Beam Profile Optimization for Thermal Noise Reduction in Advanced IFOs: Lower Bounds, Margins of Progress and Degrees of Freedom

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Outlook

- Thermal Noise Components vs. Beam Profile
  a general formula;
- Absolute Lower Bounds - Variational Solutions
  hard-clipped beams (unphysical);
- Finite Spatial Bandwidth vs. Diffraction Losses:
  degrees of freedom and effective dimension;
- A More Realistic Bound (via rLSP Expansion);
- Conclusions

Thermal Noise PSD
A General Formula

[G. Lovelace, ArXiv:gr-qc/0610041 (2007)
R. O'Shaughnessy, CQG 23 (2006) 7627]

$$S = C \int_0^\infty \kappa^{q+1} \left\{ \mathcal{H}\left[|\Phi|^2 \right](\kappa) \right\}^2 d\kappa \,, \qquad q = \begin{cases} 0 & \text{Coating (Brownian \& Thermoelastic)} \\ 1 & \text{Substrate Brownian } (SiO_2) \\ -1 & \text{Substrate Thermo-elastic } (Al_2O_3) \end{cases}$$

$$\mathcal{H}[F](\xi) \equiv \int_0^\infty F(\zeta) J_0(\xi\zeta)\zeta d\zeta$$ (Hankel transform)

$\left| \Phi(r) \right|^2 \equiv$ beam intensity distribution at mirror

Assumptions:
- axisymmetric field distribution
- infinite (thick) test-mass
- low frequency limit

Cavity Modes

[A.E. Siegman, *Lasers*, Univ. Sci. Books, Mill Valley, US, 1998]

$$\gamma\Phi(r) = \int_0^a K(r,r')\Phi(r')r'dr'$$ (integral eq., eigenvalue problem)

$$K(r,r') = \frac{\imath k}{L} J_0\left(\frac{krr'}{L}\right)\exp\left\{\imath k\left[-L + h(r) + h(r') - \frac{(r^2 + r'^2)}{2L}\right]\right\}$$

$h(r) \equiv$ mirror profile (departure from flatness)

$a \equiv$ mirror radius

$L \equiv$ cavity length; $k = 2\pi/\lambda \equiv$ wavenumber

Mapping between : a mirror profile $h(r)$

a set of eigenstates $\Omega[h] = \{\gamma_n, \Phi_n\}$

Diffraction Loss Constraint

Light spillover (diffraction) beyond mirror should be limited:

$$\mathcal{L}[\Phi] \equiv \int_a^\infty |\Phi(r)|^2 \, r dr \ \leq \mathcal{L}_T$$ (e.g., 1ppm for Adv-LIGO)

It's always possible to make
(will be assumed throughout)

$$\int_0^\infty |\Phi(r)|^2 \, r dr = 1$$

so as to rewrite the diffraction loss constraint as

$$1 - |\gamma|^2 \leq \mathcal{L}_T$$

(selects diffraction-loss *admissibile* eigenstates)

Formal Mirror Optimization Procedure

- Assume suitable (e.g., C^{∞}) functional class Λ for $h(r)$;
- Denote as $\Omega_{\mathrm{c}}[h]$ the subset of the eigenstate set $\Omega[h]$:

$$1-|\gamma|^2 \leq \mathcal{L}_T$$

- Find $h^* \in \Lambda$ such that :

$$\min_{\phi \in \Omega_c[h^*]} S[\phi] \leq \min_{\phi \in \Omega_c[h]} S[\phi], \quad \forall h \in \Lambda : h \neq h^*,$$

...A Nasty Problem

- For *most* $h(r)$, the field integral equation can only be attacked *numerically* ⟹ need to parameterize sought function $h(r)$ in terms of a *finite* number of unknowns

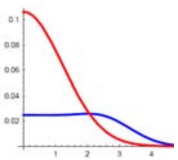
  ⟹ { "best" (minimum size) representation ?
  size of problem ? }

- Numerical solution may be *hard to obtain* due to (param-terization dependent) problem's ill-posedness and/or non-convexity (*robust* optimization algos required) .

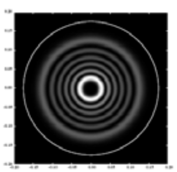
- "Exact" solution could be *technologically unfeasible .*

Available Results & Research Trends

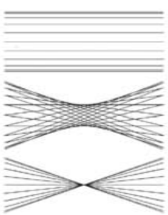
Reference solution: Gaussian Beams (GB);



Mesa-Beams (MB) (Mexican Hat (MH) mirrors)
[E. D'Ambrosio, PRD67 (2003) 102004, etc.];



Higher Order Gauss-Laguerre Modes (HOGL)
(keep std. mirrors; larger a/w; excitation issues)
[B. Mours et al., CQG 23 (2006) 5777]



Hyperboloidal-Beams and related representations
(mitigate tilt instability affecting nearly flat MH mirror cavities)
[M. Bondarescu and K. Thorne, PRD74 (2006) 082003;
V.Galdi et al., PRD73 (2006) 127101]

- *Infinite-radius mirror eigenstates used throughout in computing diffraction losses (mirror clipping approximation);*

Scalings

$$\bar{r} = r/a, \quad \bar{\kappa} = a\kappa, \quad \phi(\bar{r}) = a\Phi(a\bar{r})$$

Scaled (dimensionless) radial coordinate, wave-number and field

$$\bar{S} = \int_0^\infty \bar{\kappa}^{q+1} \left\{ \mathcal{H}\left[|\phi|^2 \right](\bar{\kappa}) \right\}^2 d\bar{\kappa}\ , \quad S = a^{-(q+2)} C\, \bar{S}$$

Scaled noise PSD

$$\bar{\gamma}\phi(\bar{r}) = \imath\pi N_D \exp\left[-\imath V(\bar{r})\right] \mathcal{H}_1 \left[\exp\left(-\imath V\right)\phi\right](\pi N_D \bar{r})$$

Scaled field equation

$$\bar{\gamma} = \gamma \exp(\imath k L)$$

Scaled half-round-trip eigenvalue

$$\mathcal{H}_1[F](\xi) \equiv \int_0^1 F(\zeta) J_0(\xi\zeta)\zeta d\zeta$$

Clipped (finite radius mirror) Hankel Tr.

$$V(\bar{r}) = kh(a\bar{r}) - \frac{\pi N_D \bar{r}^2}{2}\ , \quad N_D \equiv 2N_F = \frac{2a^2}{\lambda L}$$

Mirror-profile dependent phase (unknown)

Fresnel number of cavity

Absolute (Lower) Noise PSD Bounds

- Cope with diffraction - loss constraint by forcing $\phi(\bar{r})$ to *vanish* outside $[0,1]$ (no-diffraction, compact support beams).
- Don't care about field (eigenvalue) equation. Just seek for an *intensity* profile $f(\bar{r}) = |\phi(\bar{r})|^2 \geq 0$ for which PSD is minimum.
- Translates into simple (constrained) variational calculus problems, with unique *exact solutions* [Castaldi et al., 2007]

$$f(\bar{r}) = (q+2)\left[1-\bar{r}^2\right]^{q/2}, \quad -1 \leq q \leq 1, \; q \in \mathbb{Z}, \; 0 \leq \bar{r} \leq 1$$

yielding:

$$\bar{S}_1^{(min)} = 2^{q+1}\Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+2\right)$$

Absolute (Lower) Noise PSD Bounds, contd.

Define:

$$Q[\varphi,\mu] = \left\| \bar{\kappa}^{q/2} H_1[\varphi] \right\|^2 - 2\mu \left[\int_0^1 \bar{r}\, d\bar{r}\, \varphi(\bar{r}) - 1 \right],$$

Stationary (variational) weak solution:

$$\left.\frac{\delta Q}{\delta \varphi}\right|_{\varphi=f} = \lim_{\varepsilon \to 0} \frac{Q[f+\varepsilon\xi,\mu] - Q[f,\mu]}{\varepsilon} = 0, \quad \forall \xi \in L_1[0,1] \; : \; \int_0^1 \xi(x)\, x\, dx = 0$$

$$\delta Q = 2\varepsilon \left\langle \bar{\kappa}^{q/2} H_1[f], \bar{\kappa}^{q/2} H_1[\xi] \right\rangle + \varepsilon^2 \left\| \bar{\kappa}^{q/2} H_1[\xi] \right\|^2 + O(\varepsilon^{N>2})$$

Variational solution obtained equating this piece to zero

this being *positive, solution yields a minimum*

Absolute (Lower) Noise PSD Bounds, contd.

becomes: $$\int_0^1 \overline{r}\,d\overline{r}\left[\int_0^\infty d\overline{\kappa}\,\overline{\kappa}^{q+1} J_0(\overline{\kappa}\overline{r})\int_0^1 \overline{r}'d\overline{r}'\,f(\overline{r}')J_0(\overline{\kappa}\overline{r}')-\mu\right]\xi(\overline{r})=0,\quad 0\le\overline{r}\le 1$$

should vanish

use (see, Ryzhik & Gradhstein Tables)

$$\int_0^\infty d\overline{\kappa}\,\overline{\kappa}^{q/2} J_{q/2+1}(\overline{\kappa})J_0(\overline{\kappa}\overline{r})=2^{q/2}\Gamma\left(\frac{q}{2}+1\right),\quad 0\le\overline{r}\le 1,\quad -1\le q\le 1$$

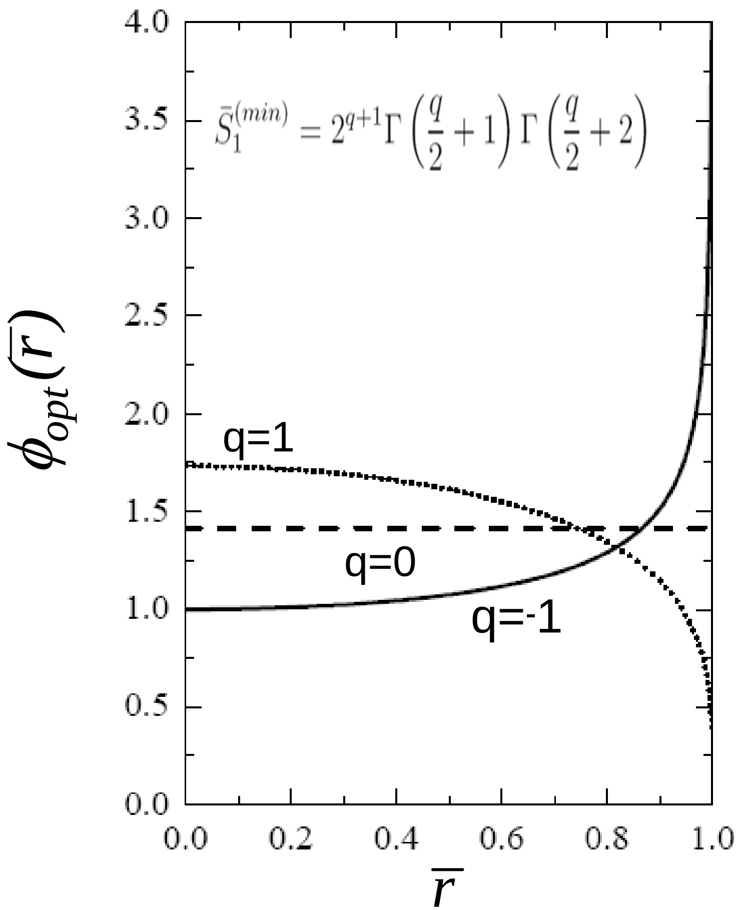
$$\overline{\kappa}^{q/2} J_{q/2+1}(\overline{\kappa})=\frac{2^{-q/2}\overline{\kappa}^{q+1}}{\Gamma(q/2+1)}\int_0^1 \overline{r}'d\overline{r}'(1-\overline{r}'^2)^{q/2} J_0(\overline{\kappa}\overline{r}'),\quad 0\le\overline{r}\le 1$$

to get: $$\int_0^\infty d\overline{\kappa}\,\overline{\kappa}^{q+1} J_0(\overline{\kappa}\overline{r})\int_0^1 \overline{r}'d\overline{r}'(1-\overline{r}'^2)^{q/2} J_0(\overline{\kappa}\overline{r}')=2^q\Gamma^{-2}(q/2+1)$$

whence: $$f=\mu\,2^{-q}(1-\overline{r}'^2)^{q/2}\Gamma^{-2}(q/2+1),\quad \|f\|=1\Leftrightarrow \mu=(q+2)2^q\Gamma(q/2+1)$$

qed

Absolute (Lower) Noise PSD Bounds, contd.



Remarks/Caveats

Optimal field-intensity profile for coating noises flat as expected; for substrate Noises, *not exactly flat,* and not obvious.

Obtained (scaled) field-intensity profiles yield absolute but likely *loose* lower bounds for the noise PSDs.

The no-diffraction field assumption made is indeed violated by *any* solution of the field equation.

How close can we go to these bounds using physically *admissible* fields ?

Spatial Band-Limitedness of Cavity Field

From the obvious properties:

$$\mathcal{H}_1[f(\bar{r})] = \mathcal{H}[\Pi(\bar{r})f(\bar{r})],\ \Pi(\bar{r}) = \begin{cases} 1, & 0 \le \bar{r} \le 1 \\ 0, & \text{elsewhere} \end{cases},\quad \mathcal{H}[\mathcal{H}[f]] = f$$

by applying $\mathcal{H}$ operator to both sides of field (eigenvalue) equation we obtain

$$\mathcal{H}\left[\phi \exp(iV)\right](\pi N_D \bar{r}) = i\frac{\pi N_D}{\bar{\gamma}}\Pi(\bar{r})\exp\left[-iV(\bar{r})\right]\phi(\bar{r})$$

⇩

The Hankel transform (wavenumber spectrum) of $\exp[\imath V(\bar{r})]\phi(\bar{r})$ has *compact support* , vanishing outside $[0, \pi N_D]$. Accordingly $\exp[\imath V(\bar{r})]\phi(\bar{r})$, and hence ϕ, *cannot* vanish identically for $\bar{r} > 1$.

The (Radial) Slepian Landau Pollack Basis

- The (real valued) eigenstates of

$$\bar{\eta}\varphi(\bar{r}) = i\pi N_D \mathcal{H}_1\left[\varphi\right]\left(\pi N_D \bar{r}\right)$$

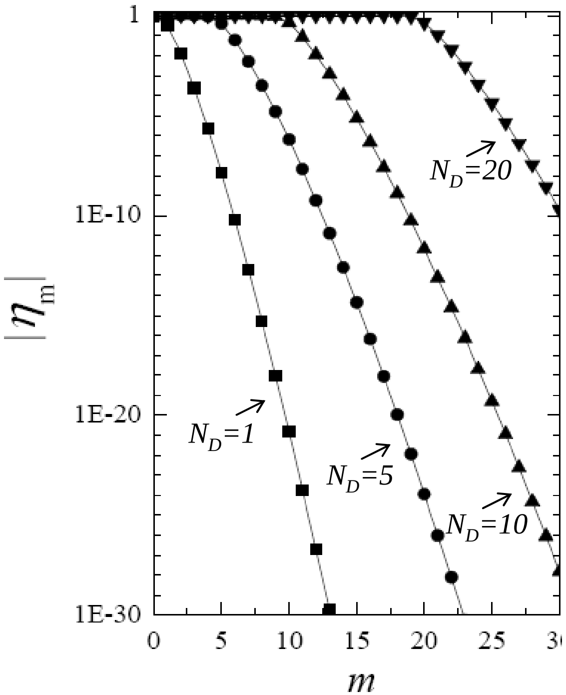
(modal fields of a confocal-spherical finite-mirror FP cavity) play *a special role* (Slepian-Landau-Pollack radial wavefunctions).

- Among *all* L^2 bases, they allow to approximate *any* exact solution of the field equations (corresponding to an arbitrary mirror profile), using the minimum number N_ε of terms for *any* prescribed L^2 error ε (*minimum-redundant basis*).

- Technically, N_ε is referred to as the number of *degrees of freedom* of our cavity fields at the *resolution level* ε.

[D. Slepian et al., Bell System Tech. Journal, 40 (1961) 43 and 65; ibid. 41 (1962) 1295]

Peculiar Properties of rSLP Eigenstates



double-orthogonality :

$$\begin{cases} \int_0^\infty \bar{r} d\bar{r} \varphi_n(\bar{r}) \varphi_m^*(\bar{r}) = \delta_{mn} \\ \int_0^1 \bar{r} d\bar{r} \varphi_n(\bar{r}) \varphi_m^*(\bar{r}) = |\eta_n|^2 \delta_{mn} \end{cases}$$

⬇

$$\int_1^\infty \bar{r} d\bar{r} \left|\varphi_n(\bar{r})\right|^2 = 1 - \left|\eta_n\right|^2$$

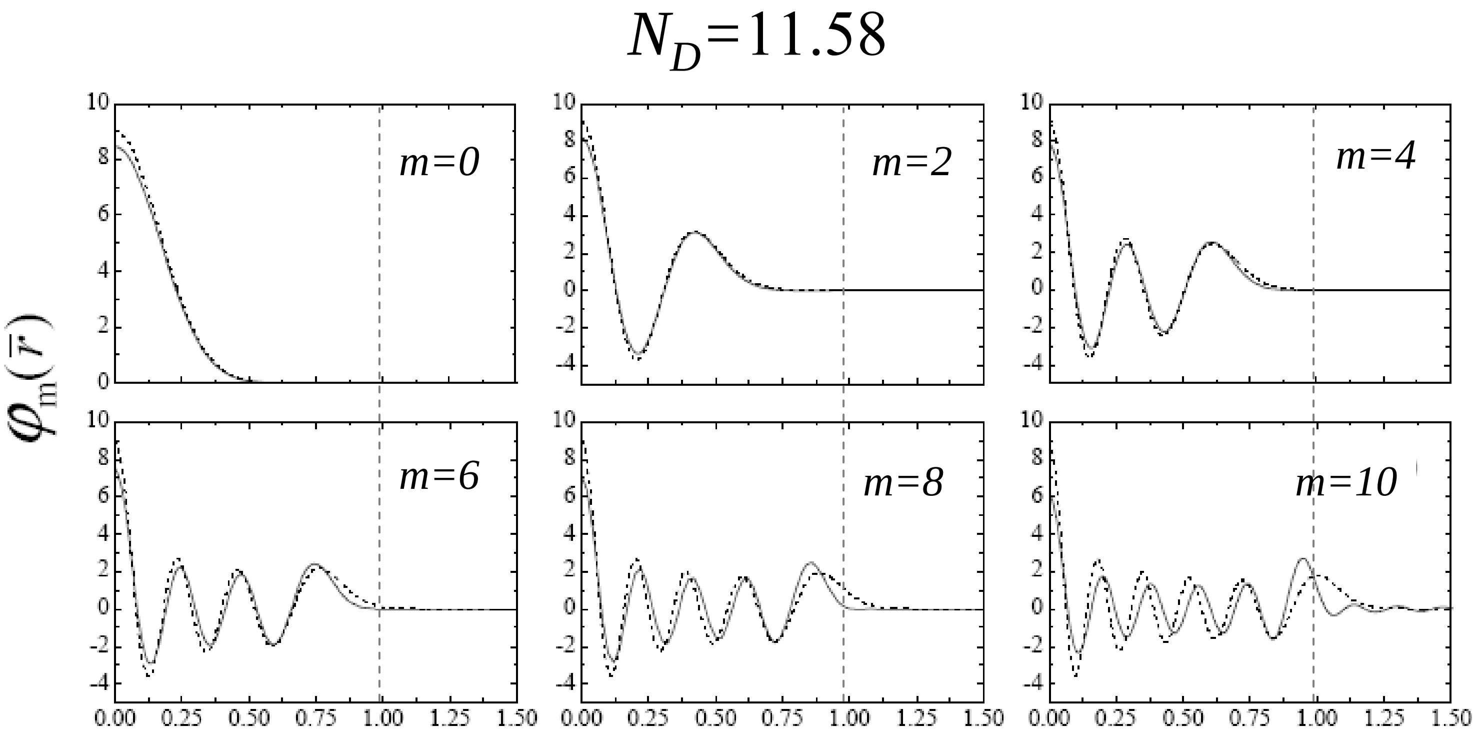
⬇

SLP eigenvalues drop *exponentially* from ~ 1 to $\sim$ 0 as order exceeds N_D

➡

SLP eigenfunctions turn from almost *perfectly localized* in $\bar{r} \leq 1$ to almost *fully delocalized* as order exceeds N_D

Peculiar Properties of rSLP Eigenstates, contd.



infinite-mirror (Gauss-Laguerre) modes also shown dashed

SLP Diffraction Loss Constraint

- Let sought field be expanded in terms of rSLP modes : $\phi(\bar{r}) = \sum_{n=1}^{N_T} b_n \varphi_n(\bar{r})$
- Diffraction loss constraint rephrases into (in view of double-orthogonality)

$$\mathcal{L}[\phi] = \int_1^{\infty} \bar{r} d\bar{r} \, |\phi(\bar{r})|^2 = \sum_{n=1}^{N_T} (1 - |\eta_n|^2) |b_n|^2 \leq$$

$$\leq \max_{n=1,2,\ldots,N_T} (1 - |\eta_n|^2) \sum_{n=1}^{N_T} |b_n|^2 \leq (1 - |\eta_{N_T}|^2)$$

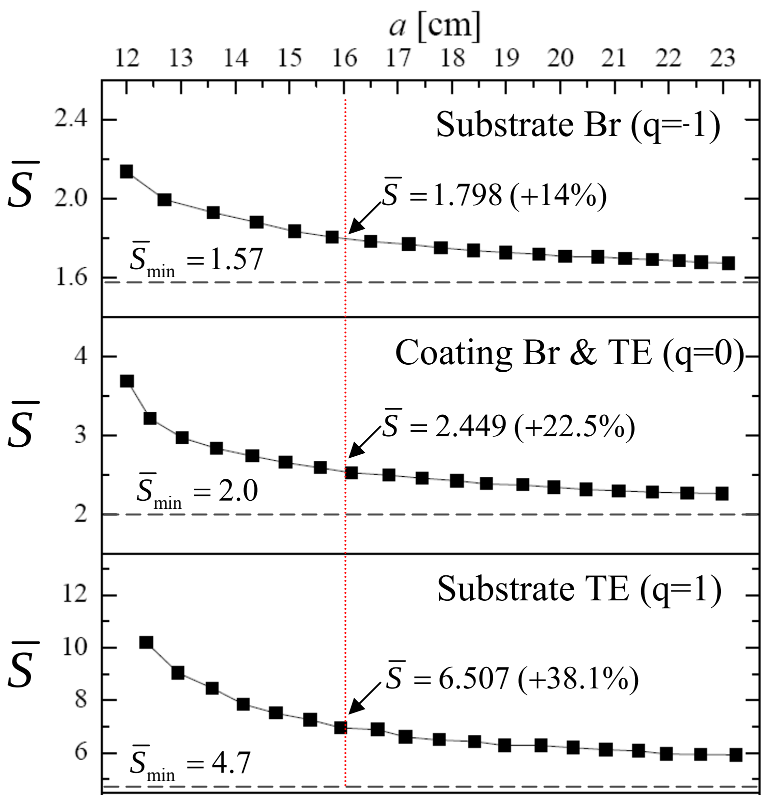
last inequality follows from : i) the fact that $\{|\eta_n|\}$ is monotonic - decreasing; ii) Parseval theorem; iii) the fact that $|| \phi || = 1$.

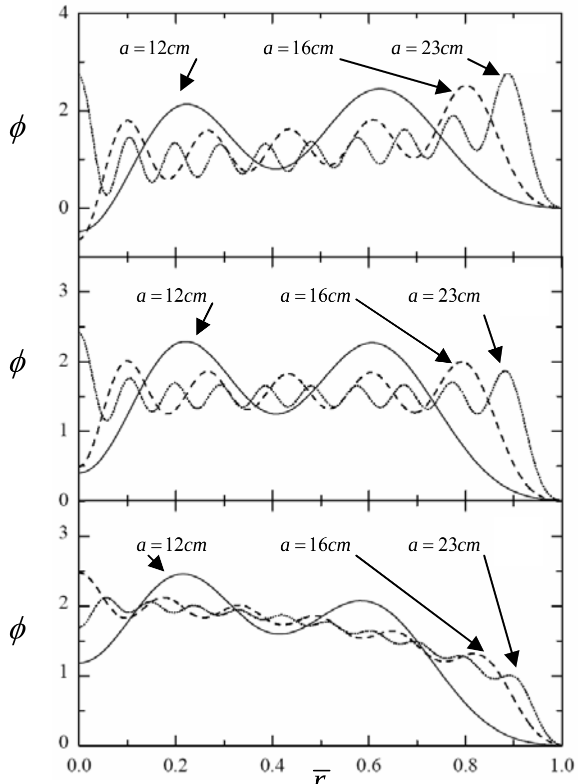
- **The diffraction loss constraint dictates the *effective dimension* $N_T \sim N_D$ of our optimization problem (number of unknown coefficients in the rSLP modal expansion of the cavity field)**

rLSP Approximants of Variational Solutions

- As a natural next step, we construct L^2 approximants of the (unphysical) fields obtained from minimal-noise variational-solutions, by suitable linear combinations of the lowest N_D rLSP-eigenstates.
- At variance of the compact-spatial-support fields deduced from the variational solutions, these fields will satisfy *both* the diffraction-loss constraint *and* the compact-spectral-support condition.
- However, there is NO guarantee that such fields may be decently approximated by the lowest (or any other pure) eigenstate corresponding to *some* mirror profile.

rLSP Approximants of Variational Solutions, contd.





$$\text{number of modes} = N_T \approx N_D = 2a^2 / \lambda L$$

How Far did We Reach ?

	$\overline{S}_{\min}$	$\overline{S}_{SLP} / \overline{S}_{\min}$	$\overline{S}_{MB} / \overline{S}_{\min}$	$\overline{S}_{GB} / \overline{S}_{\min}$
Substrate (Br)	1.5708	1.145	2.044	2.97
Coating (Br+TE)	2	1.225	3.227	6.92
Substrate (TE)	4.712	1.381	4.455	13.66

$a = 16cm$ ($N_D = 14$) ; $\mathcal{L}_T = 1ppm$; $w_{MB} = (N_D)^{-1/2}$ (minimum spreading)

…sensible possible improvement, e.g. by a factor 2.65 for the coating noise!

Conclusions

- Absolute noise lower bounds, corresponding to compact-spatial support intensity profiles have been identified, together with the intensity profiles themselves, via a variational approach;
- The effective dimension of the optimization problem has been related to the diffraction-loss constraint and found to be of the order of $N_D=2a^2/\lambda L$;
- A field solution (superposition of N_D rLSP-modes) coping w. *both* the diffraction-loss bound, *and* the compact-spectral-support property of eigenstates has been shown to get *fairly close* to the absolute noise bounds, for $N_D=2a^2/\lambda L$ *sufficiently large* (until the infinitely thick mirror approx-imation breaks down);

Conclusions, contd.

- While, there is *NO guarantee* that such fields may be the eigenstates of *some* mirror profile, the gap in terms of noise levels between the *best* currently available solutions (MB, HOGL) and the above lower bounds is *pretty large* (compared, in particular, to the expected infinitely - thick mirror assumptions related inaccuracy).

- This *suggests* that there's margin for further *substantial* noise reduction through mirror/beam optimization, and that the related conceptual/computational research effort is *worth*.

What is a *Good* “Optimized” Mirror ?

“Optimized” Mirror (and superimposed coating) must be technologically feasible (slope constraints, tolerances, etc.);

“Optimized” field (eigenstate) must be easy to launch (…ideally, a *dominant* mode…)

“Optimized” field must be robust w.r.t. cavity (& coupling) drifts/tolerances;

“Optimized” field must yield noise levels as close as possible to lower bounds stemming from (competing) diffraction loss constraint and compact support property of eigenstates.

A Possible Constructive Approach Being Explored

- Parameterize mirror profile *consistently* (to prevent ill-conditioning) to effective dimension $N_T \sim N_D = 2a^2/L\lambda$ of optimization problem;

- Derive lowest order eigenstate(s) using an *efficient* (fast and reasonably accurate) algorithm, e.g., Nystrom [J. Comp. Phys. 146 (1998) 627], or perhaps Donsker-Kac [J. Res. NBS 44 (1954) 551; V. Galdi et al., Electromagnetics, 18 (1998) 367];

- Use *robust* (e.g., *genetic*) *optimization* engine to tweak unknown mirror parameters (e.g., polynomial coefficients) to bring noise to a minimum, while coping with diffraction-loss *and suitable technological* constraints.