e. acceleration perpendicular to the wire. The tangential acceleration does not affect the wire tension and, therefore, is irrelevant for the discussion. Thus we found the acceleration of the mass

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{sp} + \frac{(\dot{\mathbf{r}} - \dot{\mathbf{r}}_{sp})^2}{L}\hat{\mathbf{L}} + \mathbf{t}.$$

The magnitude of the tension can be found from the acceleration of the test mass along the wire

$$T = \mathbf{F} \cdot \hat{\mathbf{L}} - m\,\ddot{\mathbf{r}} \cdot \hat{\mathbf{L}}.$$

From this equation we find the tension

$$T = \mathbf{F} \cdot \hat{\mathbf{L}} - m\left[\ddot{\mathbf{r}}_{sp} \cdot \hat{\mathbf{L}} + \frac{(\dot{\mathbf{r}} - \dot{\mathbf{r}}_{sp})^2}{L}\right].$$

The magnitude of the tension is

$$T = -(m\ddot{x}_{sp} - F_x)\sin\theta + (m\ddot{y}_{sp} - F_y)\cos\theta \tag{12}$$

$$+\frac{m}{L}[(\dot{x} - \dot{x}_{sp})^2 + (\dot{y} - \dot{y}_{sp})^2]. \tag{13}$$

Thus the tension is the function of all other forces, the acceleration of the suspension point, the velocity of the test mass relative to the suspension point, and the deflection angle. In the constraint dynamics the deflection angle θ is redundant. We have to eliminate it. The two trigonometric functions $\sin\theta$, $\cos\theta$ can be eliminated using the equations

$$\sin\theta = \frac{x - x_{sp}}{L}, \qquad \cos\theta = -\frac{y - y_{sp}}{L}. \tag{14}$$

Thus we obtained the general expression for the magnitude of the wire tension:

$$T = -\frac{1}{L}(m\ddot{x}_{sp} - F_x)(x - x_{sp}) - \frac{1}{L}(m\ddot{y}_{sp} - F_y)(y - y_{sp}) \tag{15}$$

$$+\frac{m}{L}[(\dot{x} - \dot{x}_{sp})^2 + (\dot{y} - \dot{y}_{sp})^2]. \tag{16}$$

This expression shows that the dynamics is non-linear. We can also obtain couplings between x and y motion from this expression.

Only x-component of the tension enters the equation of motion for the test mass

$$T_x = -T\,\frac{x - x_{sp}}{L}.$$

The expansion of the wire tension

$$T_x = -m\omega_0^2(x - x_{sp}) + \ldots$$

shows that the dominant part of wire tension generates the usual pendulum motion. The dots stand for the non-linear corrections. The frequency of the pendulum motion is $\omega_0^2 = g/L$.

The equation of motion for the test mass is

$$m\ddot{x} = F_x - T\,\frac{x - x_{sp}}{L},$$

where T is given by eq.(15). The non-linearities come from the wire tension.

The equation of motion and the equation for wire tension conclude the analysis of the non-linear dynamics. The linearized equations can be easily obtained from the above non-linear equations.

References

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