Robust Bayesian detection of unmodeled bursts of gravitational waves

Antony Searle (ANU, Visitor in Physics LIGO-Caltech)
in collaboration with
Shourov Chatterji, Albert Lazzarini, Leo Stein, Patrick Sutton (Caltech),
Massimo Tinto (Caltech/JPL),
Stephen Poprocki (Wooster),
Graham Woan (Glasgow)

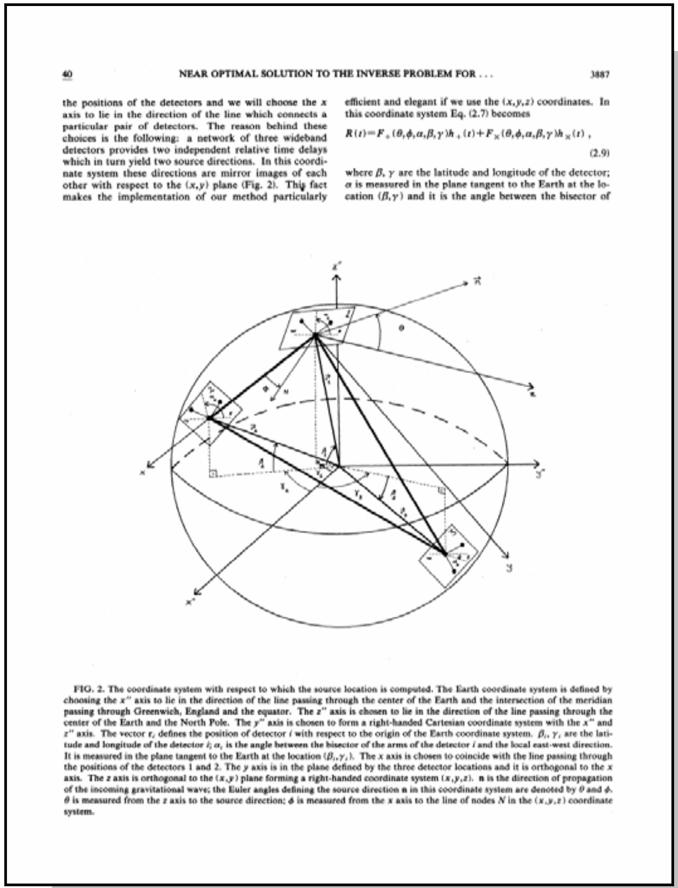
LIGO-G060502-00-Z

Unmodeled burst detection

- Detect gravitational waves of unknown waveform and short duration with ground-based interferometers
  - Sources that are unanticipated, poorly understood or whose waveforms are difficult to model
    - Supernovae are a prime candidate
  - Can’t use template searches
- Standard approach to date is excess power
- Several groups now exploring coherent burst searches
  - UFL (WaveBurst)
  - Caltech-JPL-ANU (xpipeline)
  - AEI

History and Motivation

- First addressed by Gürsel and Tinto in 1989, identifying a *null stream* as a self-consistency requirement
  - Y. Gürsel and M. Tinto, Phys. Rev. D **40**, 3884 (1989)
- Flanagan and Hughes looked at coherent detection (as opposed to consistency)
  - É. Flanagan and S Hughes, Phys. Rev. D **57**, 4566 (1998)
- The reality of the global network has spurred the creation of a number of new statistics, all owing something to the Gürsel-Tinto method
- We approach the problem from a Bayesian perspective and gain insight on the relationship between different proposed statistics
- We also note that all these statistics can be fooled by instrumental 'glitches' and propose ways to make them robust


40 NEAR OPTIMAL SOLUTION TO THE INVERSE PROBLEM FOR . . . 3887

the positions of the detectors and we will choose the x axis to lie in the direction of the line which connects a particular pair of detectors. The reason behind these choices is the following: a network of three wideband detectors provides two independent relative time delays which in turn yield two source directions. In this coordinate system these directions are mirror images of each other with respect to the (x,y) plane (Fig. 2). This fact makes the implementation of our method particularly efficient and elegant if we use the (x,y,z) coordinates. In this coordinate system Eq. (2.7) becomes

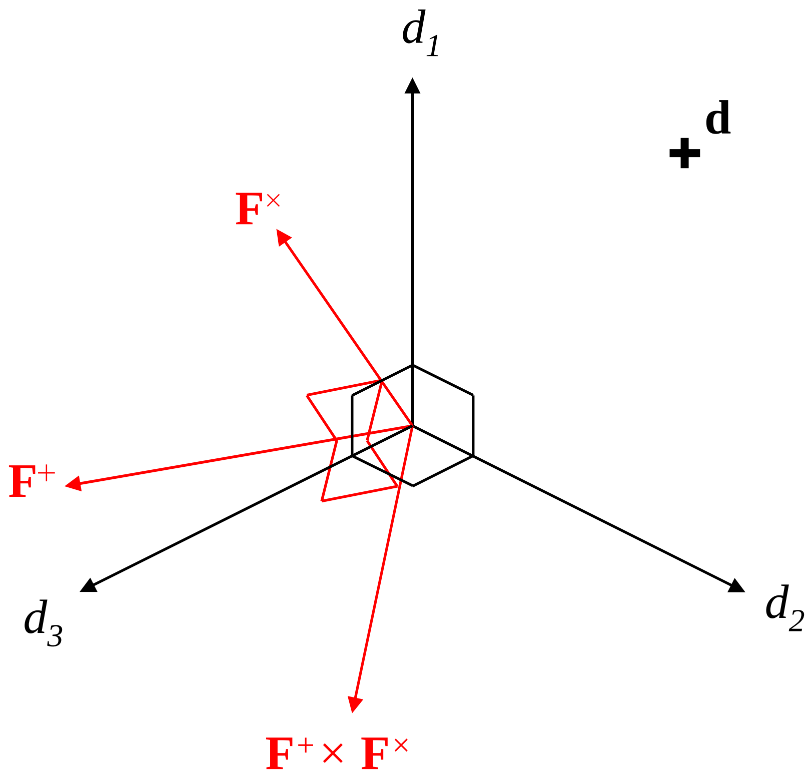
$$R(t)=F_{+}(\theta,\phi,\alpha,\beta,\gamma)h_{+}(t)+F_{\times}(\theta,\phi,\alpha,\beta,\gamma)h_{\times}(t)\ , \quad (2.9)$$

where β, γ are the latitude and longitude of the detector; α is measured in the plane tangent to the Earth at the location (β,γ) and it is the angle between the bisector of

FIG. 2. The coordinate system with respect to which the source location is computed. The Earth coordinate system is defined by choosing the x'' axis to lie in the direction of the line passing through the center of the Earth and the intersection of the meridian passing through Greenwich, England and the equator. The z'' axis is chosen to lie in the direction of the line passing through the center of the Earth and the North Pole. The y'' axis is chosen to form a right-handed Cartesian coordinate system with the x'' and z'' axis. The vector $\mathbf{r}_i$ defines the position of detector i with respect to the origin of the Earth coordinate system. β_i, γ_i are the latitude and longitude of the detector i; α_i is the angle between the bisector of the arms of the detector i and the local east-west direction. It is measured in the plane tangent to the Earth at the location (β_i,γ_i). The x axis is chosen to coincide with the line passing through the positions of the detectors 1 and 2. The y axis is in the plane defined by the three detector locations and it is orthogonal to the x axis. The z axis is orthogonal to the (x,y) plane forming a right-handed coordinate system (x,y,z). $\mathbf{n}$ is the direction of propagation of the incoming gravitational wave; the Euler angles defining the source direction $\mathbf{n}$ in this coordinate system are denoted by θ and ϕ. θ is measured from the z axis to the source direction; ϕ is measured from the x axis to the line of nodes N in the (x,y,z) coordinate system.

Dominant polarization frame

- Proposed in Klimenko *et al* Phys. Rev. D **72,** 122002 (2005)
- Consider the whitened output $d_i(f)$ of N ground-based detectors
- For each sky position Ω and frequency f we can choose a polarization basis such that the whitened vector antenna patterns $\mathbf{F}^{(+,\times)}(\Omega, f)$ satisfy
  - $|\mathbf{F}^{+}|^2 \geq |\mathbf{F}^{\times}|^2$
  - $\mathbf{F}^{+} \cdot \mathbf{F}^{\times} = 0$
- Transform d_i, with appropriate time of arrival delays $\Delta t(\Omega)$, to the new basis defined by $(\mathbf{F}^{+}, \mathbf{F}^{\times}, \mathbf{F}^{+} \times \mathbf{F}^{\times})$, for each frequency



(Generalized) Gürsel-Tinto

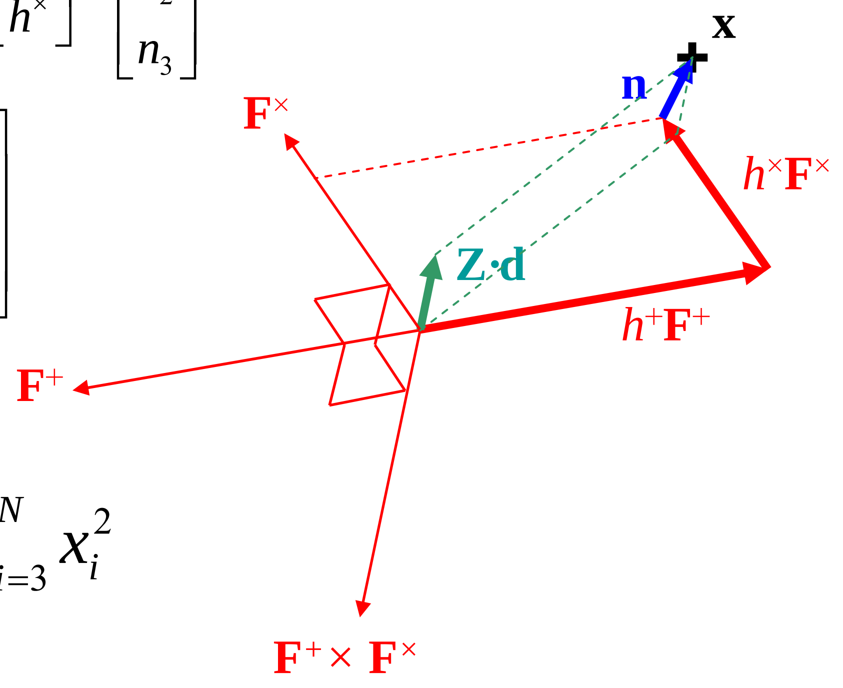
- In the dominant polarization basis, the signal is restricted to x_1 and x_2
- The remaining $(N-2)$ components are normally distributed
- Their total energy (the *null energy*) forms an elegant frequentist statistic
  - We may *reject* the hypothesis that a gravitational wave from direction Ω is present

$$\mathbf{x} = \mathbf{F}\cdot\mathbf{h} + \mathbf{n}$$

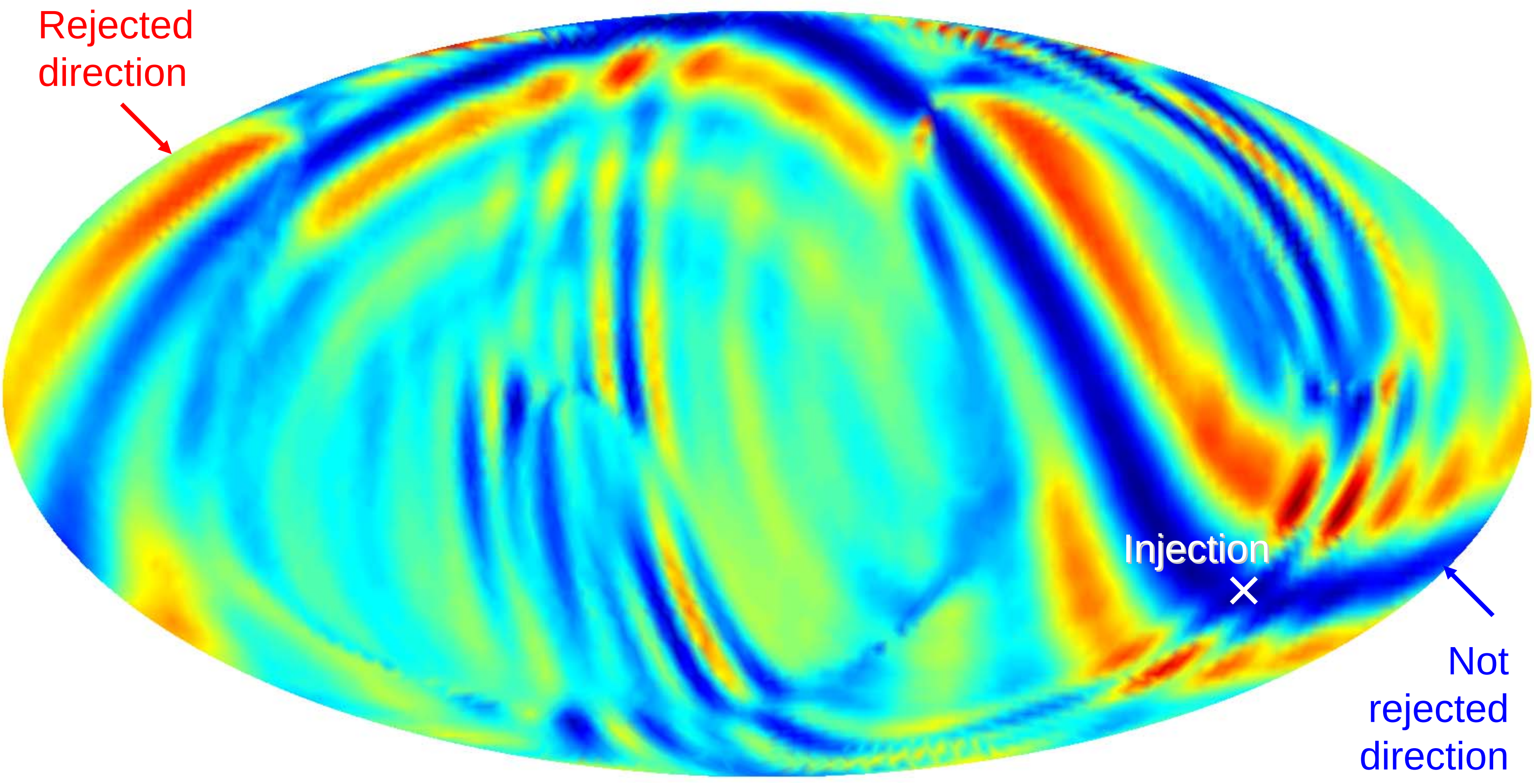
$$= \begin{bmatrix} F^+ & 0 \\ 0 & F^\times \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h^+ \\ h^\times \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$= \begin{bmatrix} F^+ h^+ + n_1 \\ F^\times h^\times + n_2 \\ n_3 \end{bmatrix}$$

$$E_{\text{null}} = \sum_{i=3}^{N} x_i^2$$



Null energy (HLV*) for SN injection



Null energy sky map features

- Three sets of rings on the sky are formed by the cross-correlations of the three detector pairs at different time-shifts
  - The details of their structure depend on the injected waveform
- The rings intersect at the source and mirrored source location
  - Amplitude tests can break the degeneracy

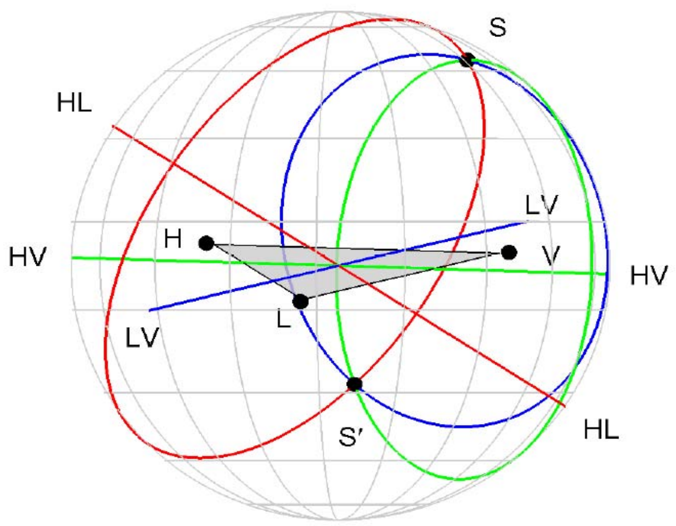


FIG. 1: Geometry of the network and travel times spent by a GWB to propagate across a three-detector network (detectors "H", "L", and "V"). The locus of constant time delay between two detectors form a ring on the sky concentric about the baseline between the two sites. For three detectors, these rings may intersect in two locations. One is the true source direction, S, while the other (S') is its mirror image with respect to the geometrical plane passing through the three sites. This two-fold ambiguity can be resolved by further considering the amplitudes of the responses. For four or more detectors there is a unique intersection point of all of the rings.

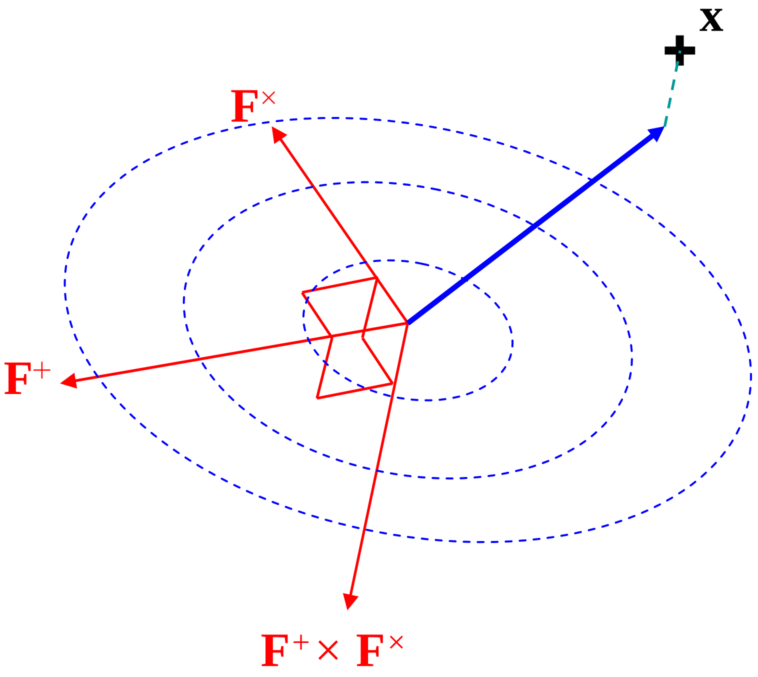
Gürsel-Tinto likelihood

- Null energy is only a consistency check, not a search
  - Noise is not rejected for all directions
- Form the likelihood ratio with the noise hypothesis

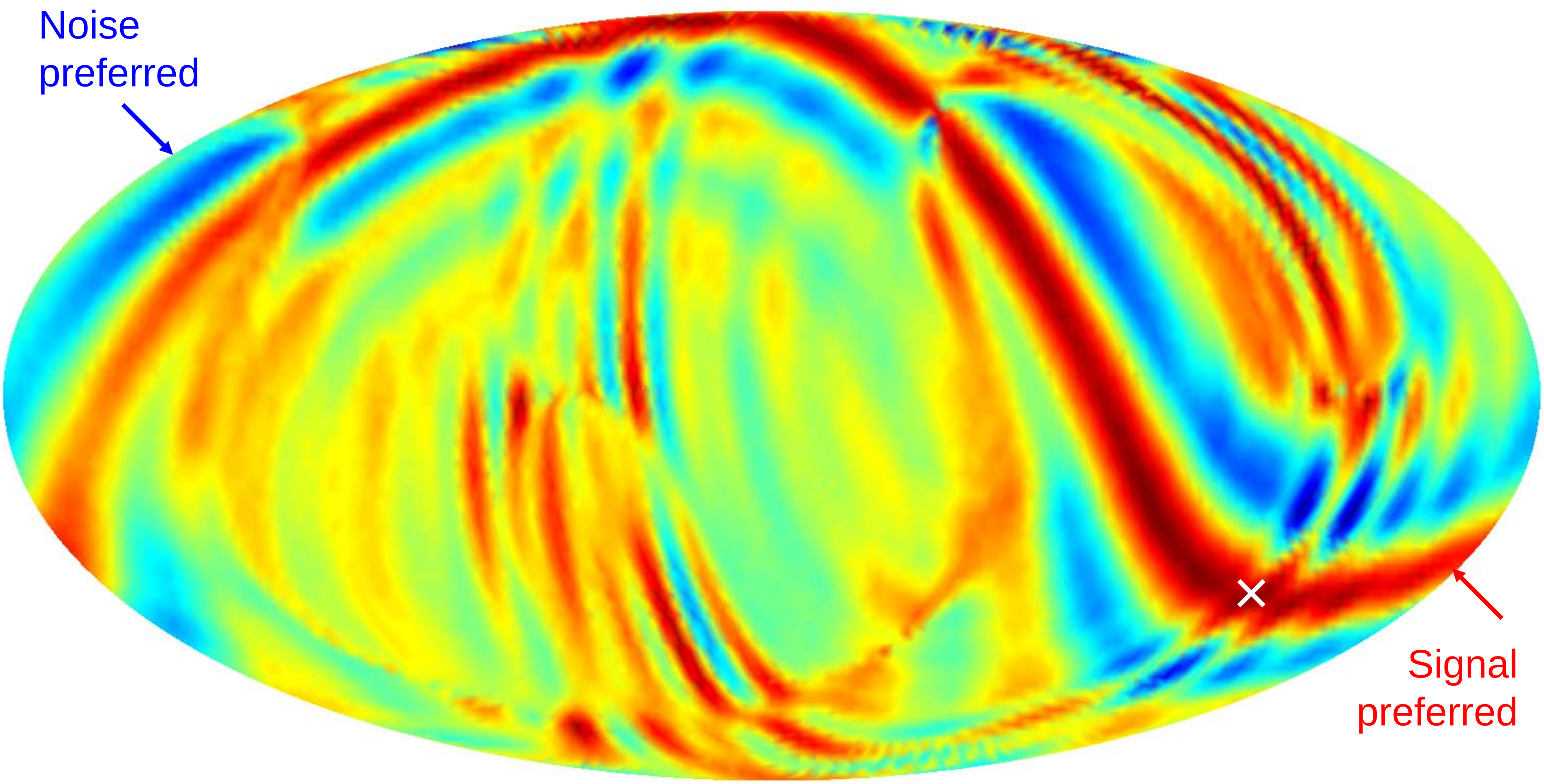
$$\Lambda \propto \frac{\exp\left(-\frac{1}{2}\sum_{i=3}^{N} x_i^2\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^{N} x_i^2\right)}$$

$$\ln\Lambda = k + \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

  - We are now measuring energy in the $(\mathbf{F}^{+}, \mathbf{F}^{\times})$ signal plane



Gürsel-Tinto log likelihood (HLV*)



Limitations of Gürsel-Tinto

- A strong consistency requirement for any one direction is weakened by considering the thousands of resolvable directions
  - In a three detector network, there exist directions compatible with *arbitrary* signals in any one or two detectors
- The likelihood test weakens the consistency requirement further
- The true direction is an extremum, but globally there are many comparable extrema
  - Poor estimator of signal direction
- The null stream varies wildly ("unphysically") when the system becomes ill-conditioned
  - Two detector "paradox"

Proposed alternative statistics

- *Constraint likelihood* methods
  - *Hard constraint* x_1^2 is power in dominant polarization
  - *Soft constraint* $x_1^2 + (F^\times/F^+)^2 x_2^2$ is power weighted by antenna patterns
  - S. Klimenko, S. Mohanty, M. Rakhmanov, and G. Mitselmakher, *Constraint likelihood analysis for a network of gravitational wave detectors*, Phys. Rev. D **72,** 122002 (2005)
- *Regularized* statistics
  - "We conclude with the reminder that regularization, by its nature, introduces a bias and therefore the optimal approach must be a trade-off between the bias and the error due to noise."
  - M. Rakhmanov, *Rank deficiency and Tikhonov regularization in the inverse problem for gravitational-wave bursts*, Class. Quantum Grav. **23** (2006) S673–S685

Constraint likelihoods

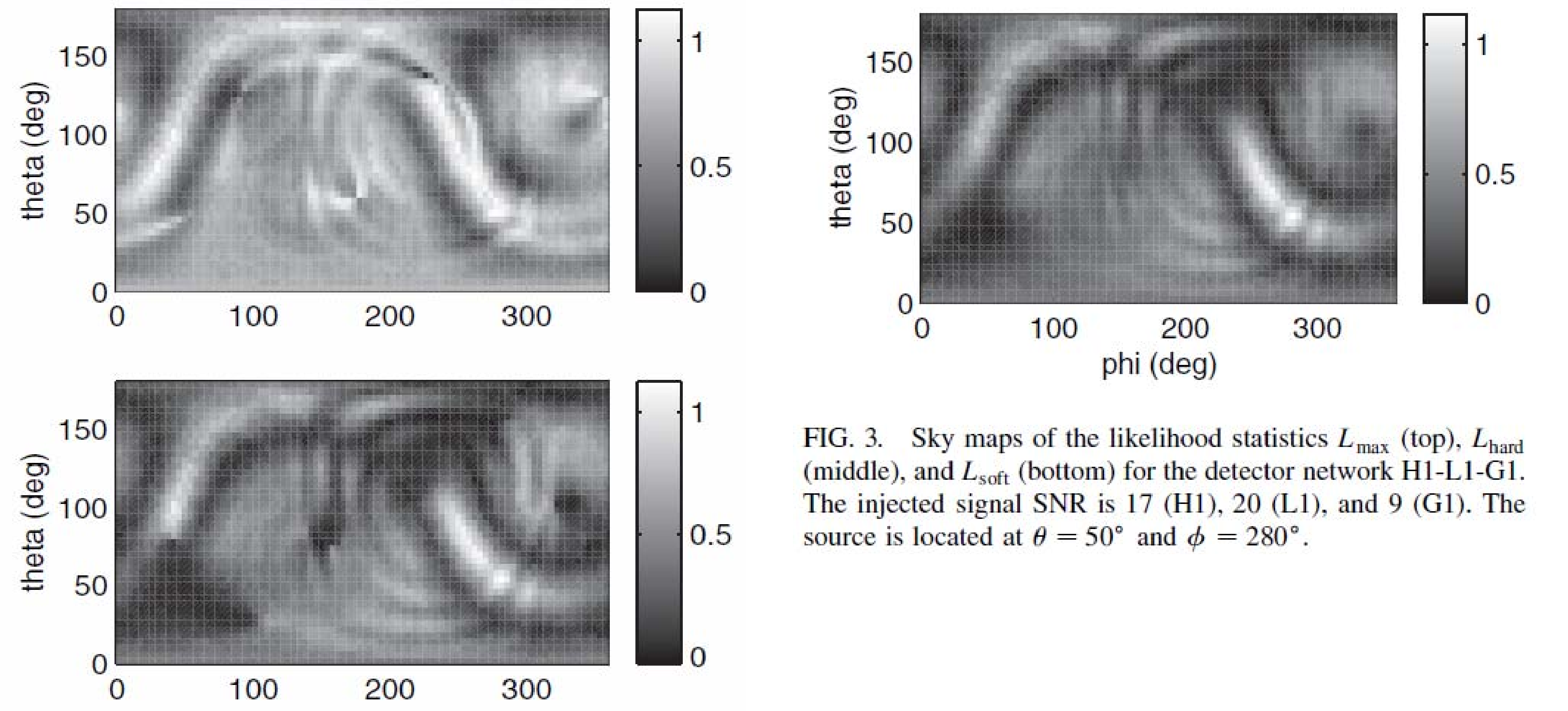


FIG. 3. Sky maps of the likelihood statistics L_{max} (top), L_{hard} (middle), and L_{soft} (bottom) for the detector network H1-L1-G1. The injected signal SNR is 17 (H1), 20 (L1), and 9 (G1). The source is located at $\theta = 50°$ and $\phi = 280°$.

Tikhonov regularization

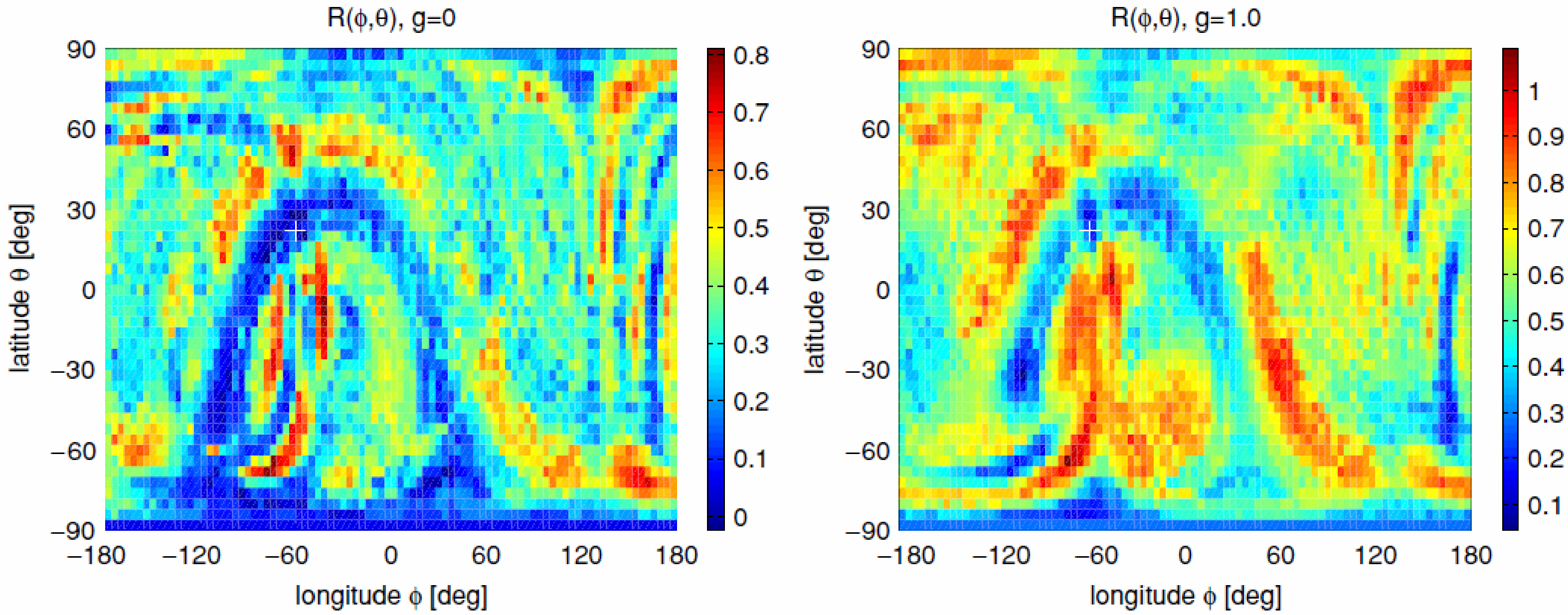


Figure 5. The residual functional (48), for the LIGO-VIRGO network (H1-H2-L-V) with no regularization (left) and full regularization (right). The injected waveforms (figure 1) are amplified to produce the signals in detectors with the SNR of 8, 8, 12 and 10 for source location (−60°, 22°) (marked by +). Note a significant reduction in the degeneracy (blue/dark grey area) introduced by the regulator. The background is removed by subtraction: $R_g \to R_g - \sigma^2 \,\mathrm{tr}(Q_g)$.

Bayesian approach

- We want the posterior plausibility $P(H_\Omega|\mathbf{d})$ that a gravitational wave came direction Ω given the data $\mathbf{d}$
- However, we can only compute the probability $P(\mathbf{d}|\mathbf{h},H_\Omega)$ of $\mathbf{d}$ arising under the less general hypothesis H_Ω for a particular waveform $\mathbf{h}$

$$P(\mathbf{d} \mid \mathbf{h}, H_\Omega) = (2\pi)^{-\frac{N}{2}} \exp\left(-\tfrac{1}{2}(\mathbf{d} - \mathbf{F}\mathbf{h})^2\right)$$

- We can marginalize over the nuisance strain $\mathbf{h}$ but to do so we need a prior plausibility distribution $P(\mathbf{h}|H_\Omega)$ for the strain

$$P(\mathbf{d} \mid H_\Omega) = (2\pi)^{-\frac{N}{2}} \int \exp\left(-\tfrac{1}{2}(\mathbf{d} - \mathbf{F}\mathbf{h})^2\right) P(\mathbf{h} \mid H_\Omega) \mathrm{d}\mathbf{h}$$

- If we require a *conjugate* prior that is independent of the choice of polarization basis, we must use a a multivariate normal distribution with some standard deviation σ

$$P(\mathbf{h} \mid H_\Omega) = (2\pi\sigma^2)^{-1} \exp\left(-\tfrac{1}{2\sigma^2}\mathbf{h}^2\right)$$

Bayesian solution

$$P(\mathbf{d} \mid H_\Omega) = (2\pi)^{-\frac{N+2}{2}} \int \exp\left(-\tfrac{1}{2}(\mathbf{d} - \mathbf{F}\mathbf{h})^2 - \tfrac{1}{2\sigma^2}\mathbf{h}^2\right) d\mathbf{h}$$

$$= (2\pi)^{-\frac{N}{2}} \det(\mathbf{C})^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\mathbf{d}^T\mathbf{C}\mathbf{d}\right)$$

$$\text{where} \quad \mathbf{C} = \mathbf{I} - \mathbf{F}\left(\mathbf{F}^T\mathbf{F} + \sigma^{-2}\mathbf{I}\right)^{-1}\mathbf{F}^T$$

- The strain prior automatically introduces a Tikhonov regularization and gives it a strong physical interpretation

$$\text{minimize error:} \quad \|\mathbf{F}\mathbf{h} - \mathbf{d}\|^2 \Rightarrow \hat{\mathbf{h}} = \left(\mathbf{F}^T\mathbf{F}\right)^{-1}\mathbf{F}^T\mathbf{d}$$

$$\text{minimize error and strain:} \quad \|\mathbf{F}\mathbf{h} - \mathbf{d}\|^2 + \sigma^{-2}\|\mathbf{h}\|^2 \Rightarrow \hat{\mathbf{h}} = \left(\mathbf{F}^T\mathbf{F} + \sigma^{-2}\mathbf{I}\right)^{-1}\mathbf{F}^T\mathbf{d}$$

$$\mathbf{d}^{\mathrm{T}}\mathbf{C}\mathbf{d} = \mathbf{d}^T\left(\mathbf{d} - \mathbf{F}\hat{\mathbf{h}}\right)$$

Bayesian odds ratio

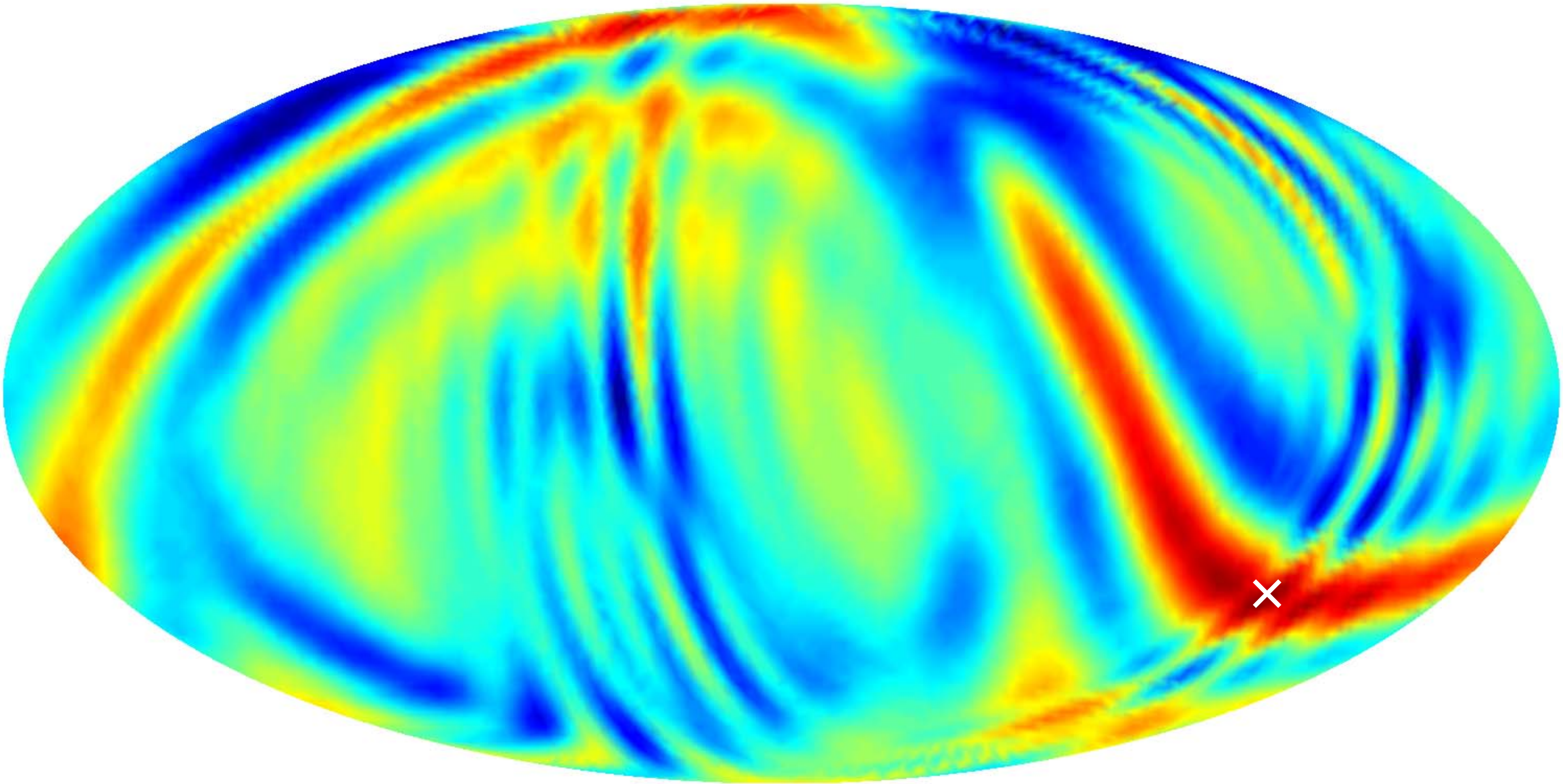
- Include the alternative hypothesis of noise only

$$P(\mathbf{d} \mid H_{\text{noise}}) = (2\pi)^{-\frac{N}{2}} \exp\left(-\tfrac{1}{2}\mathbf{d}^2\right)$$

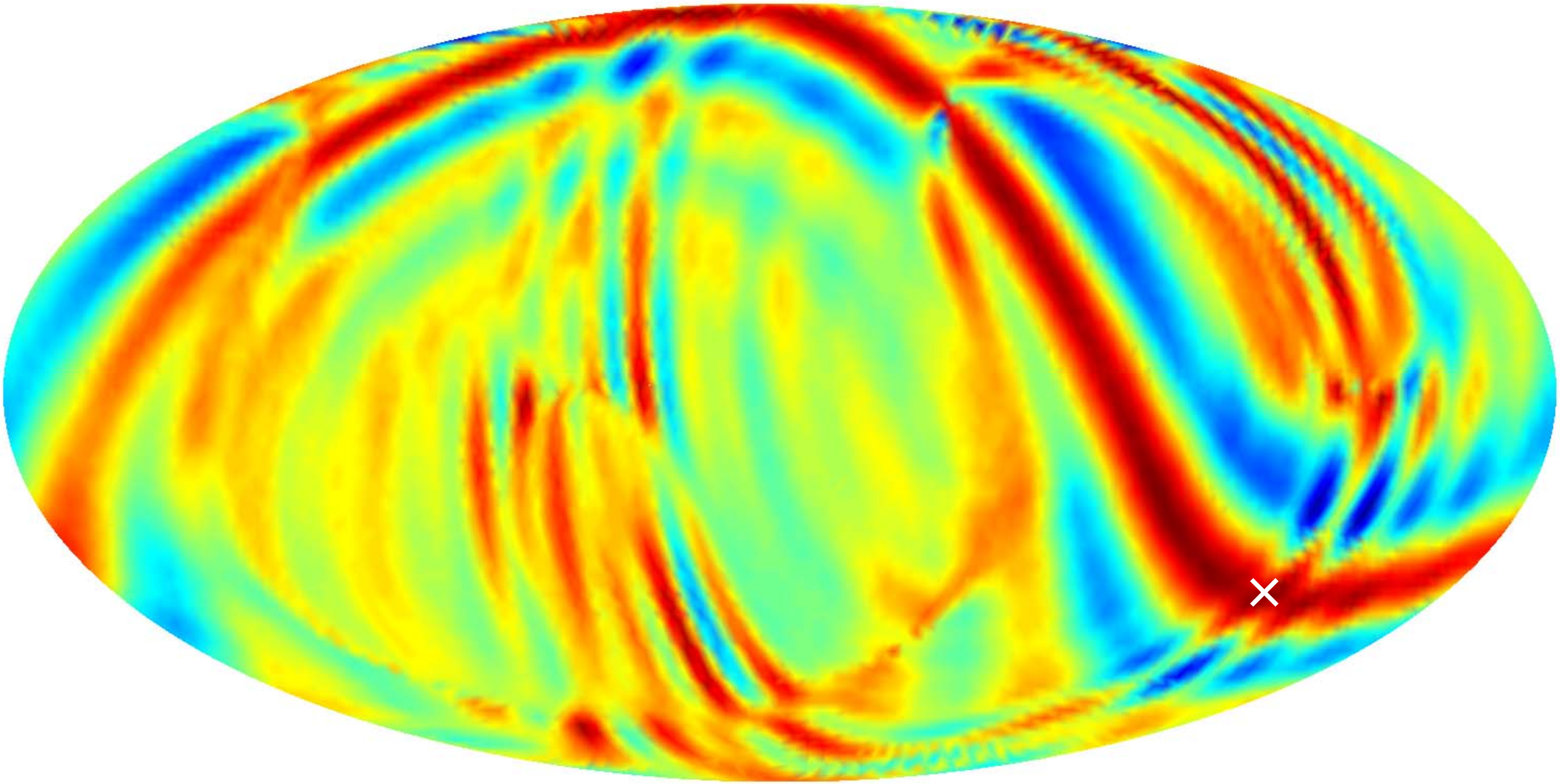
- Form the odds ratio

$$\frac{P(H_\Omega \mid \mathbf{d})}{P(H_{\text{noise}} \mid \mathbf{d})} = \frac{P(H_\Omega)}{P(H_{\text{noise}})} \det(\mathbf{C})^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\mathbf{d}^T (\mathbf{C} - \mathbf{I})\mathbf{d}\right)$$

Bayesian log odds ratio (HLV*)



Gürsel-Tinto log likelihood (HLV*)



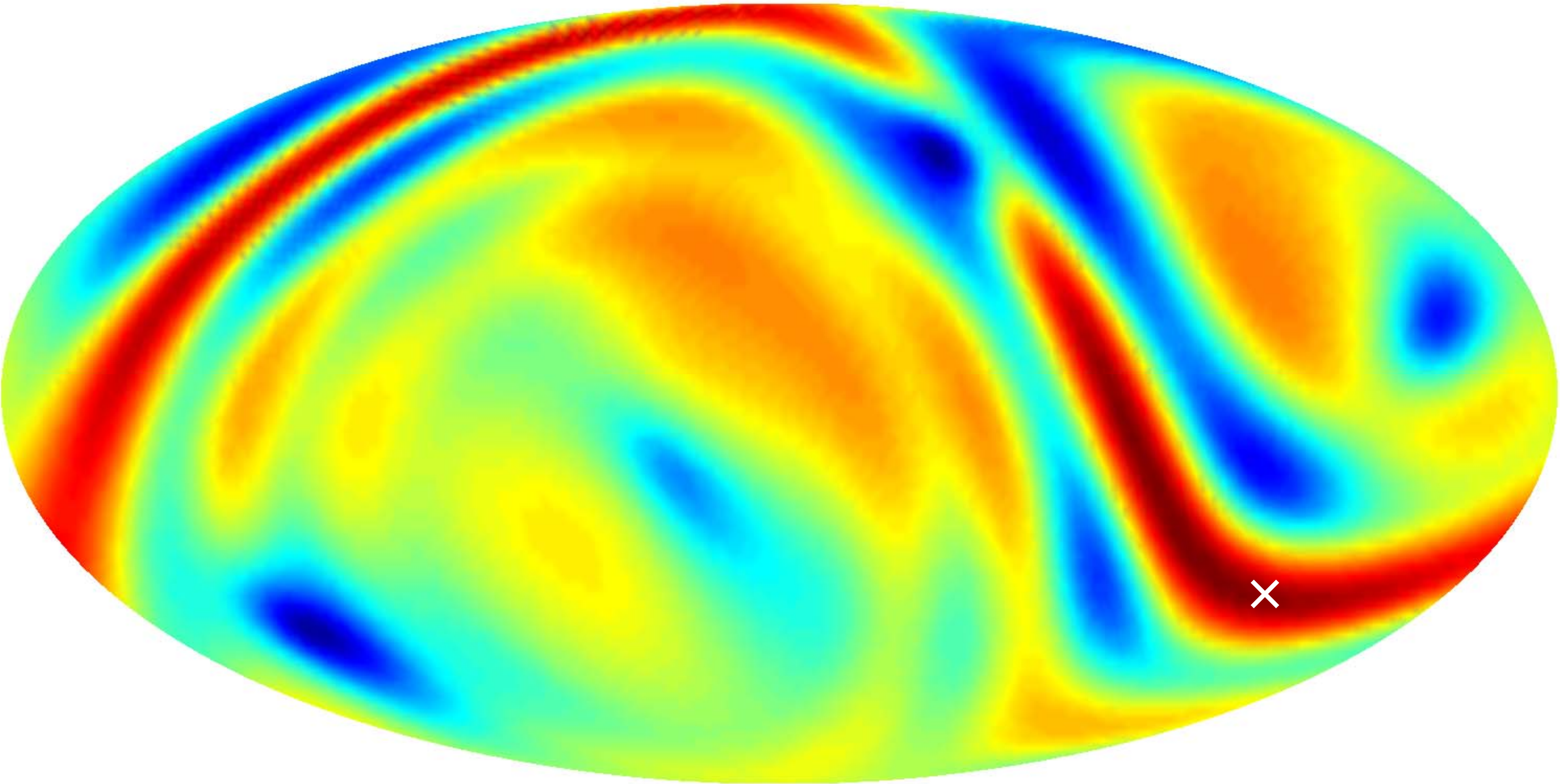
Relationship to other statistics

- When Dominant Polarization frame is valid,

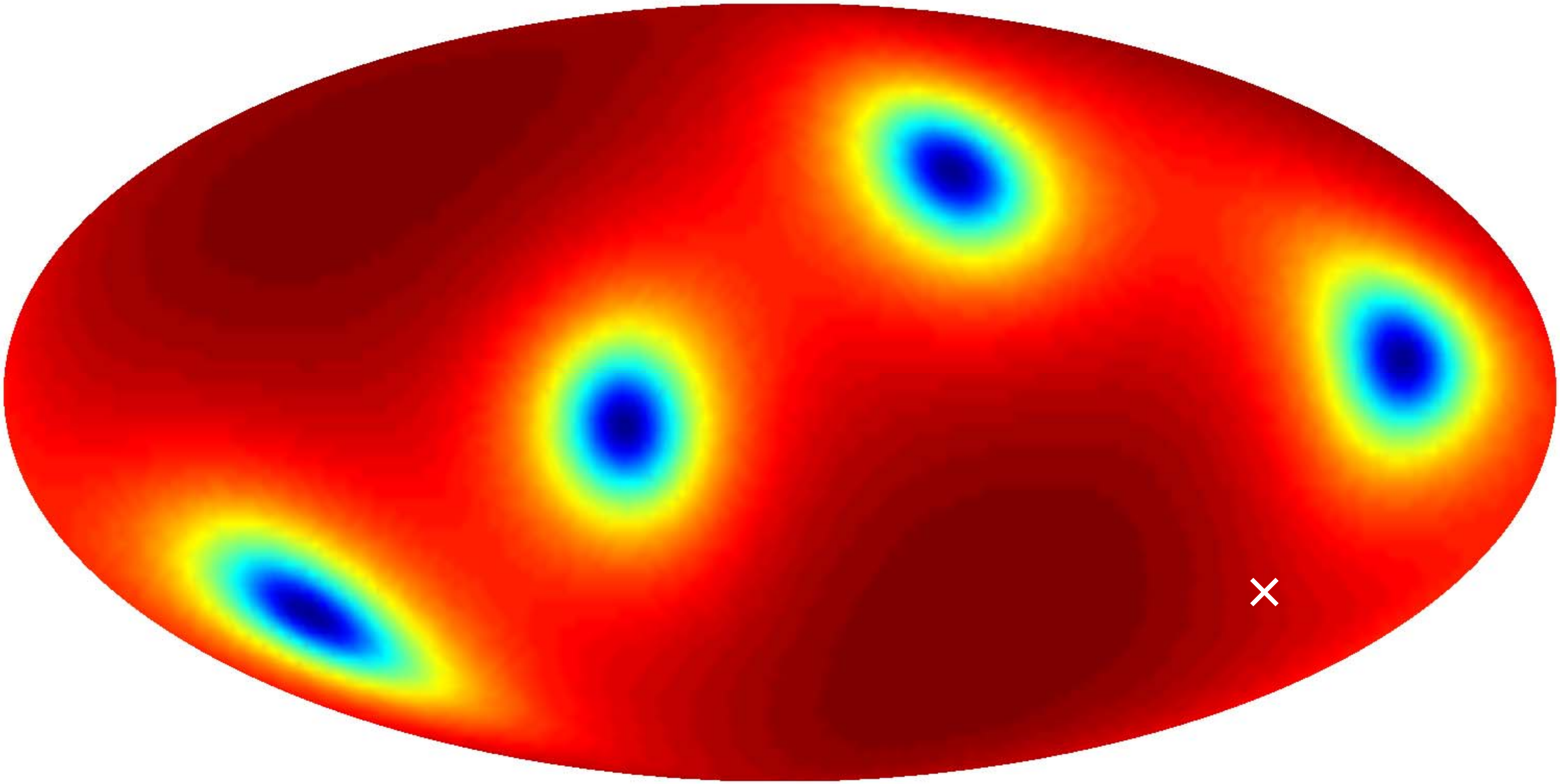
$$\frac{P(H_\Omega \mid \mathbf{x})}{P(H_{\text{noise}} \mid \mathbf{x})} = \frac{P(H_\Omega)}{P(H_{\text{noise}})} \frac{\exp \frac{1}{2}\left(\frac{F_+^2 x_1^2}{F_+^2 + \sigma^{-2}} + \frac{F_\times^2 x_2^2}{F_\times^2 + \sigma^{-2}} \right)}{\sqrt{F_+^2 \sigma^2 + 1}\sqrt{F_\times^2 \sigma^2 + 1}}$$

- In the $(\sigma \to \infty)$ limit goes to Gürsel-Tinto
  - As is common, the optimal frequentist statistic is equivalent to a flat prior
    - Incurs an infinite Occam penalty
  - Uses the pseudo-inverse (least squares) strain estimate
- In the $(\sigma \to 0)$ limit goes to something like the soft constraint
- Unlike other statistics, has a normalization term that allows us to compare or marginalize evidence for different directions
- Seems to unify and clarify a lot of the previous work

Bayesian odds ratio (HL)

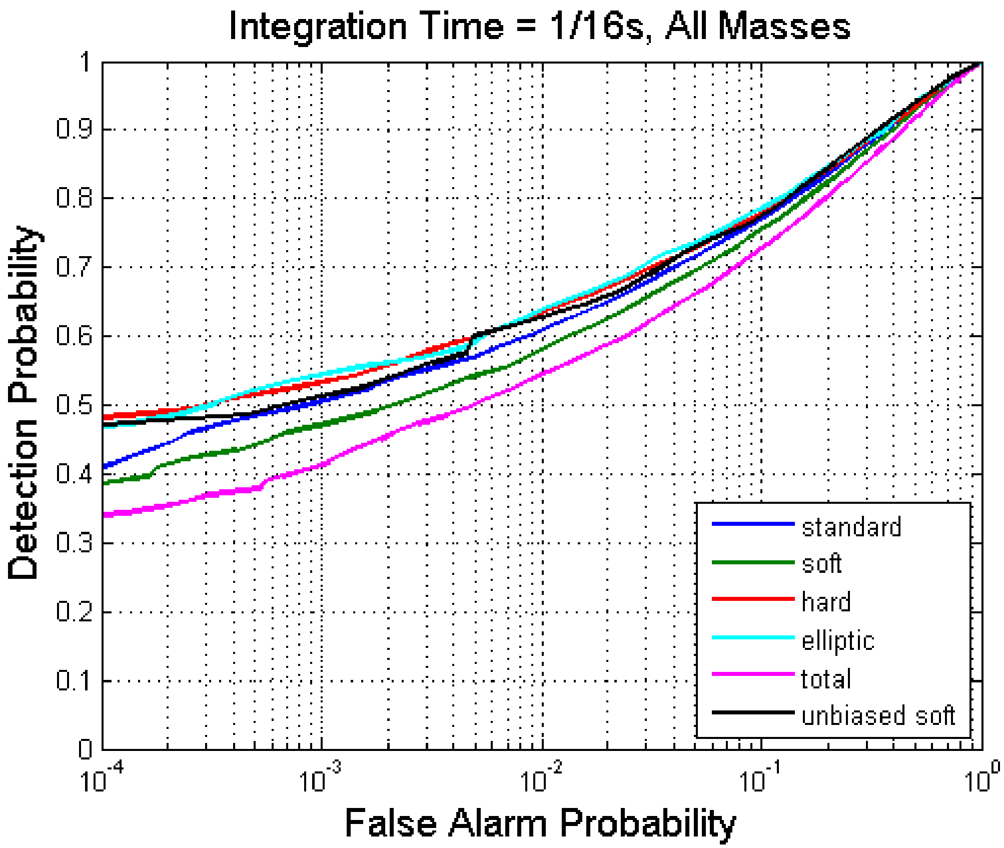


Bayesian odds ratio (H)



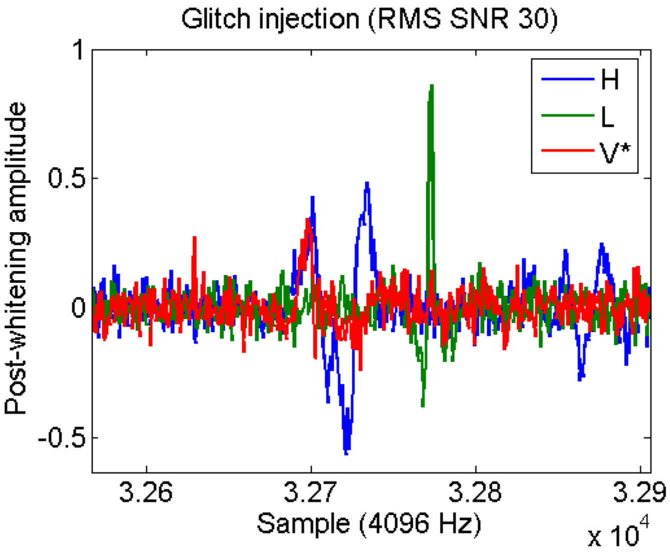
Performance

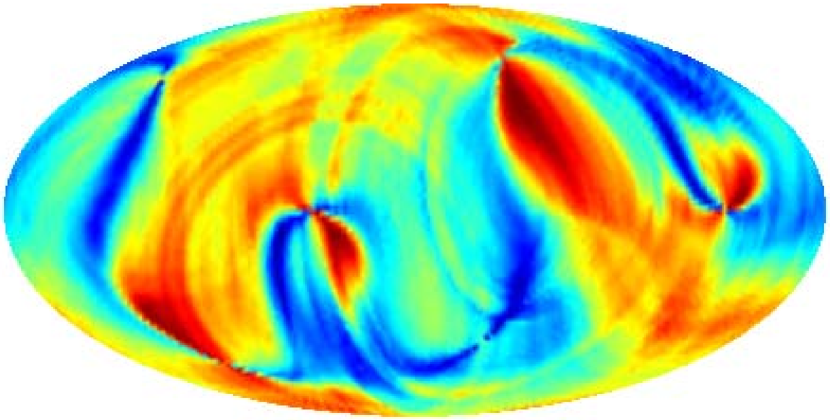
- Other statistics were tested by SURF student Stephen Poprocki over the summer:
- Bayesian statistic implemented and validated but yet to be characterized



Robustness

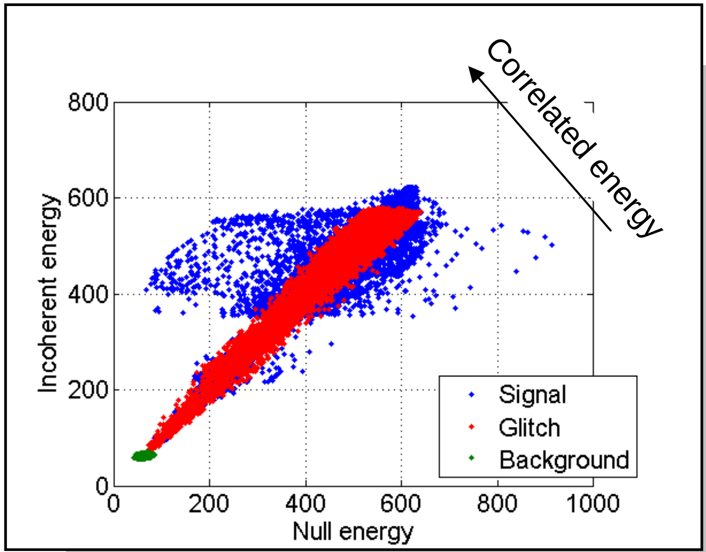
- None of the statistics mentioned above are robust against the properties of real-world noise
- In a three detector network, arbitrary measurements in any two detectors are consistent with a gravitational wave (from the null directions of the third detector)





“”Incoherent energy”

- Break the null stream into cross- and auto-correlation terms and make sure that significant correlation is present



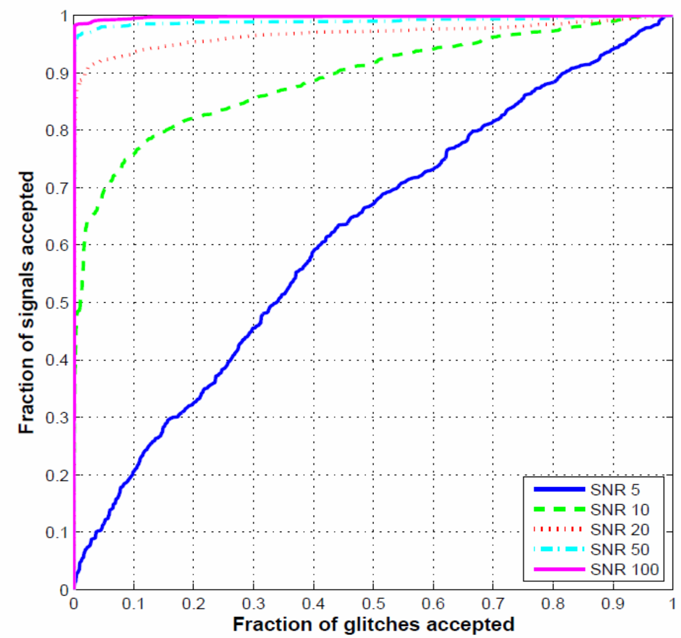
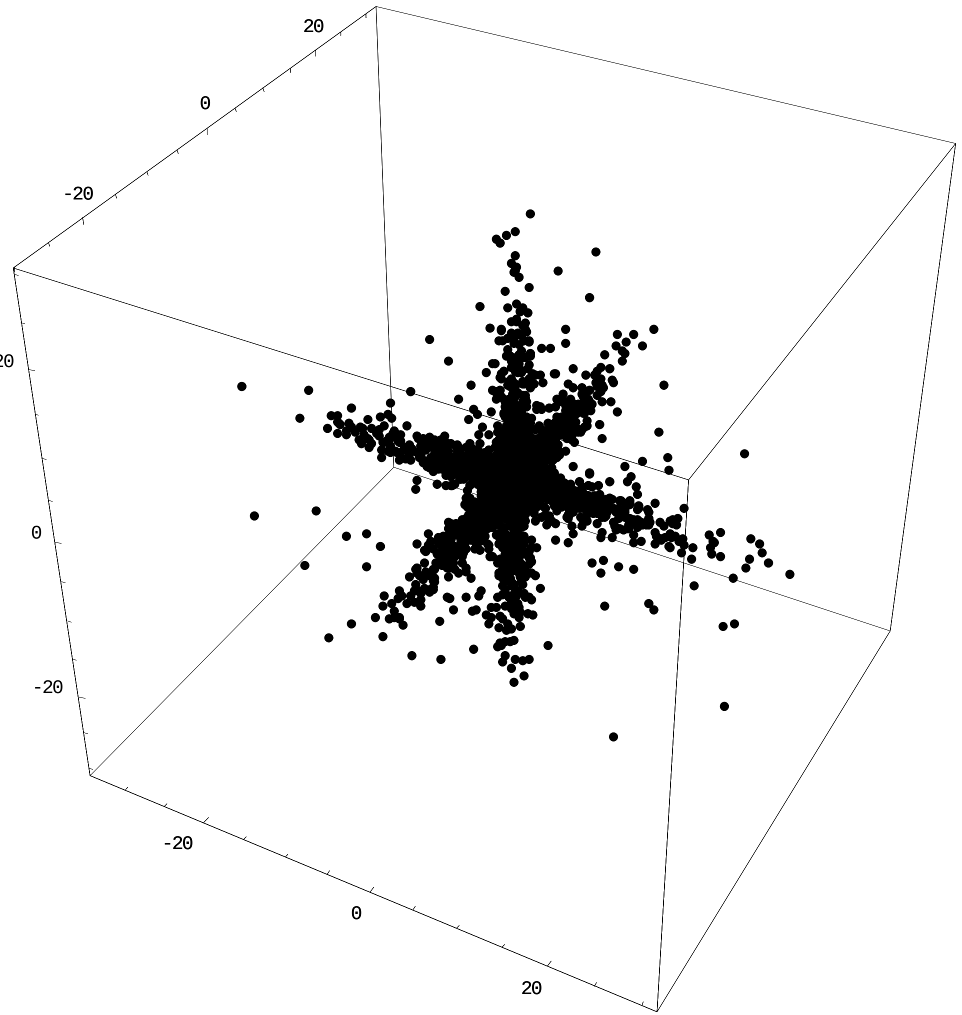


FIG. 9: Receiver-Operator Characteristic (ROC) plot for $\min(E_{\rm null}/E_{\rm inc})$ as a statistic for distinguishing GWBs from noise glitches. The ROC curve value is given by the fraction of GWBs (true acceptance) and glitches (false acceptance) of given rms SNR falling to the left of a line of constant $E_{\rm null}/E_{\rm inc}$ (a diagonal line) in Fig. 6. The rapid rise of the curves at low false acceptance is indicative of the ability of the method to confidently distinguish a significant portion of GWB signals from the glitch population.

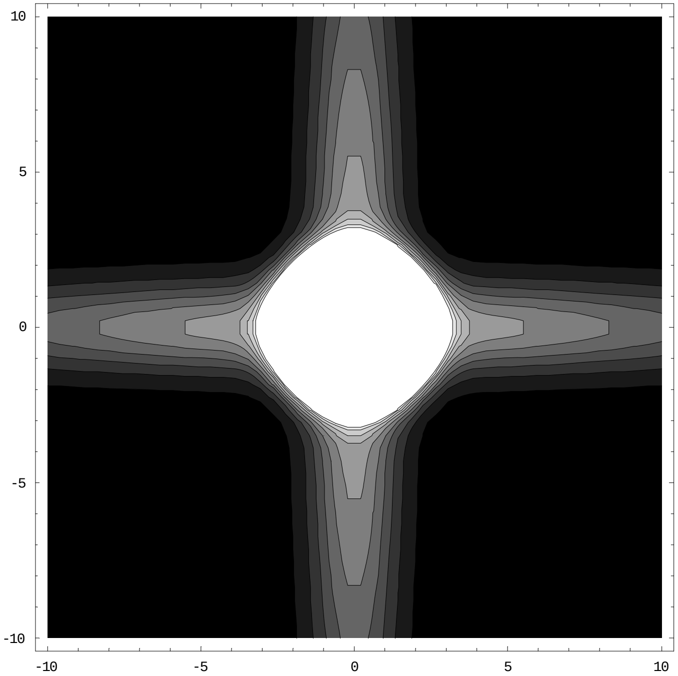
Beyond normal noise

- Both signal and noise hypotheses predict the same distribution for $x_{i>2}$ so all optimal tests ignore them
- We need a better alternative hypothesis to better model real noise and its non-Gaussian behavior



A speculative noise model

- One possible noise model has the advantage of being an “apples to apples” comparison with the Bayesian signal hypothesis:
  - With some small probability any of the detectors may “glitch” at any time, experiencing an additive glitch waveform known only up to some characteristic size ς

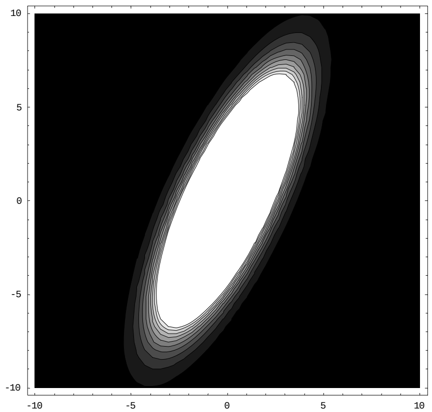


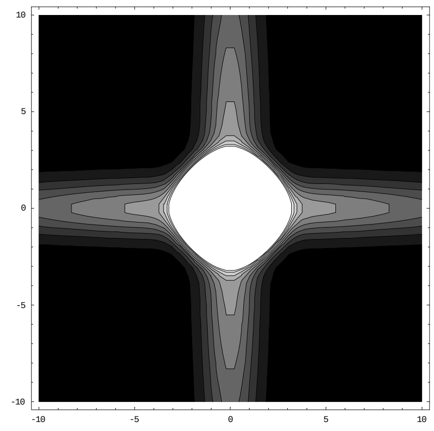
$$P\left(d_i \mid H_{\text{glitch}}\right) = \frac{1}{\sqrt{2\pi\left(1+\varsigma^2\right)}} \exp\left(-\frac{d_i^2}{2\left(1+\varsigma^2\right)}\right)$$

$$P\left(\mathbf{d} \mid H_{\text{alternative}}\right) = \prod_{i=1}^{N} \left[\frac{1-P\left(H_{\text{glitch}}\right)}{\sqrt{2\pi}} \exp\left(-\frac{d_i^2}{2}\right) + \frac{P\left(H_{\text{glitch}}\right)}{\sqrt{2\pi\left(1+\varsigma^2\right)}} \exp\left(-\frac{d_i^2}{2\left(1+\varsigma^2\right)}\right) \right]$$

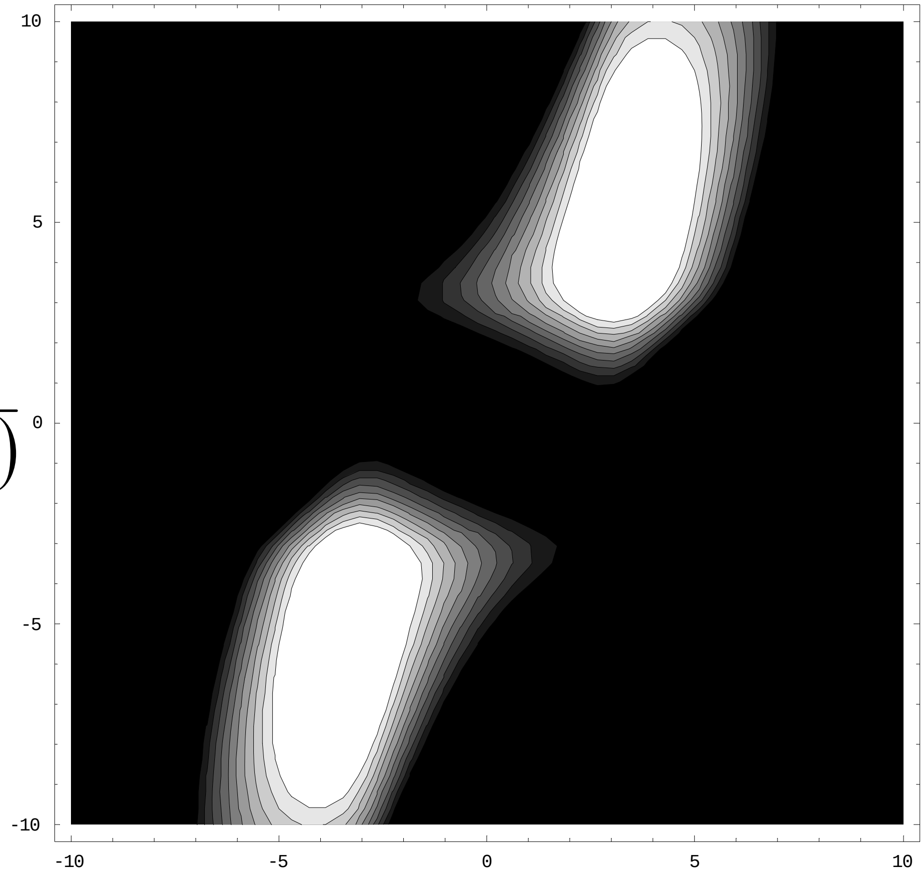
- Does it work on real data?

A robust statistic?



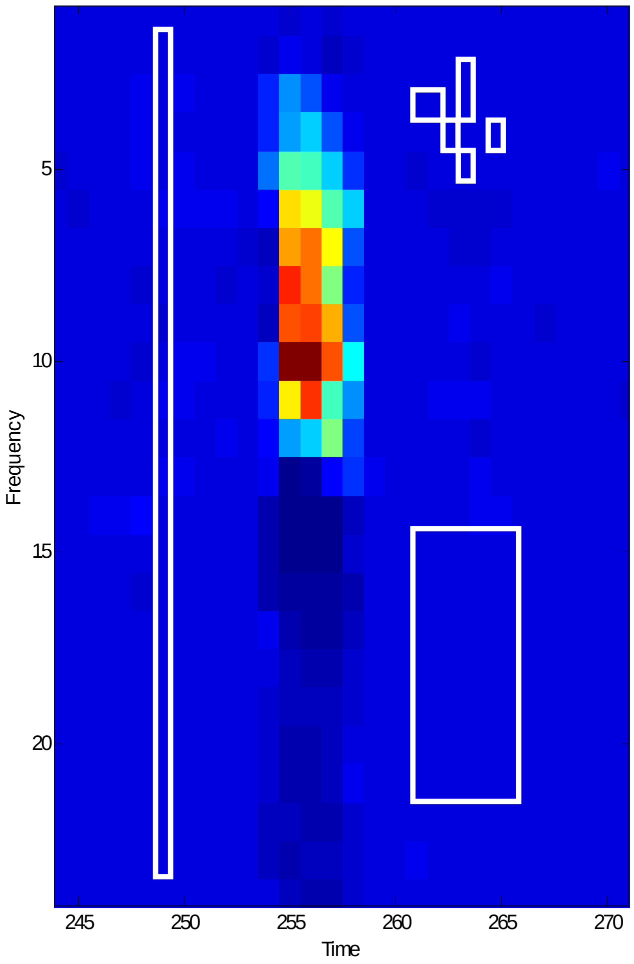


$$\frac{P(\mathbf{d} \mid H_\Omega)}{P(\mathbf{d} \mid H_{alternative})}$$



Marginalization

- Our use of Bayesian ideas is currently restricted to the derivation of an odds ratio, which we then use as a frequentist statistic
- Should we implement Bayesian ideas in more of the pipeline?
  - Marginalization rather than maximization over time, frequency and directions?
  - Signal hypotheses aggregating neighboring pixels?
- Will require rethinking of the analysis pipeline and confidence in robustness



Summary

- A large number of coherent burst detection statistics have recently been proposed
- A Bayesian approach bursts yields a new statistic and insight into existing ones
- A more realistic noise model might make these statistics robust against real-world noise

- In the short term we will concentrate on characterizing the performance of the new statistics and testing robustness against real interferometer noise

Free parameters

- The Bayesian method has several free parameters
  - Characteristic amplitude of target waveforms σ
  - Prior plausibility of signal hypothesis $P(H_\Omega)$
  - Prior plausibility of noise hypothesis $P(H_{\text{noise}})$
- These can be set by the same sort of arguments that lead to frequentist thresholds?
  - It may be possible to marginalize away σ but it is difficult to see how to achieve this in practice