Bayesian Statistics for Burst Search Results

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Keith Thorne and Sam Finn

Penn State University Relativity Group

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Bayesian Statistics

- Based on Bayes's Theorem - relates probability of hypothesis (H_i) given observed data (D_{obs}) to probability of data given the hypothesis (*I* - Information)

$$P(H_i|D_{obs}, I) = P(H_i|I)\frac{P(D_{obs}|H_i, I)}{P(D_{obs}|I)}$$

Posterior $\propto$ Prior $\times$ Likelihood

- Has explicit dependence on prior (implicit in Frequentist)
- Randomizes over hypotheses, not over data
  - Does not require multiple, identical trials on data
- Results are "degree of belief" on hypothesis, given prior

Counting Experiment Statistics

- Assume Poisson signal (mean s) and background (mean b) rates. We count N events during interval T, a Poisson of mean μ

$$\mu = (s+b)T \qquad P_N(n|\mu, I) = \frac{\mu^n}{n!}e^{-\mu}$$

- Probability that $\mu \in [z, z+dz)$, given $N=n$

$$P_\mu(z|N=n, I) = \frac{P_N(n|\mu=z, I)P_\mu(z|I)}{\int P_N(n|\mu=z, I)P_\mu(z|I)dz}$$

- Adding background rate $b \in [y, y+dy)$ which bounds μ

$$P_\mu(z|N=n, b=y, I) \propto P_\mu(z|N=n, I) \quad z > yT$$

Counting Statistics (continued)

- A separate background measurement implies

$$P_b(y|N=n,I) = P_b(y|I)$$

- This leads to joint probability on μ and b

$$P_{\mu,b}(z,y|N=n,I) \propto P_\mu(z|N=n,I)P_b(y|I) \quad z > yT$$

- These can be used to get probability on signal rate s

$$P_{s,b}(x,y|N=n,P_b,I) = TP_{\mu,b}((x+y)T,y|N=n,I)$$

$$P_s(x|N=n,P_b,I) = \int dy P_{s,b}(x,y|N=n,P_b,I)$$

- Combine the above expressions to complete P_s

Posterior Probability

- Posterior probability density of the signal rate *s* given the experiment result (# counts *N=n*, duration *T)* and background rate probability P_b is thus

$$P_s(x|N=n, P_b, T, I) = \frac{\int_0^\infty dy P_N(n|\mu=(x+y)T, I) P_b(y|I) \pi[(x+y)T|I]}{\int_0^\infty dy \int_{yT}^\infty dz P_N(n|\mu=zT, I) P_b(y|I) \pi[zT|I]}$$

- Here $\pi[\mu/I]$ is the normalized prior probability density
- The upper-limit probability that $s < s_0$ is the integral

$$p_{s_0} = \int_0^{s_0} P_s(x|N=n, P_b, T, I) dx$$

Non-informative Prior Probability

- Prior probability (density) is our knowledge about mean # of signal counts (μ) before the observation
  - » Existing constraints: $\mu \geq 0$, μ = rate(λ) × period(T) which are positive
- Non-informative prior *P(μ/I)* scales like sampling distribution
  - » Poisson invariant to time unit changes (T, $\lambda \rightarrow$ T', $\lambda' \equiv$ T/ α, $\lambda\alpha$)

$$P(\lambda'|I)d\lambda' = P(\lambda|I)d\lambda$$

$$P(\lambda'|I)\alpha d\lambda = P(\lambda|I)d\lambda$$

$$\frac{P(\lambda'|I)}{P(\lambda|I)} = \frac{1}{\alpha} = \frac{\lambda}{\alpha\lambda} = \frac{\lambda}{\lambda'}$$

$$\Rightarrow P(\lambda|I) \propto \lambda^{-1}$$

$$P(\mu|I) \propto \mu^{-1}$$

- This *P(μ/I)* can't be normalized, but is limit of proper priors

$$\pi(\mu|c,d,I) = \frac{1}{\mu \log(d/c)} \quad 0 \leq c \leq \mu \leq d$$

Bayesian Upper Limit

Assuming n observations, background b_0 so $P_b(y|I) = \delta(b-b_0)$

$$P_s(x|N=n, b=b_0, T, I) = \frac{[(x+b_0)T]^{n-1}}{\Gamma(n, b_0T)} e^{-(x+b_0)T}$$

With (upper) incomplete Gamma function $\Gamma(\alpha,\beta) = \int_\alpha^\infty t^{\alpha-1} e^{-t}$

Thus "% belief" that rate $s < s_0$ with non-informative prior

$$p_{s_0} = \frac{\gamma(n,(s_0+b_0)T)}{\Gamma(n, b_0T)}$$

(lower) incomplete Gamma

$$\gamma(\alpha,\beta) = \int_0^\alpha t^{\alpha-1} e^{-t}$$

--> Upper-limit is s_0 with desired "% belief" i.e 95% or 19:1 odds

Upper Limit with other Priors

- Use existing upper limit to create flat prior. In the limit that the existing upper limit is far above measured rate, this reduces the order

$$\text{flat} \quad \pi(\mu|d,I) = \frac{1}{d} \quad 0 \leq \mu \leq d$$

$$p_{s_0} = \frac{\gamma(n-1,(s_0+b_0)T)}{\Gamma(n-1,b_0T)}$$

- This typically provides an upper-limit of a higher value (2.2× higher for *n=0* case at 95%), which is somewhat counter-intuitive
- However this prior may not have valid behavior

Adding Posterior Vetos

- Posterior vetos can be added naturally in Bayesian statistics
- Example: “S2 Airplane Veto” of single event
  - » Assume p_{NotGW} - confidence that it was not a GW burst
  - » With b'_0- background after veto, δt - deadtime from veto, the posterior probability density is updated to

$$P_s(x|N, p_{NotGW}, b_0, T, I) = p_{NotGW} P_s(x|N-1, b'_0, T-\delta T, I) + (1-p_{NotGW}) P_s(x|N, b_0, T, I)$$

- Note that p_{NotGW} is belief about the (vetoed) event, not the efficiency or false rate from the veto procedure.
- As long as p_{NotGW} is near unity, it does not lead to significant differences in the “upper limit”

Conclusions

- Burst search results can be quoted using Bayesian statistics
- Has several advantages, such as natural incorporation of posterior vetos (which are likely given evolving understanding of the detectors)
- Use of prior probability allows incorporation of constraints based on astrophysics source distributions - can tie results to astrophysics predictions (to move beyond detection)
- We still need to educate ourselves to raise our "comfort level" with Bayesian statistics
  - » PSU workshop on Statistics for Gravitational Wave Data Analysis

http://cgwp.gravity.psu.edu/events/GravStat/index.shtml