Thermodynamical fluctuations in optical mirror coatings

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Thermodynamical (TD) fluctuations of temperature in mirrors may produce surface fluctuations not only through thermal expansion in mirror body but also through thermal expansion in mirror coating. We analyze the last "surface" effect which can be larger than the first "volume" one due to larger thermal expansion coefficient of coating material and smaller effective volume.

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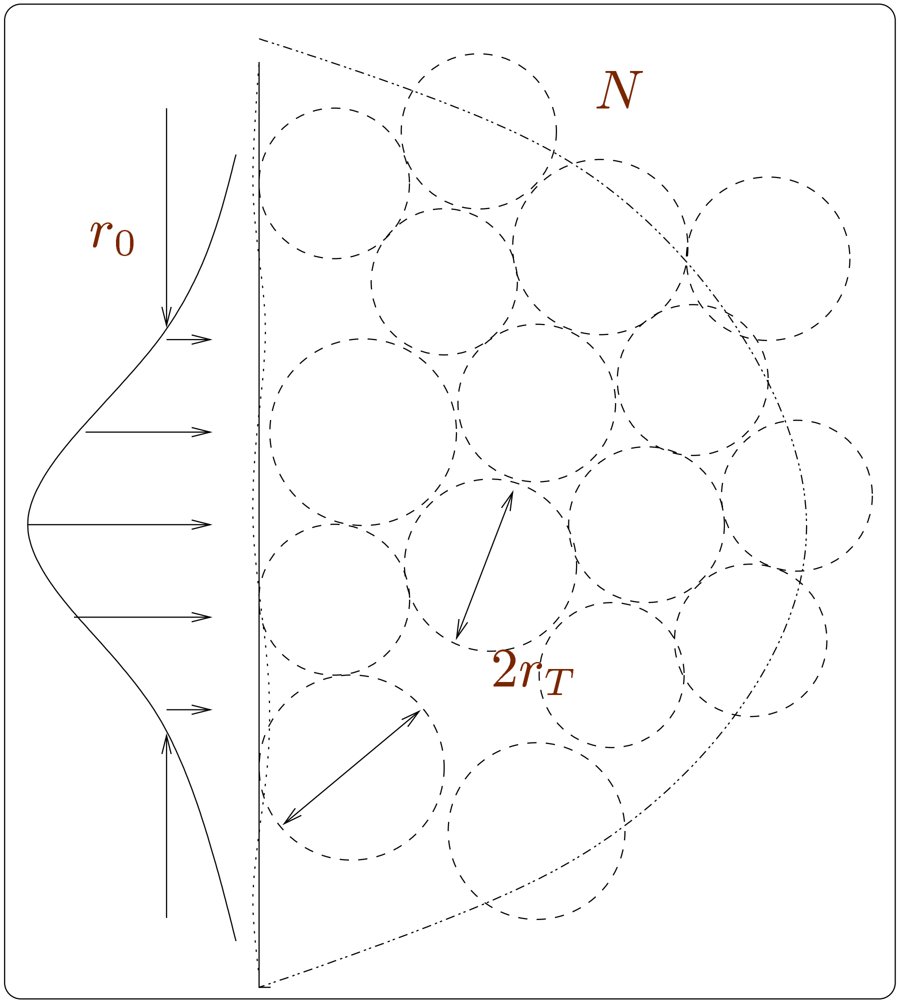
"Bulk" TD fluctuations.

Semi-qualitative consideration



$$
\begin{aligned}
r_{\mathrm{T}} &= \sqrt{\frac{\kappa\,\tau}{C_V}}, \\
r_{\mathrm{T}} &\ll r_0, \\
\tau &\simeq 0.01\,\mathrm{s}\ (\mathrm{LIGO}) \\
r_{\mathrm{T}}^{\mathrm{SiO_2}} &\simeq 2\times 10^{-3}\ \mathrm{cm}, \\
r_{\mathrm{T}}^{\mathrm{Al_2O_3}} &\simeq 7\times 10^{-3}\ \mathrm{cm}
\end{aligned}
$$

Variation of averaged displacement ΔX_{r_0}:

$$\Delta X_{r_0}^2 \simeq \underbrace{\alpha_b^2 r_T^2 \times \frac{k_B T^2}{C_V r_T^3}}_{\Delta X_{r_T}^2} \times \underbrace{\frac{r_T^3}{r_0^3}}_{\frac{1}{N}} \simeq \frac{\alpha_b^2 k_B T^2 \kappa\tau}{C_V^2 r_0^3}$$

$k_{\rm B}$ — Bolzmann constant, T is temperature, κ is thermal conductivity and C_V is specific heat capacity per unit volume.

Spectral density of displacement $\bar{X}$

Langevin or FDT approach (ν is Poisson ratio):

$$S_{\rm bulk}^{\rm TD}(\omega) = \frac{8}{\sqrt{2\pi}} \frac{\alpha_b^2 k_{\rm B} T^2 (1+\nu)^2 \kappa}{C_V^2 r_0^3 \omega^2},$$

$$\Delta X_{r_0}^2 \sim S_{\rm bulk}^{\rm TD}\big|_{\omega\simeq 1/\tau} \times \frac{1}{\tau}$$

TD temperature fluctuations in thin layer

(only expansion coefficients differs)

$$r_0 \gg r_T \gg d \tag{1}$$

d is thickness of layer.

Semiqualitative consideration:

"Surface" contribution $\alpha \Rightarrow \alpha_l - \alpha_b$:

$$\Delta T \simeq \sqrt{\frac{k_\mathrm{B} T^2}{C_V r_\mathrm{T}^3}} \times \sqrt{\frac{r_\mathrm{T}^2}{r_0^2}} = \sqrt{\frac{k_\mathrm{B} T^2}{C_V r_0^2 r_\mathrm{T}}}, \tag{2}$$

$$\overline{X}_d \simeq \alpha \Delta T d \simeq \alpha d \sqrt{\frac{k_\mathrm{B} T^2}{C_V r_0^2 r_\mathrm{T}}}. \tag{3}$$

Spectral density of displacement $\bar{X}$

(Langevin or FDT approach):

$$S_{\Delta\mathrm{T}}(\omega) \simeq \frac{\sqrt{2}k_{\mathrm{B}}T^2}{\pi r_0^2\sqrt{C_V\kappa\omega}} = \frac{\sqrt{2}k_{\mathrm{B}}T^2}{\pi r_0^2 C_V r_{\mathrm{T}}\omega} \tag{4}$$

$$S_{\mathrm{layer}}^{\mathrm{TD}}(\omega) = \frac{4\sqrt{2}(1+\nu)^2}{\pi}\frac{\alpha^2 d^2 k_{\mathrm{B}}T^2}{r_0^2\sqrt{\kappa C_V\omega}}, \tag{5}$$

$$\overline{X}_d^2 \sim S_{\mathrm{layer}}^{\mathrm{TD}}\Big|_{\omega\simeq 1/\tau} \times \frac{1}{\tau} \tag{6}$$

Different material parameters

of layer and substrate:

$$S^{\mathrm{TD}}_{\mathrm{layer}}(\omega) = \frac{4\sqrt{2}(1+\nu_b)^2}{\pi} \frac{\alpha^2 d^2 k_{\mathrm{B}} T^2}{r_0^2 \sqrt{\kappa_b C_{V,b} \omega}}, \tag{7}$$

$$\alpha = \alpha_b \Lambda, \tag{8}$$

$$\Lambda = -\frac{C_{V,\,l}}{C_{V,\,b}} + \frac{\alpha_l}{2\alpha_b} \times \tag{9}$$

$$\times \left[\frac{1+\nu_l}{(1-\nu_l)(1+\nu_b)} + \frac{E_l(1-2\nu_b)}{E_b(1-\nu_l)} \right].$$

Here E_b, E_l are Young modulus,

subscripts $_b$ — "bulk", $_l$ — "layer".

Many thanks to Martin Fejer for collaboration.

Multilayer coating

N alternating layers, $d_1 = \lambda/4n_1$ and $d_2 = \lambda/4n_2$:

$$
\begin{aligned}
d &= N(d_1 + d_2), && (10)\\
\alpha &= \frac{\alpha_b\, d_1}{d_1 + d_2}\,\Lambda_1 + \frac{\alpha_b\, d_2}{d_1 + d_2}\,\Lambda_2 && (11)\\
\Lambda_1 &= -\frac{C_{V,\,l,\,1}}{C_{V,\,b}} + \frac{\alpha_{l,\,1}}{2\alpha_b} \times && (12)\\
&\times \left[\frac{1 + \nu_{l,\,1}}{(1 - \nu_{l,\,1})(1 + \nu_b)} + \frac{E_{l,\,1}(1 - 2\nu_b)}{E_b(1 - \nu_{l,\,1})}\right],\\
\Lambda_2 &= -\frac{C_{V,\,l,\,2}}{C_{V,\,b}} + \frac{\alpha_{l,\,2}}{2\alpha_b} \times && (13)\\
&\times \left[\frac{1 + \nu_{l,\,2}}{(1 - \nu_{l,\,2})(1 + \nu_b)} + \frac{E_{l,\,2}(1 - 2\nu_b)}{E_b(1 - \nu_{l,\,2})}\right],
\end{aligned}
$$

Finite sized mirror.

"Bimetallic" effect: additionall mirror's bend through thermal expansion.

Using FDT approach developed by Liu and Thorne we have calculated numerically

$$C_{\text{fsm}} = \frac{S_{\text{layer}}^{\text{TD, finite test mass}}}{S_{\text{layer}}^{\text{TD, infinite test mass}}}. \tag{14}$$

Estimate for cylindrical test mass manufactured from fused silica with $Ta_2O_5 + SiO_2$ coating with radius $R = 19.4$ cm, height $H = 11.5$ cm and beam radius $r_0 = 6$ cm (LIGO-II):

$$C_{\text{fsm}} \simeq 1.56$$

But if $R \gg H$ then the value of C_{ftm} may be substantialy larger.

Motivation of measurement of thermal expansion of Ta_2O_5:

In existing publications there is very wide range for value of $\alpha_{Ta_2O_5}$ of thin layer:
from

$$\alpha_{\mathrm{Ta_2O_5}} \simeq -(4.43 \pm 0.05) \times 10^{-5}\ \mathrm{K}^{-1} \quad \text{(Inci)}$$

to

$$\alpha_{\mathrm{Ta_2O_5}} \simeq 3.6 \times 10^{-6}\ \mathrm{K}^{-1}, \quad \text{(Tien, Jaing, Lee, Chuang)}$$

It can be explained by not only possible experiment errors but also by the fact that properties of tantalum pentoxide may be strongly depends on procedure of layer deposition on substrate.

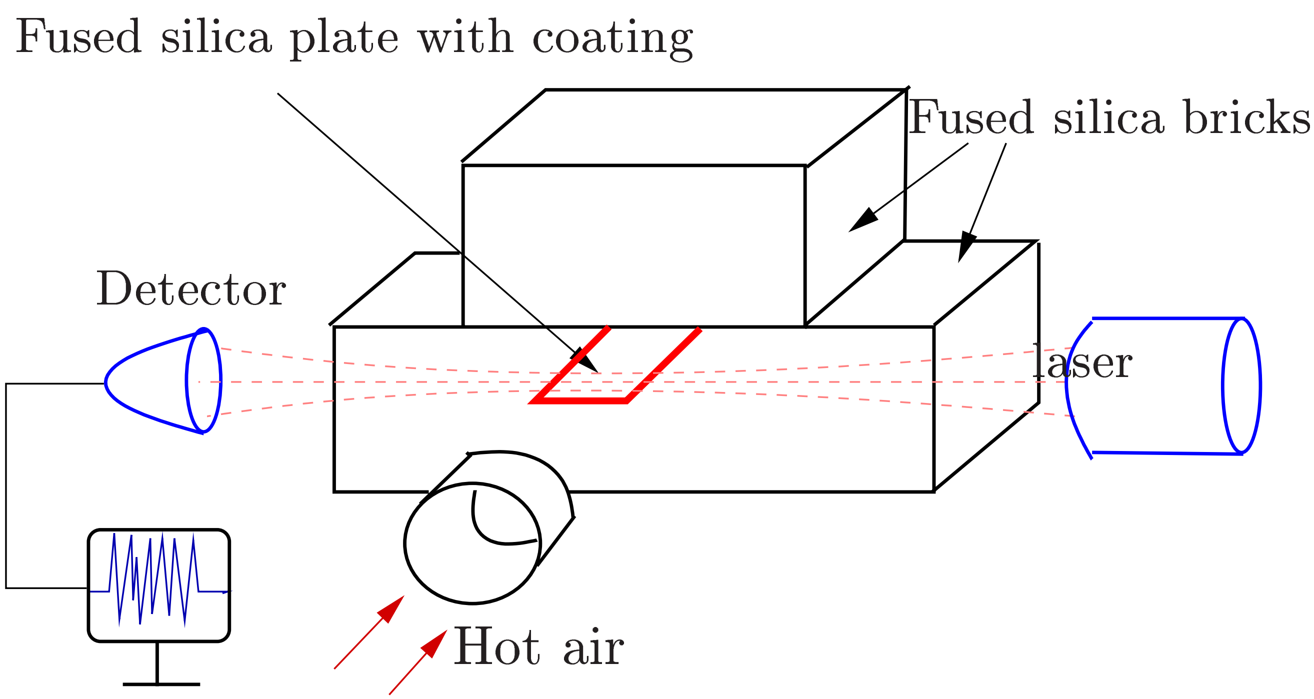
It was the motivation for us to produce independent measurement of thermal expansion of Ta_2O_5.

Measurement of $\alpha_{\mathrm{Ta_2\,O_5}}$

We measured the bend of thin (about 100 μm) substrate $\mathrm{Si\,O_2}$ plate with $Ta_2\,O_5 + Si\,O_2$ coating, when plate is heated.
The heating was produced by hot air.
The bending appears due to the difference $\alpha_{\mathrm{Ta_2\,O_5}} - \alpha_{\mathrm{Si\,O_2}}$.
The bending was measured by laser beam (knife and slot scheme).

Important: the coating was deposited by *the same technology* as coating of LIGO mirrors.
(Many thanks to Helena Armandula.)

Scheme of measurement



Result of measurement of $\alpha_{\mathrm{Ta_2\,O_5}}$:

$$\alpha_{\mathrm{Ta_2\,O_5}} = \left(5 \pm 1\right) \times 10^{-6}\,\mathrm{K}^{-1}$$

Taking into account that Young modulus $E_{Ta_2O_5}$ may substantialy depend on the deposition procedure (as weel as value of $\alpha_{Ta_2O_5}$) it is better to use for estimates the more wide confidence limits:

$$\alpha_{\mathrm{Ta_2\,O_5}} = \left(5 \pm 2\right) \times 10^{-6}\,\mathrm{K}^{-1}$$

Thermorefractive noise:

TD temperature fluctuations produce fluctuations of refraction indices n_1 and n_2 in coating:

$$\beta_1 = \frac{dn_1}{dT} \neq 0, \quad \beta_2 = \frac{dn_2}{dT} \neq 0,$$

$$S_{\rm trefr}^{\rm TD}(\omega) = \frac{\sqrt{2}\beta^2\lambda^2\kappa T^2}{\pi r_0^2\sqrt{\rho C\kappa\omega}}, \qquad (15)$$

$$\beta = \frac{n_1 n_2(\beta_1+\beta_2)}{4(n_1^2-n_2^2)}. \qquad (16)$$

For estimates we use "optimistic" (smallest) value of $\beta_{\rm Ta_2O_5} \simeq 2.3\times 10^{-6}$ K^{-1} (Chu, Lin, Cheng). For example, Inci reports unexpectedly high value $\beta_{\rm Ta_2O_5} \simeq 1.4\times 10^{-4}$ K^{-1}.

Structural losses (Brownian)

Loss angle ϕ does does not depend on frequency:

$$S_x^{\mathrm{B}}(\omega) \simeq \frac{4k_{\mathrm{B}}T}{\omega}\,\frac{(1-\nu^2)}{\sqrt{2\pi}\,Er_0}\,\phi. \tag{17}$$

We use new (*smallest*) value

$$\phi \simeq 0.5 \times 10^{-8}$$

for very pure fused silica obtained by Ageev, Penn and Saulson.

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Recalculation

to the perturbation of metric $h(\omega)$:

$$h(\omega) = \frac{\sqrt{N_0 S_x(\omega)}}{L} \tag{18}$$

We use $N_0 = 4$,

for TD layer fluctuations – $N_0 \simeq 2$ (only two end mirrors are taken into account).

The standard quantuim limit (SQL):

$$h_{\rm SQL} = \sqrt{\frac{8\hbar}{m\omega^2 L^2}}. \tag{19}$$

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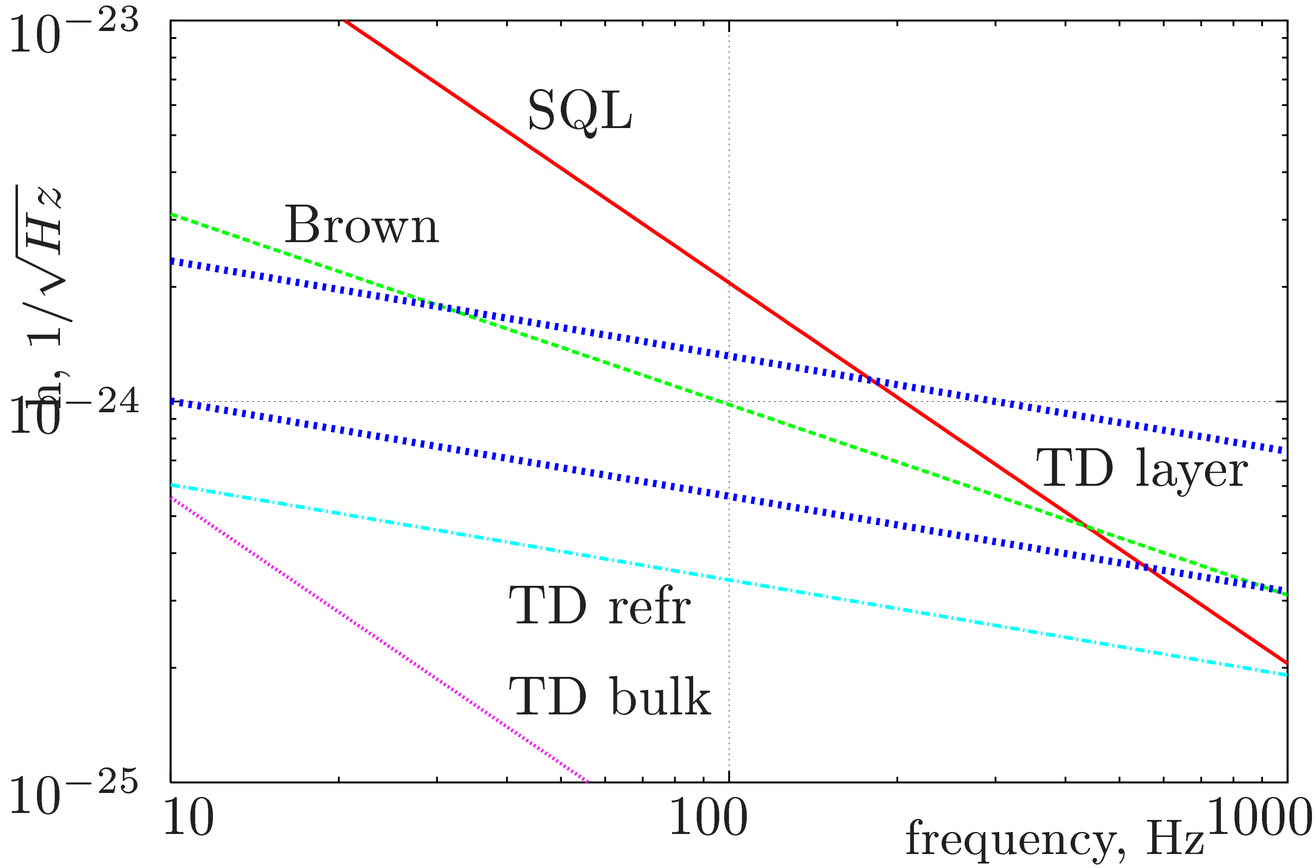
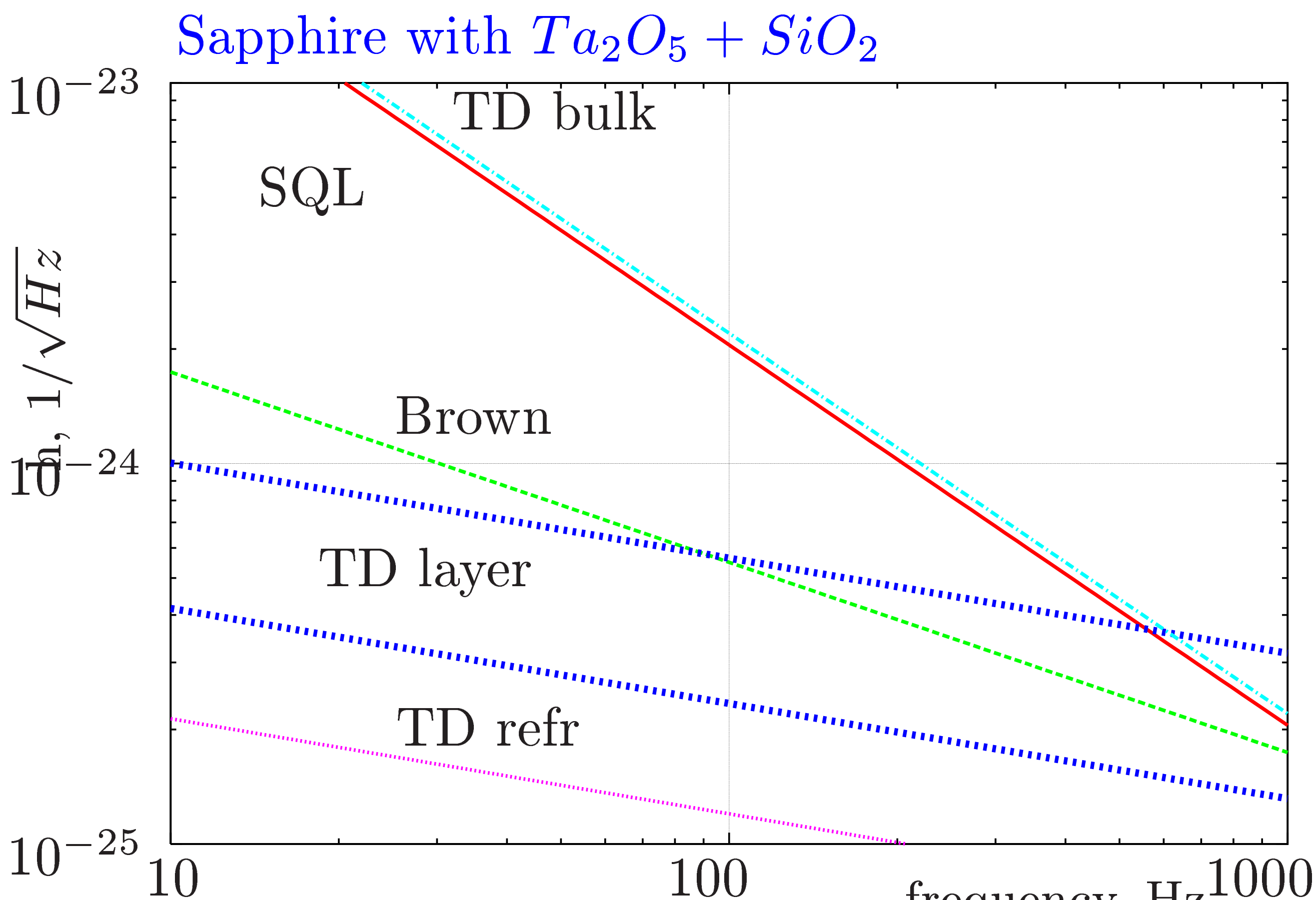
Fused silica with $Ta_2O_5 + SiO_2$



"SQL"(red) — $h_{\rm SQL}(\omega)$, "Brown"(green) — Brownian (structural) fluctuations, "TD layer"(blue) — upper and lower limits of surface thermoelastic noise, "TD bulk"(rose) — volume thermoelastic noise, "TD refr"(light blue) — thermorefractive noise.



"SQL"(red) — $h_{\rm SQL}(\omega)$, "Brown"(green) – Brownian (structural) fluctuations, "TD layer"(blue) – upper and lower limits of surface thermoelastic noise, "TD bulk"(light blue) — volume thermoelastic noise, "TD refr"(rose) — thermorefractive noise.

Conclusion

The slow dependence of interferometric coating TD noise on frequency $\sim \omega^{-1/4}$ and on the radius of the beam spot $\sim r_0^{-1}$ is worth noting.

TD noise in interferometric layer is a little smaller than SQL and it is not an obstacle for projects LIGO-II (SQL sensitivity) but it may be a problem for LIGO-III (sensitivity better than SQL).

The resume: it is important to measure *in situ* the value of effective thermal expansion coefficient α for interferometric multilayer coatings of high quality mirrors for precision measurements.

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Parameters

General:

$$\omega = 2\pi \times 100 \text{ s}^{-1}, \quad T = 300 \text{ K}, \quad r_0 \simeq 6 \text{ cm}$$

$$m = 3 \times 10^4 \text{ g}, \quad \lambda = 1.06\ \mu\text{m}, \quad L = 4 \times 10^5 \text{ cm};$$

Fused silica:

$$\alpha = 5.5 \times 10^{-7} \text{ K}^{-1}, \quad \kappa = 1.4 \times 10^5 \frac{\text{erg}}{\text{cm s K}},$$

$$\rho = 2.2 \frac{\text{g}}{\text{cm}^3}, \quad C = 6.7 \times 10^6 \frac{\text{erg}}{\text{g K}},$$

$$E = 7.2 \times 10^{11} \frac{\text{erg}}{\text{cm}^3}, \quad \nu = 0.17,$$

$$\phi = 0.5 \times 10^{-8} \text{ (Ageev et al);}$$

$$\frac{dE}{dT} = -1.5 \times 10^8 \frac{\text{erg}}{\text{K cm}^3}.$$

$$n = 1.45, \quad \frac{dn}{dT} = 1.5 \cdot 10^{-5} \text{ K}^{-1},$$

Sapphire:

$$
\begin{aligned}
\alpha &= 5.0 \times 10^{-6}\ \mathrm{K}^{-1}, \quad \kappa = 4.0 \times 10^{6}\ \frac{\mathrm{erg}}{\mathrm{cm\ s\ K}}, \\
\rho &= 4.0\ \frac{\mathrm{g}}{\mathrm{cm}^3}, \quad C = 7.9 \times 10^{6}\ \frac{\mathrm{erg}}{\mathrm{g\ K}}, \\
E &= 4 \times 10^{12}\ \frac{\mathrm{erg}}{\mathrm{cm}^3}, \quad \nu = 0.29, \quad \phi = 3 \times 10^{-9}, \\
\frac{dE}{dT} &= -4 \times 10^{8}\ \frac{\mathrm{erg}}{\mathrm{K\ \ cm}^3}, \\
n &= 1.76, \quad (\lambda = 1\ \mu m\ )
\end{aligned}
$$

Tantal pentoxide Ta_2O_5

$$
\begin{aligned}
\alpha &= -4.4 \cdot 10^{-5}\ \mathrm{K}^{-1}\ \text{(Inci)}, \\
\alpha &= 3.6 \cdot 10^{-6}\ \mathrm{K}^{-1}\ \text{(Tien et al)}, \\
\alpha &= (5 \pm 2) \cdot 10^{-5}\ \mathrm{K}^{-1}\ \text{(Braginsky, Samoilenko)} \\
E &= 1.4 \times 10^{12}\ \frac{\mathrm{erg}}{\mathrm{cm}^3}, \quad \nu = 0.23, \\
\kappa &= 4.6 \times 10^{6}\ \frac{\mathrm{erg}}{\mathrm{cm\ \ s\ \ K}}, \quad \rho = 6.85\ \frac{\mathrm{g}}{\mathrm{cm}^3}\ , \\
C &= 3.06 \times 10^{6}\ \frac{\mathrm{erg}}{\mathrm{g\ K}}, \\
n &= 2.1, \\
dn/dT &= 1.21 \cdot 10^{-4}\ \mathrm{K}^{-1}\ (\lambda \simeq 1.4\,\mu m)\ \text{(Inci)} \\
dn/dT &= 2.3 \cdot 10^{-6}\ \mathrm{K}^{-1}\ (\lambda \simeq 0.63\,\mu m)\text{(Chu)}, \\
dn/dT &= 4.7 \cdot 10^{-6}\ \mathrm{K}^{-1}\ \text{(Scobey)}
\end{aligned}
$$

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