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Thermal noise in coupled harmonic oscillators

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ABSTRACT

LIGO I uses a single pendulum arrangement to suspend its test masses, and there are good reasons to consider moving to a triple-pendulum scheme for LIGO II. I calculated the thermal noise in a triple pendulum to see how it differed from the single-pendulum case and found the triple pendulum to be noisier than the single pendulum. However, the two cases are asymptotically the same at high frequencies, where the thermal noise of the triple pendulum is dominated by its last stage. The increase in thermal noise does not appear to be a problem with expected LIGO II parameters, since the expected single and triple pendulum thermal noises are both less than the expected mirror thermal noise above the expected seismic wall frequency, $10Hz$.

The method I used for calculating the thermal noise is extended to a chain of harmonic oscillators with an arbitrary number of elements, and some general observations can be made about the thermal noise in such a chain. A reasonably simple method of calculating the trasfer function between any two elements is also presented.

KEYWORDS

Thermal noise; LIGO II; triple pendulum

1 Introduction

The test masses in LIGO I are suspended as simple, one-stage pendula, and we would like to move to a triple-pendulum design for LIGO II, much like what is used in GEO. The idea is that the triple pendulum design greatly improves the seismic isolation of the test mass. I was curious to see if the thermal noise would change when the single pendulum was replaced by a triple pendulum, so I did a simple, one dimensional calculation to compare the two cases.

This excercise revealed to me a number of useful things about thermal noise in multimode systems. There is quite a lot one can say in general about the behavior of the thermal noise in the 'tails' far from any resonances, regardless of the relative Q's of each stage or the total number of stages.

I turns out that the thermal noise gets worse when you move from a single pendulum to a triple pendulum. This is probably not going to be a problem for LIGO II, since it appears to happen at frequencies lower than the anticipated seismic wall.

Throughout this paper I will use standard notation: ϕ_i is the loss angle for the i-th stage, $Q_i \equiv 1/\phi_i$, etc. ω_0 is the resonant frequency of one stage in isolation, i.e. not coupled to any other stages.

2 General method for 1d

2.1 Thermal noise

According to the fluctuation dissipation theorem, the thermal noise of one mass in a chain of masses is

$$x_{th}^2 = \frac{4k_B T}{\omega^2} \mathrm{Re}\left\{Y_n\right\}.$$

Where $Y_n = v_n/f_n$ is the mechanical admittance of mass n, v_n is the velocity of the n-th mass in the chain in response to a driving force f_n, applied directly to that mass. For a sinusoidal driving force, the admittance is just

$$Y_n = \frac{i\omega x_n}{f_n},$$

where x_n is the displacement of that mass from its equilibrium point.

The mechanical admittance is straightforward to calculate for the case of a 1d chain of harmonic oscillators. The equations of motion for N coupled oscillators, with a force applied to each one, can be written in matrix form and solved for x_n. The equation of motion for the n-th mass is

$$-m\omega^2 x_n = -k(1+i\phi_n)(x_n - x_{n-1}) + k(1+i\phi_{n+1})(x_{n+1} - x_n) + f_n,$$

which can be rearranged to

$$-\left(\frac{\omega}{\omega_0}\right)^2 x_n = (1+i\phi_n)x_{n-1} - (2 + i(\phi_n + \phi_{n+1}))x_n + (1+i\phi_{n+1})x_{n+1} + \frac{f_n}{m\omega_0^2},$$

where $\omega_0^2 \equiv k/m$.

The equations of motion can be written quite simply in matrix form

$$-\left(\frac{\omega}{\omega_0}\right)^2 \mathbf{x} = \mathbf{M}\mathbf{x} + \frac{1}{m\omega_0^2}\mathbf{f}.$$

We can solve for the positions of the masses by simple linear algebra.

$$\mathbf{x} = -\frac{1}{m\omega_0^2}\mathbf{\Omega}^{-1}\mathbf{f},$$

where

$$\mathbf{\Omega} \equiv \mathbf{M} + \left(\frac{\omega}{\omega_0}\right)^2 \mathbf{I}.$$

If we only apply a force to one mass, then the position of that mass will be determined by a single entry on the diagonal of the matrix $\mathbf{\Omega}^{-1}$, and that entry is given by a cofactor divided by the determinant of $\mathbf{\Omega}$.

$$\begin{aligned} x_n &= -\left[\mathbf{\Omega}^{-1}\right]_{nn} \frac{f_n}{m\omega_0^2} \\ &= -\frac{C_{nn}}{|\mathbf{\Omega}|}\frac{f_n}{m\omega_0^2} \end{aligned}$$

Finding the position of one mass with a force applied to just that mass then reduces to a calculation of two determinants, one of an $(N-1)\times(N-1)$ matrix (the cofactor C_{nn}), and the other of an $N\times N$ matrix ($\mathbf{\Omega}$). These determinants are not too difficult if the losses are small and you only retain first order terms in each ϕ_i.

2.2 Transfer functions

Transfer functions are even easier, since they only involve taking determinants of $(N-1)\times(N-1)$ matrices.

$$\frac{x_m}{x_n} = \frac{C_{mn}}{C_{nn}}$$

3 Result for 3 stages

For a three mass system as described above, the thermal noise in the last mass is

$$\begin{aligned} x_{th}^2 &= \frac{4k_BT}{m\omega_0^2\omega} \\ &\times \frac{\phi_3\zeta^8 - 6\phi_3\zeta^6 + (\phi_2 + 11\phi_3)\zeta^4 - 2(\phi_2 + 3\phi_3)\zeta^2 + (\phi_1 + \phi_2 + \phi_3)}{\left[\zeta^6 - 5\zeta^4 + 6\zeta^2 - 1\right]^2 + \left[(\phi_1 + 2\phi_2 + 2\phi_3)\zeta^4 - (3\phi_1 + 4\phi_2 + 5\phi_3)\zeta^2 + (\phi_1 + \phi_2 + \phi_3)\right]^2}, \end{aligned}$$

where I have defined $\zeta \equiv \omega/\omega_0$ so that the formula would fit on the page.

4 Some useful observations

4.1 Asymptotic behavior

At high frequencies (well above the highest resonance), the thermal noise falls off as $\omega^{-5/2}$, just like a single oscillator, and it is dominated by the last stage, regardless of the relative values of the loss angles in each stage.

$$\omega >> \omega_0 \Rightarrow x_{th}^2 \approx \frac{4k_B T}{m\omega^5}\omega_0^2\phi_3$$

At low frequencies, the thermal noise goes as $\omega^{-1/2}$, also just like a single oscillator, but here the losses in each stage contribute equally.

$$\omega << \omega_0 \Rightarrow x_{th}^2 \approx \frac{4k_B T}{m\omega_0^2\omega}(\phi_1 + \phi_2 + \phi_3)$$

It is fairly easy to show that this scaling behavior ($\omega^{-1/2}$ at low frequencies, $\omega^{-5/2}$ at high frequencies) is independent of the number of stages in the system. I suspect that the loss-dependencies (all losses at low frequencies, last stage only at high frequencies) are also independent of the number of stages, but I haven't carried through the proof yet.

4.2 Peaks and modal Q's

The thermal noise has peaks at each of the normal modes, given by the roots of the loss-independent polynomial in the denominator of the thermal noise formula. For the three stage system treated above: $(\omega/\omega_0)^6 - 5(\omega/\omega_0)^4 + 6(\omega/\omega_0)^2 - 1 = 0$, which gives

$$\begin{aligned}\omega &= 0.45\omega_0 \\ \omega &= 1.24\omega_0 \\ \omega &= 1.80\omega_0\end{aligned}$$

This result, that the location of the peaks in the thermal noise is the same as the normal modes of the system, is trivial to prove in the general case of any number of stages.

If the loss in the first stage is much greater than the losses in subsequent stages ($\phi_1 >> \phi_2$,ϕ_3), then the heights of the thermal noise peaks are suppressed by the first lossy stage. For our three stage system, the heights of the peaks would be, for a lossy first stage

$$x_{th}^2 = \frac{4k_B T}{m\omega_0^3}\frac{Q_1}{[\left(\frac{\omega}{\omega_0}\right)^4 - 3\left(\frac{\omega}{\omega_0}\right)^2 + 1]^2}.$$

For comparison, the height of the peak in the thermal noise of a single oscillator is

$$x_{th}^2 = \frac{4k_B T}{m\omega_0^3}Q$$

The weighting polynomial in the denominator is 0.22 for the lowest frequency mode, and 1.54 and 3.17 for the other two.

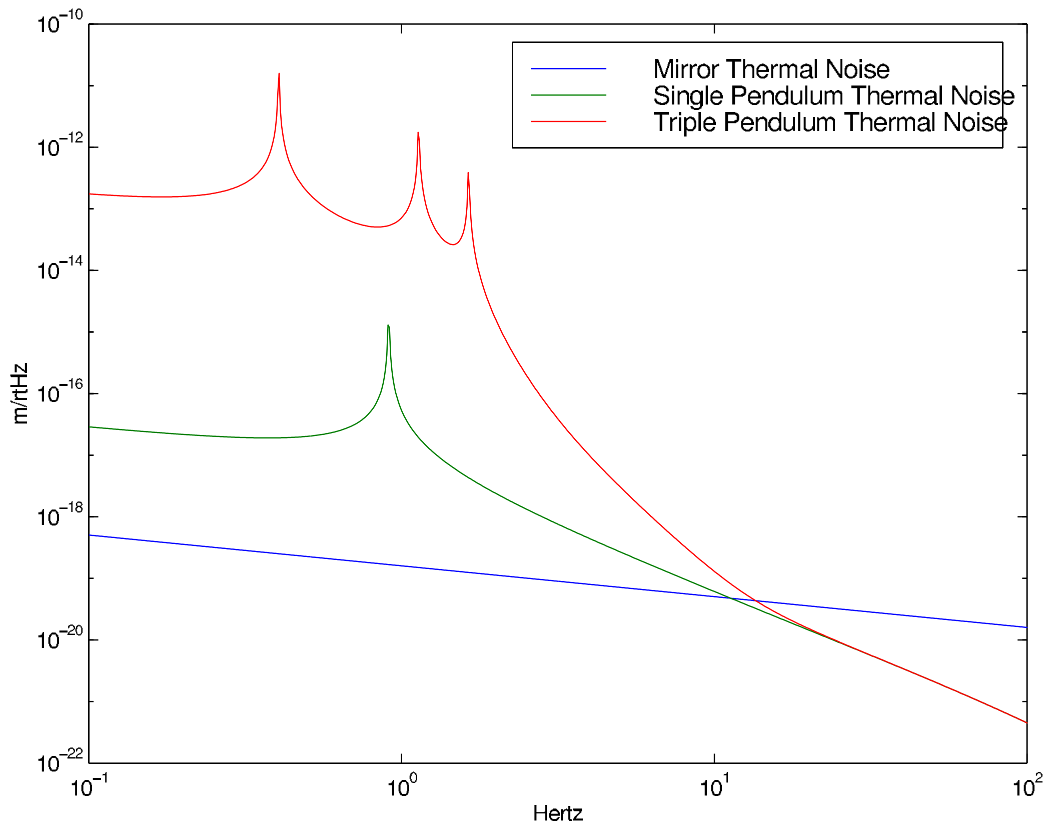

Figure 1: Thermal noise for a triple pendulum versus that of a single pendulum. The single pendulum is identical to the last stage of the triple pendulum. Parameters are those presented by the suspensions working group at the MIT design summit, May 13-16, 1999 for evaluating advanced seismic isolation schemes. Also included is an estimate of the internal mirror noise, also from the design summit. Note that at low frequencies the triple pendulum is much noisier than the single pendulum.

It is important to note that this weighting factor depends only on the number of stages in the system, not on the relative values of the Q's of each stage. This indicates that measuring the Q's of the individual modes would probably only give you information about the lossiest element, and about the number of resonances. It wouldn't tell you anything about what to expect at high frequencies, where the thermal noise is dominated by the last stage. This is an example of how measuring the modal Q's and summing over independent normal modes would get you into trouble.

5 An example

As an example of the above procedure, I estimated the pendulum thermal noise in a GEO-style triple pendulum, using the parameters provided on the suspensions working group website. Figure 1 shows the thermal noise of a triple pendulum, compared with that of a single pendulum identical to the last stage of the triple pendulum.

The single-stage resonance is split into three normal mode resonances by the coupling between the stages, and two of the modes have resonant frequencies higher than ω_0. This makes the triple pendulum noise much greater than that of the single pendulum below about $10Hz$.

6 Conclusion

Going from a single pendulum to a triple pendulum increases the thermal noise at low frequencies. Well above the highest mode frequency, the single and triple pendulum thermal noises are essentially the same. The increased thermal noise of the triple pendulum does not appear to be a problem for LIGO II, since the greatest effect occurs at frequencies below $10Hz$, the expected seismic wall.

APPENDIX I
Matlab code for a triple pendulum

The following is a modification of the matlab code (ReqtsSummit.m) developed by the suspensions working group and posted on its website, (http://fiji.nirvana.phys.psu.edu/s̃wg/) for evaluating advanced seismic isolation schemes.

```
%SWG summit tool to evaluate designs.
%Modified to include a triple pendulum June 8, 1999.

f=logspace(-1,2,5e4);
w=2*pi*f;

%parameters

kBT = 1.38e-23*290;


%internal modes thermal noise
%scale initial LIGO just with Q
Q = 3e7/2.9;         %PRS correction to internal noise
f0 = 1e4;
M=11;
xIM = 4*kBT/(8*pi^3*Q*M*f0^2);
xIM = sqrt(4.5*xIM./f);


%pendulum thermal noise
M=30; %mass
Y=4e8;  %breaking stress
rho=2.2e3; %density
c=772; %thermal conductiviy
k=1.38; %thermal transmission?
T=290; %temperature
alpha=5.1e-7; %linear expansion
E=7e10; %Young's modulus
g=9.8;

%ribbon cross section for 1/3 of yield stress in each of 4 fibers
tB=M*g/(4*3*Y);
t=1e-4; %reasonable thickness
B=tB/t;
tau=rho*c*t^2/(pi^2*k); %thermoelastic relaxation time
nabla=E*alpha^2*T/(rho*c); %thermoelastic peak amplitude
phi=nabla*w*tau./(1+(w*tau).^2);
phi=4e-8+phi;

l=.30;
w0=sqrt(g/l);
```

```
I=B*t^3/12;
Qp=(1./phi)*M*g*l/(4*sqrt(M*g*E*I/4));
phip=1./Qp;

%triple pendulum upper stage losses
phi1 = 10e-4;
phi2 = 10e-6;
phi3 = phip;

%calculate triple pendulum thermal noise

alice = phi3.*((w./w0).^8) - ...
        6.*phi3.*((w./w0).^6) + ...
        (phi2 + 11.*phi3).*((w./w0).^4) -...
        2.*(phi2 + 3.*phi3).*((w./w0).^2) + ...
        (phi1 + phi2 + phi3);

betty = ((w./w0).^6 - 5.*(w./w0).^4 + 6.*(w./w0).^2 - 1).^2 +...
        ((phi1 + 2.*phi2 + 2.*phi3).*(w./w0).^4 -...
        (3.*phi1 + 4.*phi2 + 5.*phi3).*(w./w0).^2 +...
        (phi1 + phi2 + phi3)).^2;

xTrip=sqrt((4*kBT./(M*w0^2*w)).*alice./betty);

xPend=sqrt(4*kBT*w0^2*phip./(M*w.*(w0^4*phip.^2+(w.^2-w0^2).^2)));

xReq=sqrt(xPend.^2+xIM.^2);

loglog(f,xIM,f,xPend,f,xTrip)
legend('Mirror Thermal Noise','Single Pendulum Thermal Noise','Triple Pen-
dulum Thermal Noise')
%grid on
axis([1e-1 1e2 1e-22 1e-7])
xlabel('Hertz')
ylabel('m/rtHz')
shg
```