Flat Parameter-Space Metric for Continuous Gravitational Waves

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Abstract

The parameter-space metric describes the local distance structure of a template family for gravitational-wave (GW) signals. This is a fundamental building block for constructing efficient template banks. Here we focus on the coherent metric for continuous GWs, e.g. from spinning neutron stars (ground-based detectors) or from white-dwarf binaries (LISA). We show that this metric is approximately flat, and we discuss the relation of the local metric structure to the global "circles in the sky" structure in parameter space.

Continuous Gravitational Waves

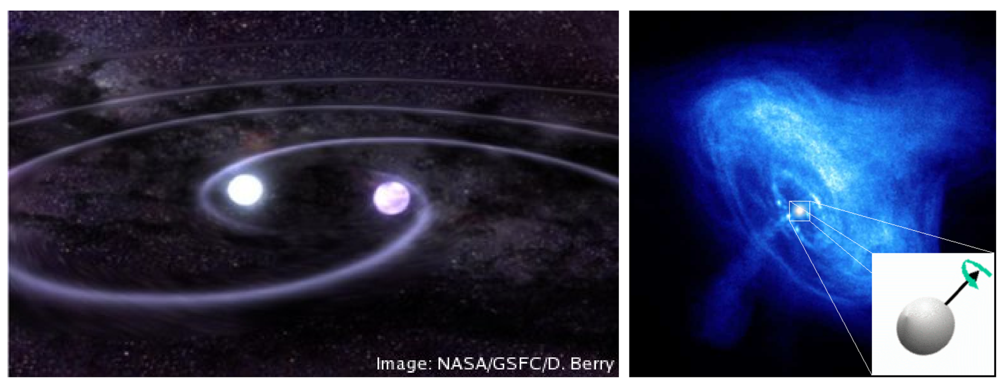

Figure 1: *Sources of continuous GWs. Left: white-dwarf binary systems (LISA, $f \sim \mathrm{mHz}$). Right: spinning deformed neutron stars (LIGO/GEO/Virgo etc, $f \sim 100\,\mathrm{Hz}$).*

A continuous GW signal $s(t)$ with intrinsic frequency $f(\tau)$, located at longitude α and latitude δ is characterized by its Doppler parameters $\theta = \{\alpha, \delta, f, \dot{f}, \ldots\}$ and amplitude parameters $\mathcal{A} = \{h_0, \cos\iota, \psi, \phi_0\}$. A reparametrization $\{\mathcal{A}^\mu\}_{\mu=1}^4$ of the amplitude parameters allows the factorization of the signal

$$s(t; \mathcal{A}, \theta) = \sum_{\mu=1}^{4} \mathcal{A}^\mu \, h_\mu(t; \theta) \,. \tag{1}$$

The optimal detection statistic is the *likelihood ratio* Λ:

$$\ln \Lambda(x; \mathcal{A}, \theta) \equiv (x\|s) - \frac{1}{2}(s\|s) \,, \tag{2}$$

where $x(t)$ is the measured strain data from the detector, and $(.\|.)$ is the (Wiener) matched-filtering scalar product. By analytically maximizing $\ln\Lambda$ over the unknown $\mathcal{A}^\mu$, we obtain the "$\mathcal{F}$-statistic" [4]:

$$2\mathcal{F}(x; \theta) = \sum_{\mu,\nu=1}^{4} (x\|h_\mu) \, \mathcal{M}^{\mu\nu} \, (x\|h_\nu) \,, \tag{3}$$

where $\mathcal{M}^{\mu\nu}$ is the matrix inverse of $\mathcal{M}_{\mu\nu} \equiv (h_\mu\|h_\nu)$.

☞ Using the $\mathcal{F}$-statistic, we only need to search over the unknown Doppler-parameters $\theta = \{\alpha, \delta, f, \dot{f}, \ldots\}$.

Parameter-space Metric for Continuous GWs

In the presence of a signal (1) with parameters $\{\mathcal{A}, \theta\}$, the expectation value of $2\mathcal{F}$, *targeting* an offset Doppler position $\theta_{\mathrm{t}} = \theta + \Delta\theta$ is

$$E\left[2\mathcal{F}(x; \theta_{\mathrm{t}})\right] = 4 + \mathrm{SNR}^2(\mathcal{A}, \theta; \Delta\theta) \,, \tag{4}$$

where SNR is the "signal-to-noise ratio". The SNR has a maximum $\mathrm{SNR}(\mathcal{A}, \theta; 0)$ when exactly targeting the signal. We define the *mismatch* m as the relative loss of SNR^2 due to the Doppler offset $\Delta\theta$:

$$\begin{aligned} m(\mathcal{A}, \theta; \Delta\theta) &\equiv \frac{\mathrm{SNR}^2(0) - \mathrm{SNR}^2(\Delta\theta)}{\mathrm{SNR}^2(0)} \\ &= g_{ij}(\mathcal{A}, \theta) \, \Delta\theta^i \, \Delta\theta^j + \mathcal{O}(\Delta\theta^3) \,, \end{aligned} \tag{5}$$

defining the metric tensor g_{ij} (first introduced in [1, 2]) of the Doppler space θ. As shown in [3], for long observation times $T \gg 1$ day the Doppler metric $g_{ij}(\mathcal{A}, \theta)$ can be approximated by the "orbital metric" $g^{\mathrm{orb}}_{ij}(\theta)$, defined as

$$g^{\mathrm{orb}}_{ij}(\theta) \equiv \langle \partial_i \phi \, \partial_j \phi \rangle - \langle \partial_i \phi \rangle \langle \partial_j \phi \rangle \equiv \left[\partial_i \phi, \, \partial_j \phi\right] \,, \tag{6}$$

where $\partial_i \equiv \partial/\partial\theta^i$, and the time-average $\langle Q \rangle \equiv \frac{1}{T}\int_0^T Q(t)\, dt$. The phase $\phi(t; \theta)$ in this expression is the GW phase *at the detector*, neglecting the spin-motion of the Earth, namely

$$\frac{\phi(t; \theta)}{2\pi} = f\,\left[t + \tilde{r}_{\mathrm{orb}}(t) \cdot \vec{n}\right] + \frac{1}{2}\dot{f}\,\left[t + \tilde{r}_{\mathrm{orb}}(t) \cdot \vec{n}\right]^2 + \ldots \,, \tag{7}$$

where $\tilde{r}_{\mathrm{orb}}(t)$ is the light-travel time between the solar-system barycenter (SBB) and the Earth (or LISA), and $\vec{n}$ is the unit-vector pointing to the sky-position α, δ of the signal (see Fig. 2).

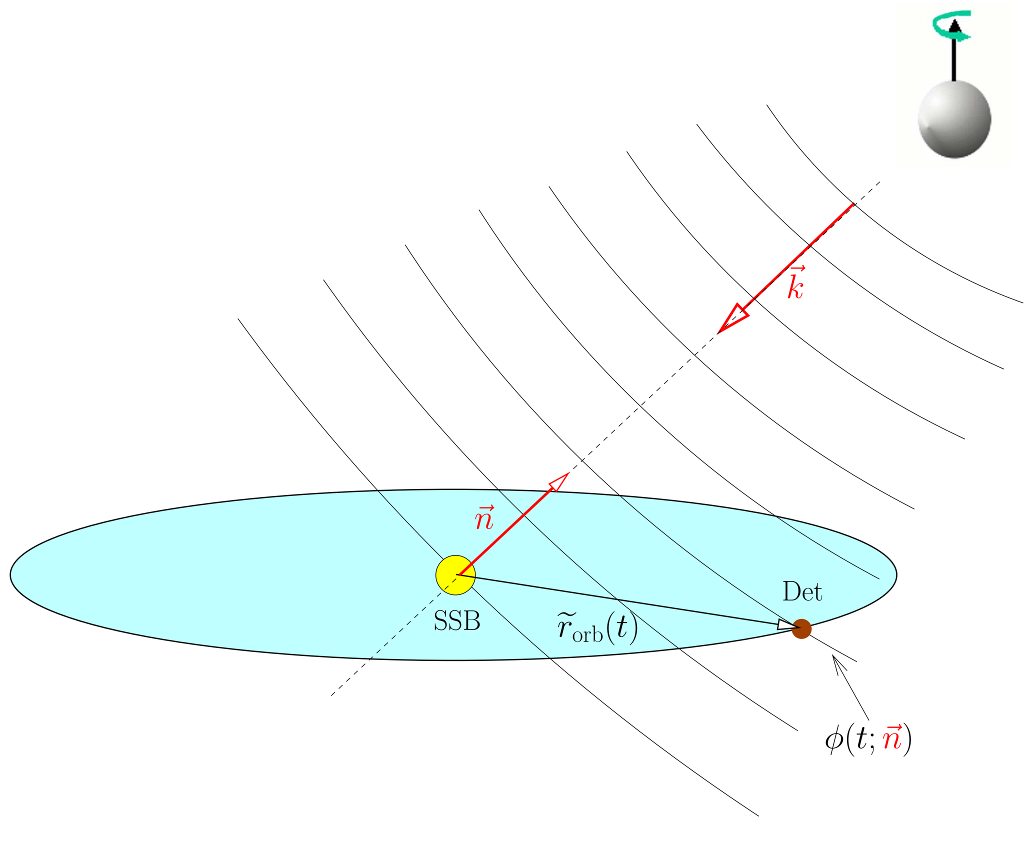

Figure 2: *Orbital motion of the detector ('Det') in the ecliptic plane (neglecting spin-motion of the earth) and Doppler evolution of the GW-phase $\phi(t; \vec{n})$ (constant along wavefronts) for a signal from sky-direction $\vec{n}$.*

The parameter-space metric is the *local* description of the *global* parameter-space structure. For moderately short observation times $T \ll 1$ year, the global parameter-space structure can be approximately described [5] by the "circles in the sky":

$$f\left(1 + \vec{\beta} \cdot \vec{n}\right) = \mathrm{const.} \,, \tag{8}$$

where $\vec{\beta}$ is the average orbital velocity (in units of c) during the observation time. Fig. 3 shows the circles in the sky and the metric ellipses for fixed f in equatorial coordinates α, δ, which are commonly used for ground-based detectors.

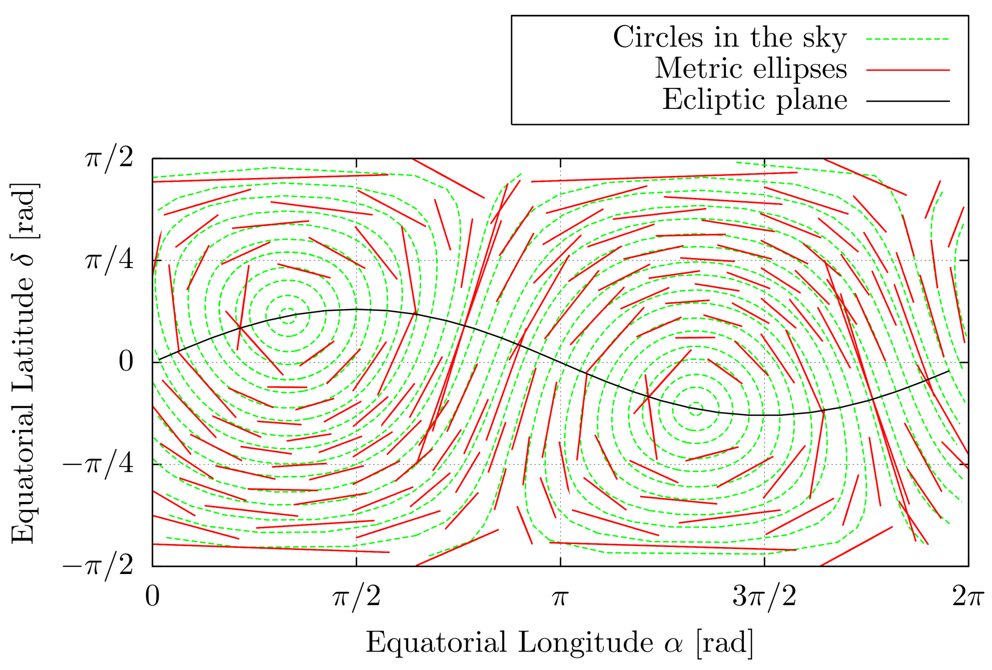

Figure 3: *Skymap of "Circles in the Sky" (8) and metric (iso-mismatch) ellipses for fixed spin-parameters $f, \dot{f}, \ldots$. The ellipses are "needle"-like and tangential to the global structure. The orientation and size of the ellipses depend on the sky position α, δ.*

The same global and local parameter-space structure is illustrated in Fig. 4, translated into *ecliptic* coordinates λ, β.

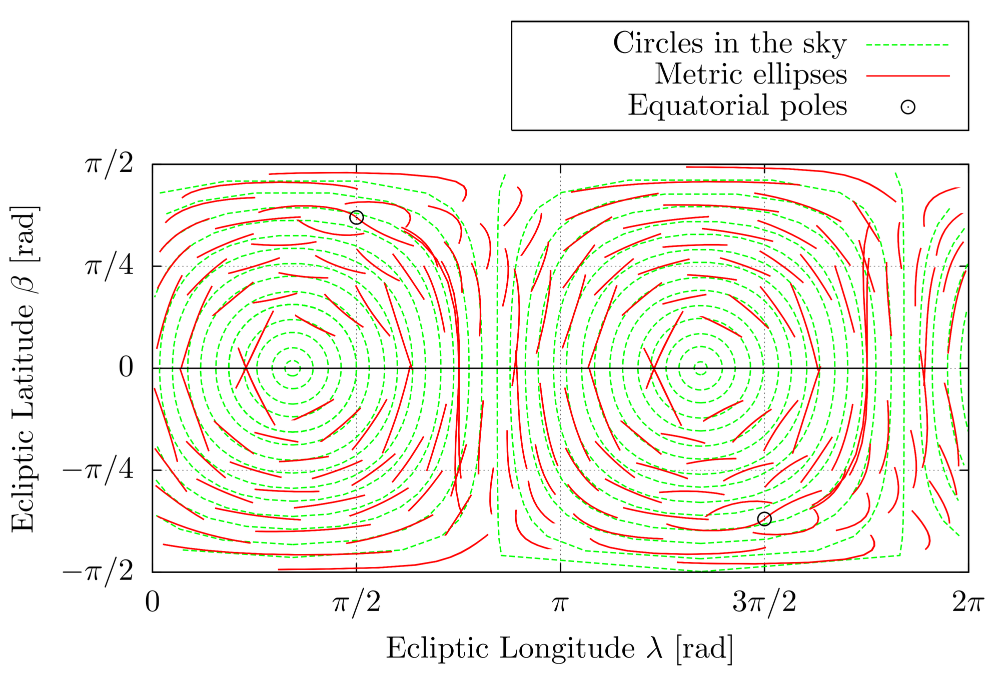

Figure 4: *Skymap of "Circles in the Sky" (8) and metric (iso-mismatch) ellipses as shown in Fig. 3, translated into ecliptic coordinates λ, β. (The apparent "deformation" of the metric ellipses is an artifact of their finite size and the coordinates-transformation.) The orientation and size of the ellipses still depends on the sky position λ, β.*

(Approximate) Flatness of the Orbital Metric

As shown in [4], the phase (7) can be approximated as

$$\frac{\phi(t; \theta)}{2\pi} \approx f\,\tilde{r}_{\mathrm{orb}}(t) \cdot \vec{n} + f\,t + \frac{1}{2}\dot{f}\,t^2 + \ldots \,, \tag{9}$$

and introducing new variables for the sky-position α, δ:

$$k_x \equiv -2\pi f\, n_x \,, \quad k_y \equiv -2\pi f\, n_y \,, \tag{10}$$

where n_x, n_y are the components of $\vec{n}$ in the *ecliptic plane*. Because of $\vec{n}^2 = 1$, the range of $\{k_x, k_y\}$ is restricted to the disc ${k_x}^2 + {k_y}^2 \leq (2\pi f)^2$. In these new Doppler-variables $\Theta \equiv \{k_x, k_y, f, \dot{f}, \ldots\}$, the variation of the orbital phase (9) can be approximated as

$$d\phi(t; \Theta) \approx \tilde{r}^x_{\mathrm{orb}}(t)\, dk_x + \tilde{r}^y_{\mathrm{orb}}(t)\, dk_y + 2\pi t\, df + \pi t^2\, d\dot{f} + \ldots \,, \tag{11}$$

and using this the orbital metric (6) is found as

$$g^{\mathrm{orb}} \approx \begin{array}{c} \begin{array}{cccc} \quad k_x \quad & \quad k_y \quad & \quad f \quad & \ldots \end{array} \\ \begin{pmatrix} \left[\tilde{r}^x_{\mathrm{orb}}, \tilde{r}^x_{\mathrm{orb}}\right], & \left[\tilde{r}^x_{\mathrm{orb}}, \tilde{r}^y_{\mathrm{orb}}\right], & \left[\tilde{r}^x_{\mathrm{orb}}, 2\pi\,t\right], & \ldots \\ \left[\tilde{r}^y_{\mathrm{orb}}, \tilde{r}^x_{\mathrm{orb}}\right], & \left[\tilde{r}^y_{\mathrm{orb}}, \tilde{r}^y_{\mathrm{orb}}\right], & \left[\tilde{r}^y_{\mathrm{orb}}, 2\pi\,t\right], & \ldots \\ \left[2\pi\,t, \tilde{r}^x_{\mathrm{orb}}\right], & \left[2\pi\,t, \tilde{r}^y_{\mathrm{orb}}\right], & \left[2\pi\,t, 2\pi\,t\right], & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \,, \tag{12}$$

which *does not depend* on the Doppler point Θ!

☞ this metric is explicitly seen to be *flat*. □

Fig. 5 shows the iso-mismatch ellipses of the flat metric (12) (for fixed $f, \dot{f}, \ldots$) and the "circles in the sky" (8), in sky-coordinates n_x, n_y (which are proportional to k_x, k_y (10)).

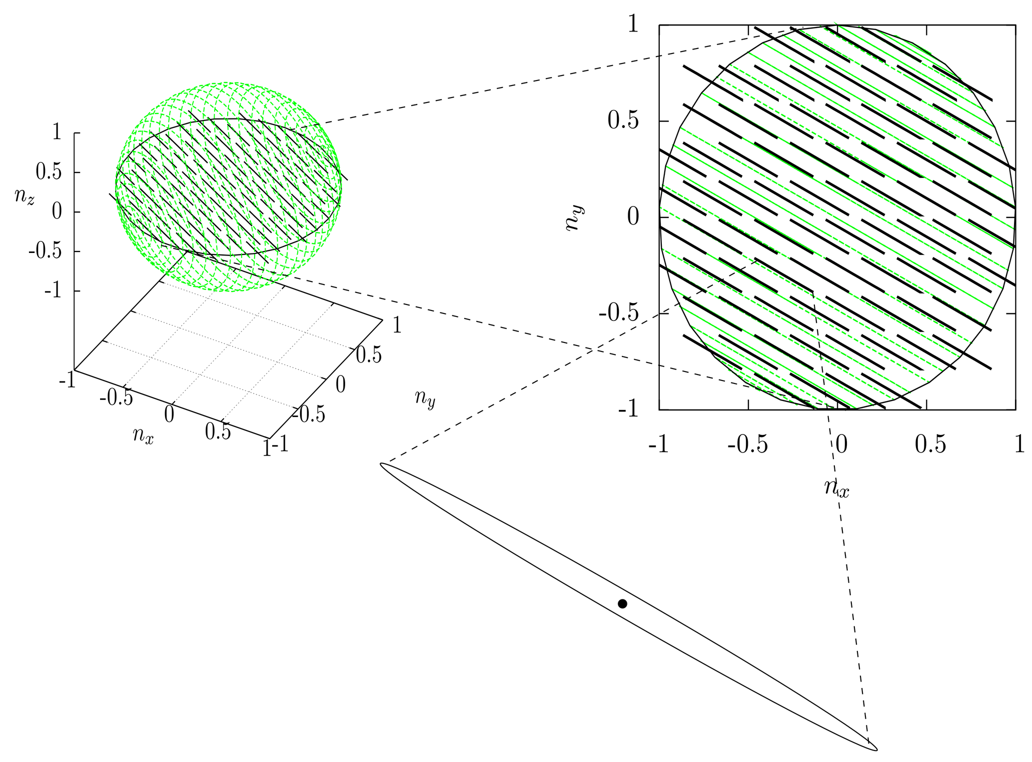

Figure 5: *With $\{n_x, n_y\}$ in the ecliptic plane as sky-coordinates, the "circles in the sky" reduce to parallel straight lines. The iso-mismatch ellipses of the orbital metric (12) are independent of sky-position n_x, n_y and parallel to the global circles.*

Note that the choice of k_x, k_y (10) instead of n_x, n_y as sky coordinates has the advantage of rendering the metric (12) independent of the frequency f. With this choice the metric is constant, but the parameter-space corresponding to the sky grows with frequency, as ${k_x}^2 + {k_y}^2 \leq (2\pi f)^2$: the parameter space has the form of a cone with constant metric. Alternatively one could use $\{n_x, n_y\}$ as sky-coordinates, resulting in a fixed "sky" ${n_x}^2 + {n_y}^2 \leq 1$ and a parameter-space in the form of a cylinder, but with a metric that scales with frequency. In both cases the number of templates will therefore grow as $\propto f^2$, as expected.

Summary

We have shown that a series of suitable approximations together with the choice of the ecliptic components k_x, k_y of the "wave-vector" (10) as "sky"-coordinates renders the metric constant over the whole parameter space. This is an important step for the construction of more efficient template banks for the search for continuous GWs, such as emitted from spinning neutron stars or white-dwarf binary systems.

References

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