Optimal covering of template parameter-spaces

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Plan

- Metric parameter-spaces and optimal grids
- What is known about the covering problem?
- Advantage of the best lattice-covering A_n^* over $\mathbb{Z}^n$
- LAL-implementation and results.

Metric parameter space

Matched filtering: templates parametrized by $\vec{\lambda} = \{\lambda_1, \lambda_2, ...\lambda_n\} \in \mathbb{P}_n$, plus some detection statistic: $\mathcal{F}(\vec{\lambda})$.

If signal in $\vec{\lambda}_s \implies \mathcal{F}(\vec{\lambda}_s)$ local maximum $\frac{\partial}{\partial \vec{\lambda}} \mathcal{F}\Big|_{\vec{\lambda}_s} = 0$.

In a "nearby" point $\vec{\lambda} = \vec{\lambda}_s - \Delta\vec{\lambda}$:

$$\mathcal{F}(\vec{\lambda}_s - \Delta\vec{\lambda}) = \mathcal{F}(\vec{\lambda}_s) + \underbrace{\frac{1}{2}\,\frac{\partial^2}{\partial\lambda_i \partial\lambda_j}\mathcal{F}\Big|_{\vec{\lambda}_s}}_{-\mathcal{F}_s\, g_{ij}} \Delta\lambda_i\, \Delta\lambda_j + \mathcal{O}(\Delta\lambda^3)$$

"mismatch": $m = \frac{\mathcal{F}(\vec{\lambda}_s) - \mathcal{F}(\vec{\lambda})}{\mathcal{F}(\vec{\lambda}_s)} = g_{ij}(\vec{\lambda}_s)\, \Delta\lambda_i\, \Delta\lambda_j + \mathcal{O}(3)$

$\implies \sqrt{m}$ is the "distance" corresponding to $\Delta\vec{\lambda}$

Requirements for a template bank

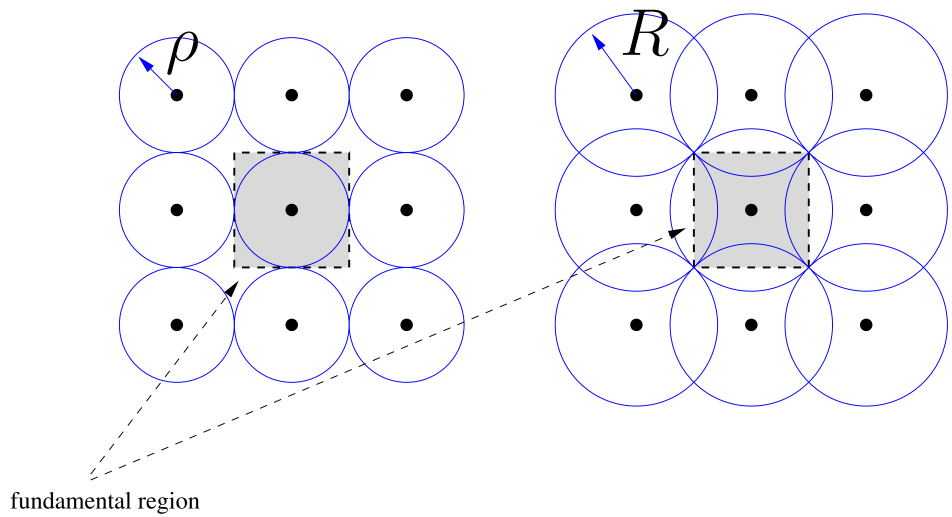
I) Find a set of "templates" $\vec{\lambda}_{(i)} \in \mathbb{P}_n$ such that for any point in $\mathbb{P}_n$ the mismatch m to the *nearest* grid-point is $m \leq m_{\max}$.
$\Longrightarrow$ sphere covering with covering radius $R = \sqrt{m_{\max}}$.

II) If search is computationally limited: find covering with *minimal* number of templates $\vec{\lambda}_{(i)}$ $\Longrightarrow$ "covering problem".
('dual' of sphere-packing problem!)

For general non-flat metric $g_{ij}(\vec{\lambda})$: no known solutions. hard!
But: partial results available for *Euklidean* space! (flat metric)

$\Longrightarrow$ Strategy: if we can find a *flat* approximation for g_{ij}, at least in patches of $\mathbb{P}_n$ small compared to curvature-radius, we can use Euklidean covering on these patches!

Sphere "Packing" versus "Covering"



Packing density: $\Delta \equiv \dfrac{\text{Volume of sphere}}{\text{fundamental volume}} < 1$

Covering thickness: $\Theta \equiv \dfrac{\text{Volume of sphere}}{\text{fundamental volume}} > 1$

Δ: fraction of space occupied by spheres
Θ: average number of spheres covering a point

"Packing problem": given ρ, *maximize* density Δ
"Covering problem": given R, *mimimize* thickness Θ

The Euklidean Covering Problem

Conway, Sloane, *Sphere packings, lattices and groups* (1998)

in 2D:

- thinnest covering = hexagonal lattice, A_2^* (Kershner 1939)
- densest packing = hexagonal lattice (Thue 1892/1910)

in 3D and higher: unsolved, *but*

- Thinnest *lattice* coverings: A_3^* (bcc), A_4^*, A_5^*
- Densest *lattice* packings: A_3 (fcc), D_4, D_5, E_6, E_7, E_8

for $n > 5$:
Thinnest coverings *known*: A_n^* up to $n \leq 21$ (except $n = 8, 9$).

$\Longrightarrow A_n^*$ provides the best (known) covering up to $n = 21$!

Properties of the A_n^* lattice

A_2^*: hexagonal, A_3^*: body-centered-cubic
A_n^*: Generating matrix ($n \times (n+1)$):

$$M_{A*} = \begin{pmatrix} \vec{l}_1 \\ \vec{l}_2 \\ \\ \ldots \\ \vec{l}_n \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & -1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 1 & 0 & 0 & \ldots & -1 & 0 \\ \frac{-n}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \ldots & \frac{1}{n+1} & \frac{1}{n+1} \end{pmatrix}$$

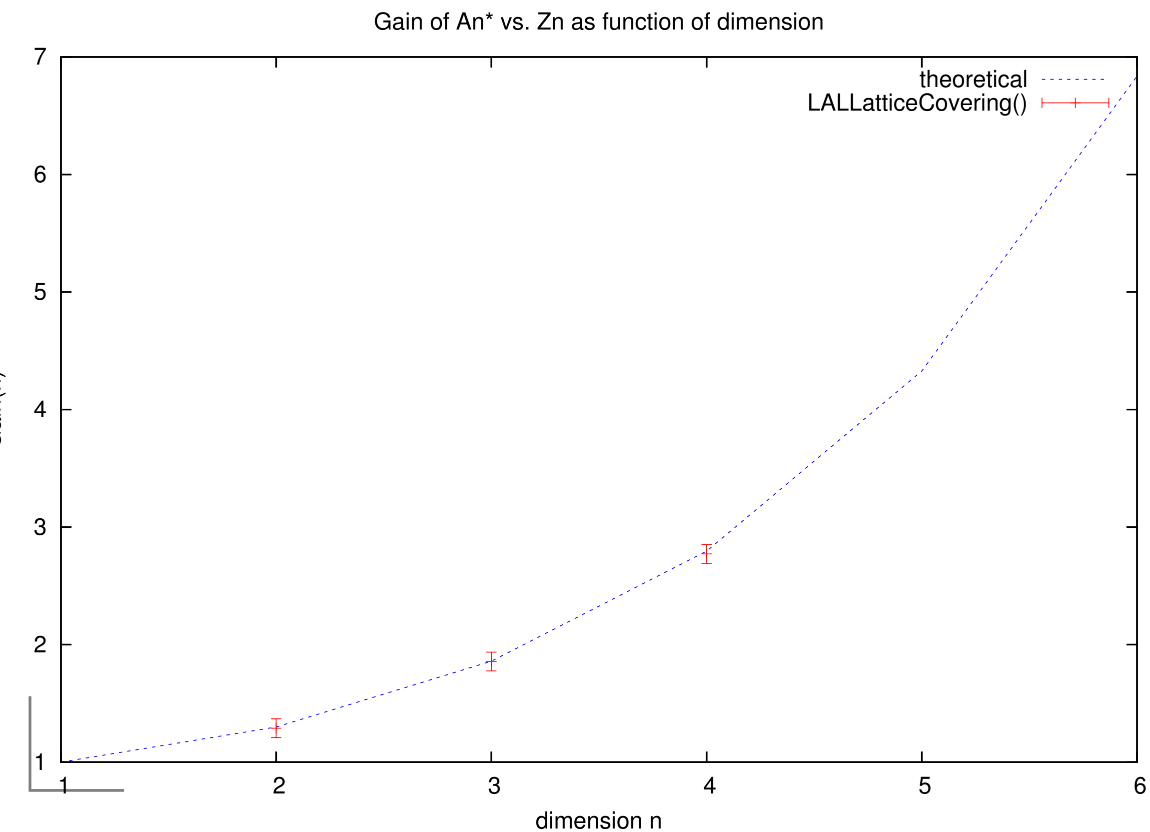
Covering radius $R_{A*} = \sqrt{\frac{n(n+2)}{12(n+1)}}$.

(Normalized) thickness $\theta_{A*} = \sqrt{n+1}\left(\frac{n(n+2)}{12(n+1)}\right)^{n/2}$

Advantage of A_n^* covering over $\mathbb{Z}^n$

Cubic lattice $\mathbb{Z}^n$ has generating matrix $M_\square = \mathbb{I}_n$, covering radius $R_\square = \sqrt{n}/2$, (normalized) thickness $\theta_\square = 2^{-n}\, n^{n/2}$

$$\Longrightarrow \text{“gain” factor } \kappa(n) \equiv \frac{\theta_\square(n)}{\theta_{A*}(n)} = \frac{3^{n/2}}{\sqrt{n+1}} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^{n/2} \overset{n\to\infty}{\sim} \frac{3^{n/2}}{\sqrt{en}}$$



n	$\kappa(n)$	$\Theta_{A*}(n)$
2	1.30	1.21
3	1.86	1.46
4	2.80	1.77
5	4.33	2.12
11	78.23	6.28
17	1691.6	18.3

Implementation

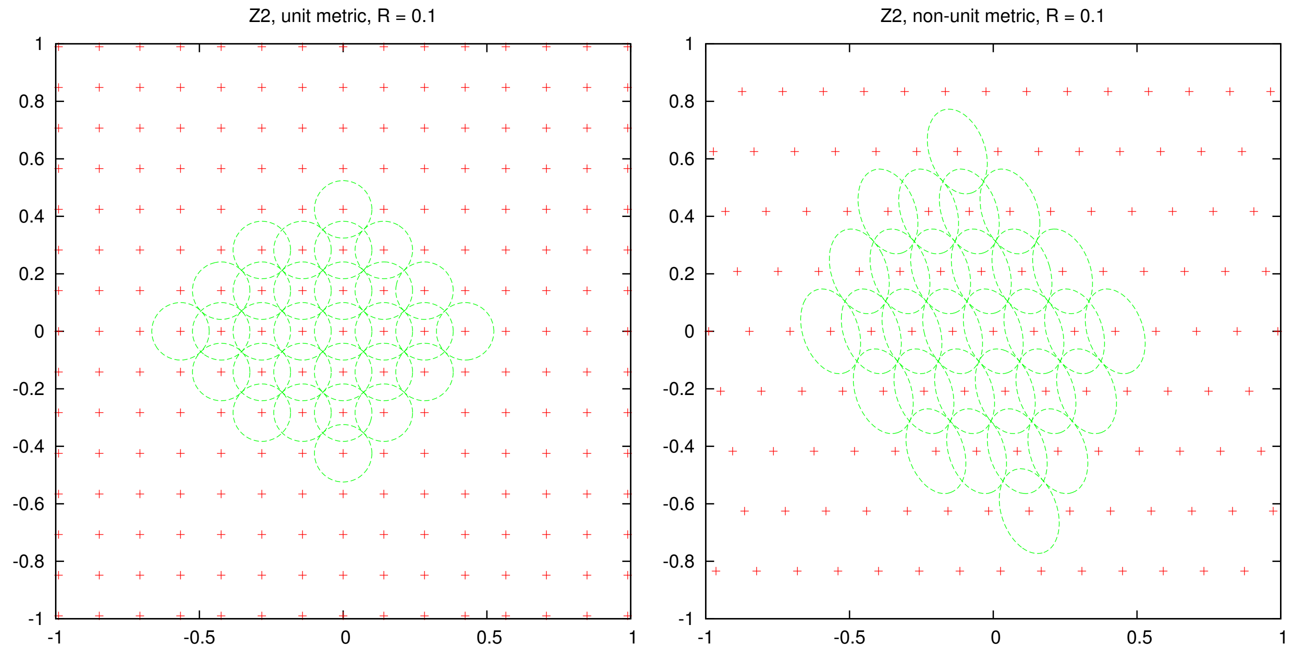
`LALLatticeCovering()` in `LAL`

Specification: given a covering-radius R, a *constant-coefficient* metric g_{ij}, a start-point and boundary-condition (and lattice-type), returns the corresponding lattice-covering.

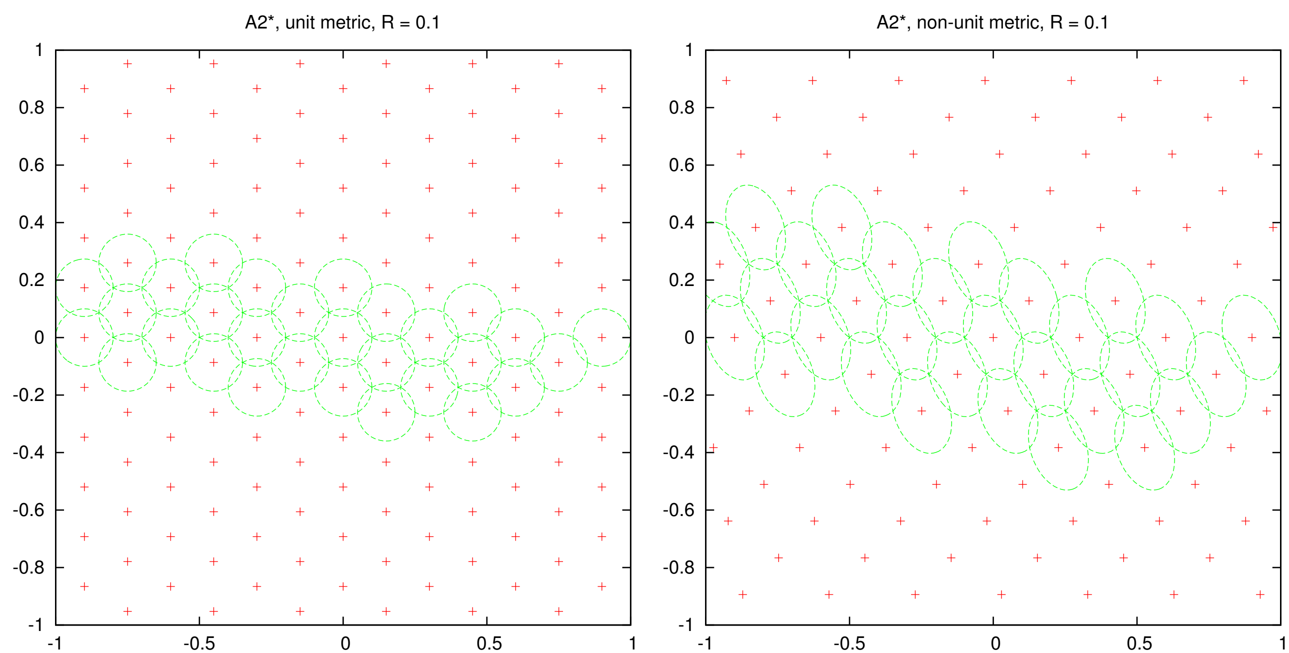
```
void LALLatticeCovering (LALStatus *status,
                         REAL8VectorList **covering,
                         REAL8 coveringRadius,
                         const REAL8Vector *metric,
                         const REAL8Vector *startPoint,
                         BOOLEAN (*isInside)(const REAL8Vector *point),
                         LatticeType latticeType);
```

Implementation: 2D results

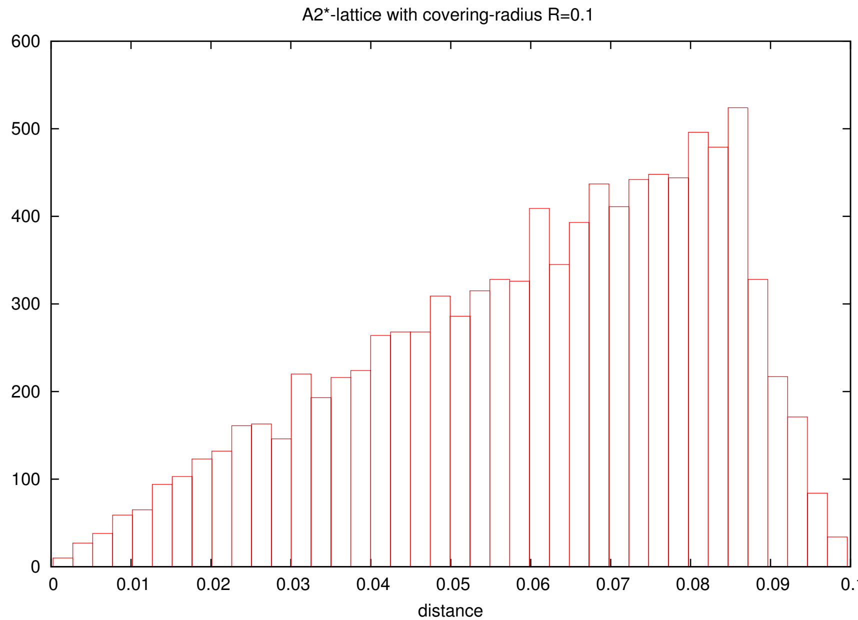
$\mathbb{Z}^2$:

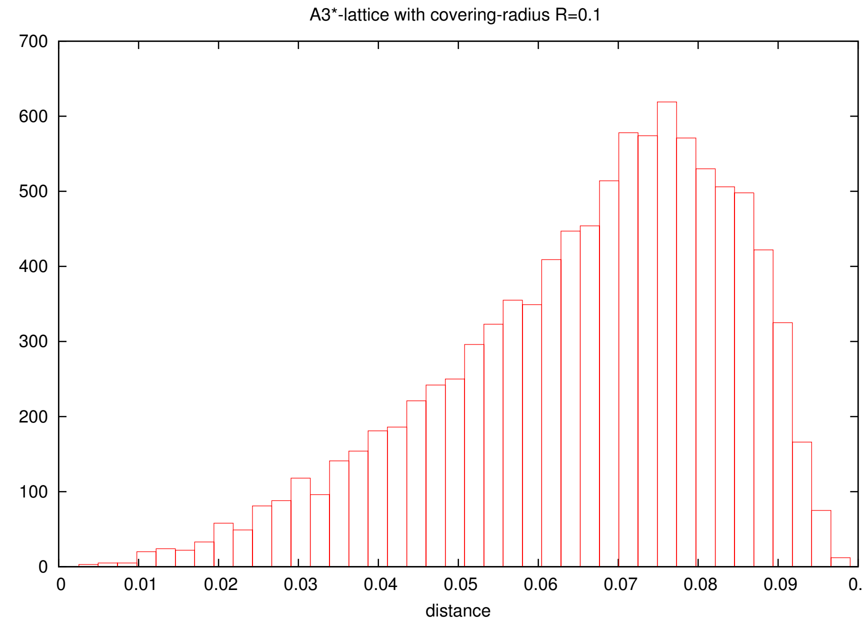


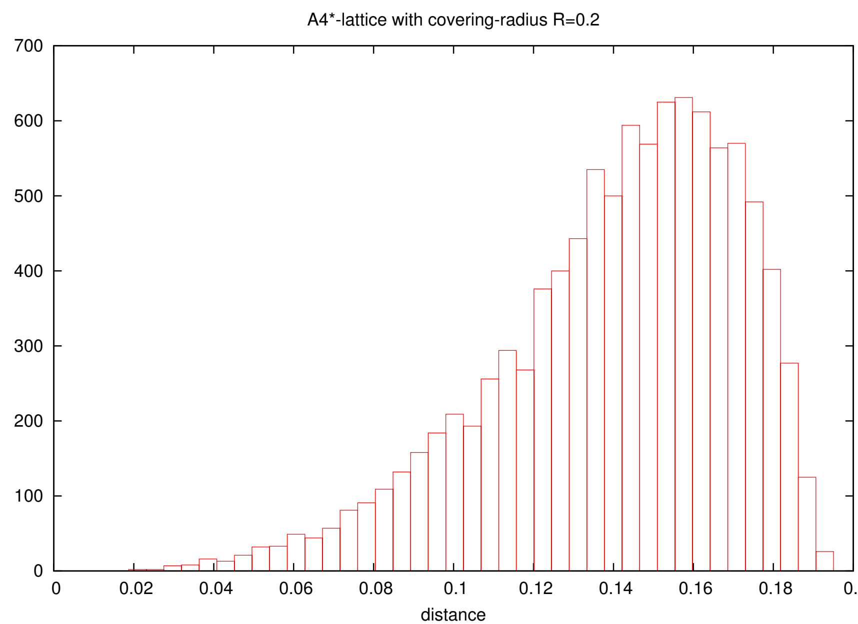
A_2^*:



MC-check of covering-radius







Plans

- Provide interface for use in other matched-filtering searches (inspiral?)
- Generate covering-lattice “point-by-point” instead of “all-at-once” (Pulsar-search: 10^{10} templates...)
- Implement & validate the (Krolak-Jaranowski) flat-metric approximation for pulsar-metric.
- Integrate flat pulsar-matric and A_n^*-lattice covering into $\mathcal{F}$-statistic.