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# *Perspectives on Beam-Shaping Optimization for Thermal-Noise Reduction in Advanced LIGO: Bounds, Profiles, and Critical Parameters*

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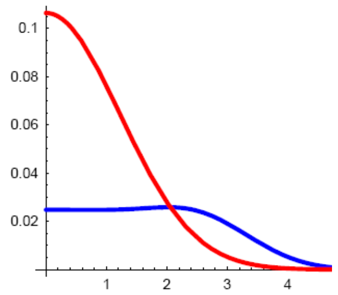
# Background, motivations, and approach

- Laser-beam shape
  - Affects thermal noise
    [Levin, *Phys. Rev. D* **57**, 659, 1998; Bondu *et al.*, *Phys. Lett. A* **246**, 227, 1998; Liu and Thorne, *Phys. Rev. D* **62**, 122002, 2000]
  - Can be controlled to a certain extent

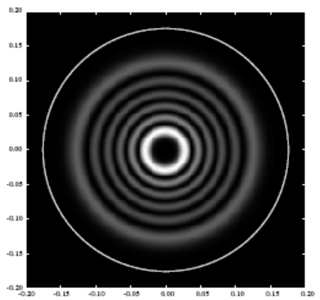
- Basic question
  - What is the "***optimal***" (i.e., minimum noise) beam shape?

- Some answers
  - Formulation of the optimization problem – a nasty one!
  - Derivation of some (absolute and realistic) bounds
  - Assessments
    - Potential noise reductions w.r.t current status
    - Goodness of suboptimal solutions (e.g., nearly-Bessel-Gauss beams [Bondarescu, PhD Thesis, 2007])

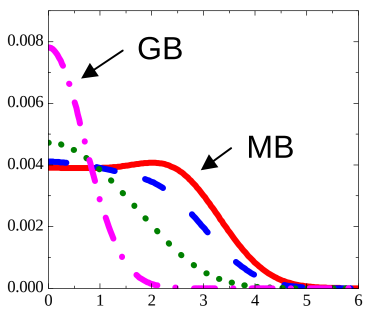
# Current status and research trends




- Reference solution: **Gaussian beams** (GBs)
- **Mesa beams** (MBs), Mexican Hat (MH) mirrors
  - Reduction (coating noise) of a factor ~2 w.r.t. GBs
    [D'Ambrosio, *Phys. Rev. D* **67**,102004, 2003]



- **Higher-order Gauss-Laguerre** (HOGL) modes
  - Keep standard (spherical) mirrors
  - Excitation issues
    [Mours *et al.*, *Class. Quantum. Grav.* **23**, 5777, 2006]




- **Hyperboloidal-beams**, nearly-spheroidal mirrors
  - Span from nearly-flat to nearly-concentric MBs
    [Bondarescu and Thorne, *Phys. Rev. D* **74** 082003, 2006]
  - Analytic representations [Galdi *et al.*, *Phys. Rev. D.* **73**, 127101, 2006]
  - Potential noise reduction of ~30% w.r.t. MBs [Lundgren *et al.*, *Phys. Rev. D* **77,** 042003, 2008]

# Theoretical framework

[O'Shaughnessy, *Class. Quantum. Grav.* **23**, 7627, 2006; Lovelace, *Class. Quantum Grav.* **24**, 4491, 2007]

## Thermal noise vs. beam shape

$$S = C \int_0^{\infty} \kappa^{q+1} \left\{ \mathcal{H}\left[ |\Phi|^2 \right](\kappa) \right\}^2 d\kappa \,, \qquad q = \begin{cases} 0 & \text{Coating (Brownian \& Thermoelastic)} \\ 1 & \text{Substrate Brownian } (SiO_2) \\ -1 & \text{Substrate Thermo-elastic } (Al_2O_3) \end{cases}$$

$$\mathcal{H}[F](\xi) \equiv \int_0^{\infty} F(\zeta) J_0(\xi\zeta)\zeta d\zeta$$

Hankel transform (HT)

$\left| \Phi(r) \right|^2 \equiv$ beam intensity distribution at mirror

**Assumptions:**
- axisymmetric field distribution
- infinite (thick) test-mass
- low frequency limit

# Theoretical framework (cont'd)

[Siegman, *Lasers*, Univ. Sci. Books, Mill Valley, US, 1998]

$$\gamma\Phi(r) = \int_0^a K(r,r')\Phi(r')r'dr'$$ (integral eq., eigenvalue problem)

$$K(r,r') = \frac{\imath k}{L} J_0\left(\frac{krr'}{L}\right) \exp\left\{\imath k\left[-L + h(r) + h(r') - \frac{(r^2 + r'^2)}{2L}\right]\right\}$$

$h(r)$ $\equiv$ mirror profile (departure from flatness)

$a$ $\equiv$ mirror radius

$L \equiv$ cavity length; $k = 2\pi / \lambda$ $\equiv$ wavenumber

Mapping between : a mirror profile $h(r)$
a set of eigenstates $\Omega[h] = \{\gamma_n, \Phi_n\}$

# Theoretical framework (cont'd)

Light spillover (diffraction) beyond mirror should be limited:

$$\mathcal{L}[\Phi] \equiv \int_a^\infty |\Phi(r)|^2 \, r dr \;\; \leq \mathcal{L}_T$$ (e.g., 1ppm for Adv-LIGO)

It is always possible to make
(will be assumed throughout)

$$\int_0^\infty |\Phi(r)|^2 \, r dr = 1$$

so as to rewrite the diffraction loss constraint as

$$1 - |\gamma|^2 \leq \mathcal{L}_T$$

**(selects diffraction-loss *admissibile* eigenstates)**

# Optimal (minimum-noise) beam/mirror profile

## *Formal* mirror optimization procedure

- Assume suitable (e.g., $C^{\infty}$) functional class $\Lambda$ for $h(r)$

- Denote as $\Omega_c[h]$ the subset of the eigenstate set $\Omega[h]$:

$$1 - |\gamma|^2 \leq \mathcal{L}_T$$

- Find $h^* \in \Lambda$ such that :

$$\min_{\phi \in \Omega_c[h^*]} S[\phi] \leq \min_{\phi \in \Omega_c[h]} S[\phi], \quad \forall h \in \Lambda : h \neq h^*$$

# …A nasty problem

- For *most* $h(r)$ , the field integral equation can only be attacked *numerically* $\Rightarrow$ need to parameterize sought function $h(r)$ in terms of a *finite* number of unknowns

  $\Rightarrow$ $\begin{cases} \text{“best” (minimum size) representation ?} \\ \text{size of problem ?} \end{cases}$

- Concerns about numerical optimization
  - Parameterization-dependent problem’s ill-posedness
  - Non-convexity
    - Local minima (*robust* optimization algorithms required)

- “Exact” solution could be *technologically unfeasible*

# Scalings

$$\bar{r} = r/a, \quad \bar{\kappa} = a\kappa, \quad \phi(\bar{r}) = a\Phi(a\bar{r})$$

Scaled (dimensionless) radial coordinate, wave-number and field

$$\bar{S} = \int_0^{\infty} \bar{\kappa}^{q+1} \left\{ \mathcal{H}\left[|\phi|^2\right](\bar{\kappa}) \right\}^2 d\bar{\kappa} \quad , \quad S = a^{-(q+2)} C\,\bar{S}$$

Scaled noise PSD

$$\bar{\gamma}\phi(\bar{r}) = \imath\pi N_D \exp\left[-\imath V(\bar{r})\right] \mathcal{H}_1\left[\exp\left(-\imath V\right)\phi\right](\pi N_D \bar{r})$$

Scaled field equation

$$\bar{\gamma} = \gamma \exp\left(\imath k L\right)$$

Scaled half-round-trip eigenvalue

$$\mathcal{H}_1[F](\xi) \equiv \int_0^1 F(\zeta) J_0(\xi\zeta)\zeta d\zeta$$

Clipped (finite radius mirror) Hankel Tr.

$$V(\bar{r}) = kh(a\bar{r}) - \frac{\pi N_D \bar{r}^2}{2} \quad , \quad N_D \equiv 2N_F = \frac{2a^2}{\lambda L}$$

Mirror-profile dependent phase (unknown)

Fresnel number of cavity

# Absolute (lower) noise PSD bounds

- Cope with diffraction-loss constraint by forcing $\phi(\bar{r})$ to *vanish* outside [0,1] (no-diffraction, compact support beams)
- Don't care about field (eigenvalue) equation. Just seek for an *intensity* profile $f(\bar{r}) = |\phi(\bar{r})|^2 \geq 0$ for which PSD is minimum
- Translates into simple (constrained) variational calculus problems, with unique *exact solutions*

  [Pierro *et al.*, *Phys. Rev. D* **76,** 122003, 2007]

$$f(\bar{r}) = (q+2)\left[1-\bar{r}^2\right]^{q/2}, \quad -1 \leq q \leq 1, \ q \in \mathbb{Z}, \ 0 \leq \bar{r} \leq 1$$

yielding:

$$\boxed{\bar{S}_{abs}^{(\min)} = 2^{q+1}\Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+2\right)}$$

# Absolute (lower) noise PSD bounds (cont'd)

[Pierro *et al.*, *Phys. Rev. D* **76,** 122003, 2007]

Define:

$$Q[\varphi,\mu]=\left\|\overline{\kappa}^{q/2}H_1[\varphi]\right\|^2-2\mu\left[\int_0^1 \overline{r}\,d\overline{r}\,\varphi(\overline{r})-1\right],$$

Stationary (variational) weak solution:

$$\left.\frac{\delta Q}{\delta\varphi}\right|_{\varphi=f}=\lim_{\varepsilon\to 0}\frac{Q[f+\varepsilon\xi,\mu]-Q[f,\mu]}{\varepsilon}=0,\quad \forall\,\xi\in L_1[0,1]\ :\ \int_0^1\xi(x)x\,dx=0$$

$$\delta Q=2\varepsilon\boxed{\left\langle\overline{\kappa}^{q/2}H_1[f],\overline{\kappa}^{q/2}H_1[\xi]\right\rangle}+\varepsilon^2\boxed{\left\|\overline{\kappa}^{q/2}H_1[\xi]\right\|^2}$$

Variational solution obtained equating this piece to zero

this being *positive, solution yields a minimum*

# Absolute (lower) noise PSD bounds (cont'd)

[Pierro *et al.*, *Phys. Rev. D* **76,** 122003, 2007]

should vanish

becomes: $$\int_0^1 \bar{r}\, d\bar{r} \boxed{\left[\int_0^\infty d\bar{\kappa}\, \bar{\kappa}^{q+1} J_0(\bar{\kappa}\bar{r}) \int_0^1 \bar{r}' d\bar{r}' f(\bar{r}') J_0(\bar{\kappa}\bar{r}') - \mu\right]} \xi(\bar{r}) = 0, \quad 0 \le \bar{r} \le 1$$

use (see, Ryzhik & Gradhstein Tables)

$$\int_0^\infty d\bar{\kappa}\, \bar{\kappa}^{q/2} J_{q/2+1}(\bar{\kappa}) J_0(\bar{\kappa}\bar{r}) = 2^{q/2}\Gamma\left(\frac{q}{2}+1\right), \quad 0 \le \bar{r} \le 1, \quad -1 \le q \le 1$$

$$\bar{\kappa}^{q/2} J_{q/2+1}(\bar{\kappa}) = \frac{2^{-q/2}\,\bar{\kappa}^{q+1}}{\Gamma(q/2+1)} \int_0^1 \bar{r}' d\bar{r}' (1-\bar{r}'^2)^{q/2} J_0(\bar{\kappa}\bar{r}'), \quad 0 \le \bar{r} \le 1$$

to get: $$\int_0^\infty d\bar{\kappa}\, \bar{\kappa}^{q+1} J_0(\bar{\kappa}\bar{r}) \int_0^1 \bar{r}' d\bar{r}' (1-\bar{r}'^2)^{q/2} J_0(\bar{\kappa}\bar{r}') = 2^q \Gamma^{-2}(q/2+1)$$

whence: $$f = \mu\, 2^{-q} (1-\bar{r}'^2)^{q/2} \Gamma^{-2}(q/2+1), \quad \|f\| = 1 \Leftrightarrow \mu = (q+2)2^q \Gamma(q/2+1)$$

*qed*

# For more details

PHYSICAL REVIEW D **76**, 122003 (2007)

## Perspectives on beam-shaping optimization for thermal-noise reduction in advanced gravitational-wave interferometric detectors: Bounds, profiles, and critical parameters

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Suitable shaping (in particular, *flattening* and *broadening*) of the laser beam has recently been proposed as an effective device to reduce internal (mirror) thermal noise in advanced gravitational-wave interferometric detectors. Based on some recently published analytic approximations (valid in the infinite-test-mass limit) for the Brownian and thermoelastic mirror noises in the presence of *arbitrary-shaped* beams, this paper addresses certain preliminary issues related to the *optimal beam-shaping* problem. In particular, with specific reference to the Laser Interferometer Gravitational-wave Observatory (LIGO) experiment, absolute and realistic *lower bounds* for the various thermal-noise constituents are obtained and compared with the current status (Gaussian beams) and trends (mesa beams), indicating fairly ample margins for further reduction. In this framework, the effective dimension of the related optimization problem, and its relationship to the critical design parameters are identified, physical-feasibility and model-consistency issues are considered, and possible additional requirements and/or prior information exploitable to drive the subsequent optimization process are highlighted.

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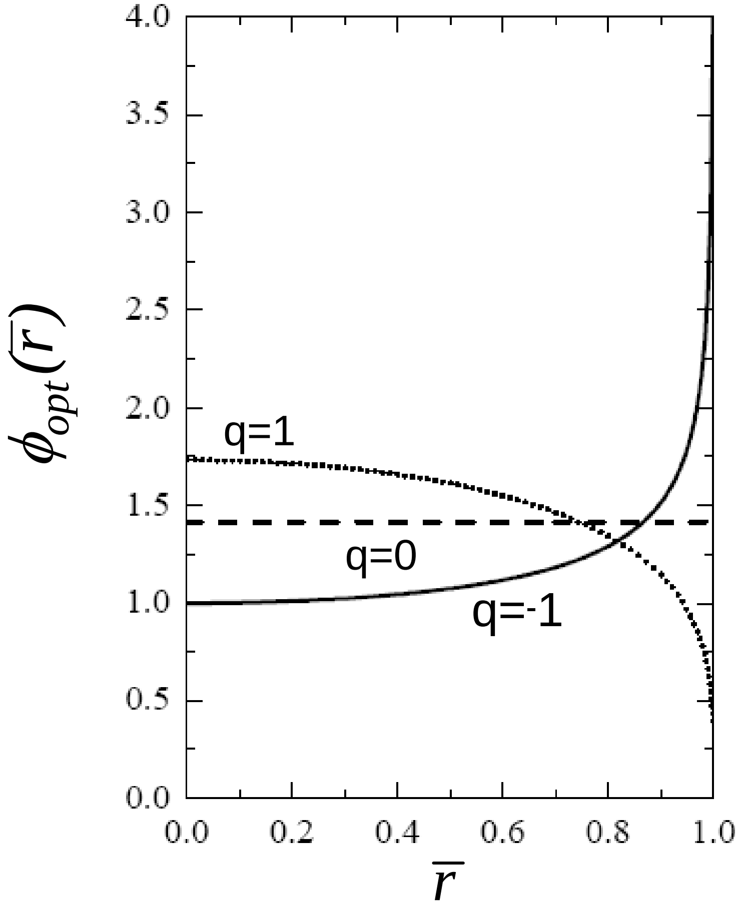
### I. INTRODUCTION

In all currently operating (and possibly future) interferometric gravitational-wave detectors, the overall limit sensitivity of the instrument is bounded by the noise floor, which, in the most interesting observational frequency

led to the development of a cavity prototype with non-spherical "Mexican hat" (MH) profile mirrors [11,12]. Alternative (nearly concentric [13], nearly spheroidal [14–16]) designs have been subsequently proposed to cope with the inherent tilt-instability of the originally conceived nearly flat configuration. Also, use of higher-

# Absolute (lower) noise PSD bounds (cont'd)

[Pierro *et al.*, *Phys. Rev. D* **76,** 122003, 2007]




## Remarks/caveats

Optimal field-intensity profile for coating noises flat as expected; for substrate noises, *not exactly flat,* and not obvious

Obtained (scaled) field-intensity profiles yield absolute but likely *loose* lower bounds for the noise PSDs

The zero-diffraction field assumption made is clearly violated by *any* solution of the field equation

How can we introduce the diffraction-loss and other *physical-feasibility* constraints ?

# Spatial band-limitedness of cavity eigenmodes

[Pierro *et al.*, *Phys. Rev. D* **76,** 122003, 2007]

From the obvious properties:

$$\mathcal{H}_1[f(\bar{r})] = \mathcal{H}[\Pi(\bar{r})f(\bar{r})],\ \Pi(\bar{r}) = \begin{cases} 1, & 0 \le \bar{r} \le 1 \\ 0, & \text{elsewhere} \end{cases},\quad \mathcal{H}[\mathcal{H}[f]] = f$$

by applying $\mathcal{H}$ operator to both sides of field (eigenvalue) equation we obtain

$$\mathcal{H}\left[\phi \exp\left(iV\right)\right]\left(\pi N_D \bar{r}\right) = i\frac{\pi N_D}{\bar{\gamma}} \Pi\left(\bar{r}\right) \exp\left[-iV\left(\bar{r}\right)\right]\phi\left(\bar{r}\right)$$

$\Downarrow$

The Hankel transform (wavenumber spectrum) of $\exp[\imath V(\bar{r})]\phi(\bar{r})$ has *compact support ,* vanishing outside $[\,0, \pi N_D]$. Accordingly $\exp[\imath V(\bar{r})]\phi(\bar{r})$, and hence $\phi$, *cannot* vanish identically for $\bar{r} > 1$.

# The PSWF basis

[Slepian *et al.*, *Bell System Tech. Journal* **40**, **43**, **65**, 1961; *ibid.* **41**, 1295, 1962]

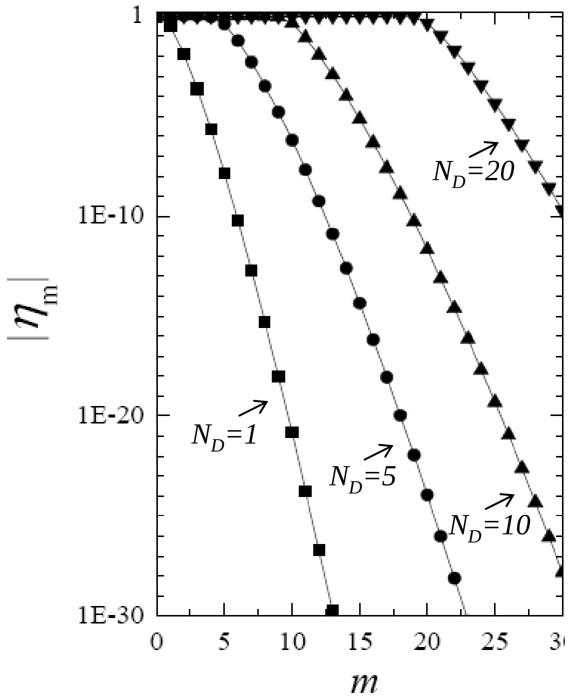
- The (real valued) eigenstates of a confocal-spherical finite-mirror cavity) play *a special role*
  - Prolate-spheroidal wave-functions (PSWFs)

$$\bar{\eta}\varphi(\bar{r}) = i\pi N_D \mathcal{H}_1 \left[\varphi\right] \left(\pi N_D \bar{r}\right)$$

- Among *all* $L^2$ bases, they allow to approximate *any* exact solution of the field equations (corresponding to an arbitrary mirror profile), using the minimum number $N_\varepsilon$ of terms for *any* prescribed $L^2$ error $\varepsilon$ (*minimum-redundant basis*)

- Technically, $N_\varepsilon$ is referred to as the number of *degrees of freedom* of our cavity fields at the *resolution level* $\varepsilon$

# Peculiar properties of PSWFs




*double*-orthogonality :

$$\begin{cases} \displaystyle\int_0^\infty \bar{r}d\bar{r}\varphi_n(\bar{r})\varphi_m^*(\bar{r}) = \delta_{mn} \\ \displaystyle\int_0^1 \bar{r}d\bar{r}\varphi_n(\bar{r})\varphi_m^*(\bar{r}) = |\eta_n|^2\delta_{mn} \end{cases}$$

$\Downarrow$

$$\int_1^\infty \bar{r}d\bar{r}\,|\varphi_n(\bar{r})|^2 = 1 - |\eta_n|^2$$

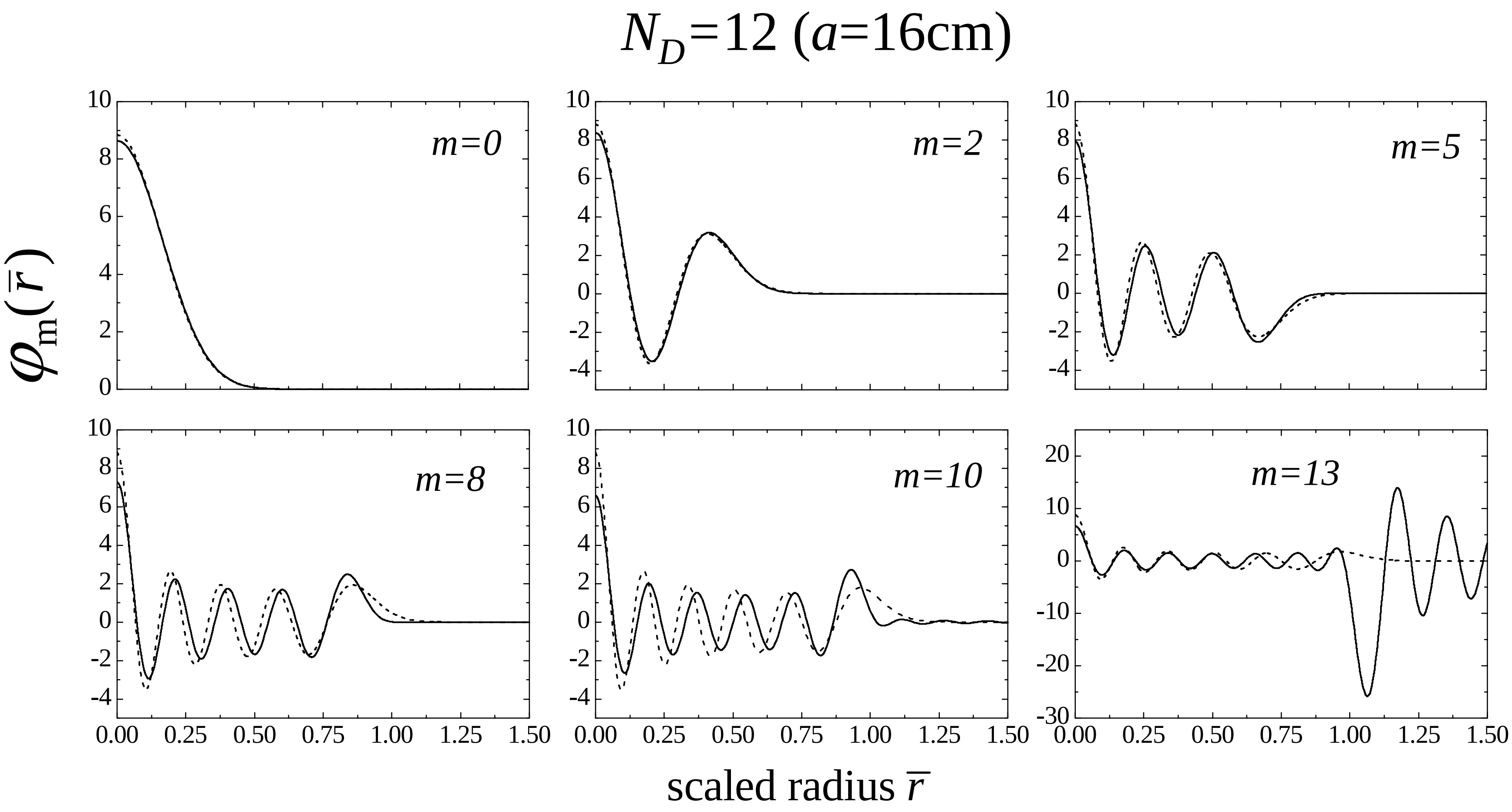
$\Downarrow$

PSWF eigenvalues drop *exponentially* from $\sim 1$ to $\sim$ 0 as order exceeds $N_D$

$\Rightarrow$

PSWFs turn from almost *perfectly localized* in $\bar{r} \leq 1$ to almost *fully delocalized* as order exceeds $N_D$

# Peculiar properties of PSWFs




infinite-mirror (Gauss-Laguerre) modes also shown dashed

# Diffraction-loss constraint

- PSWF expansion (bandlimited approximation)

$$\phi_{\mathrm{BL}}(\bar{r}) = \sum_{m=0}^{M_T-1} c_m \varphi_m(\bar{r})$$

- Diffraction loss constraint rephrases into (in view of double-orthogonality)

$$\mathcal{L}[\phi_{\mathrm{BL}}] = \sum_{m=0}^{M_T-1} (1 - |\bar{\eta}_m|^2)|c_m|^2$$

$$\leq (1 - |\bar{\eta}_{M_T-1}|^2) \underbrace{\sum_{m=0}^{M_T-1} |c_m|^2}_{=1} = (1 - |\bar{\eta}_{M_T-1}|^2)$$

Step behavior
(=0 for $M_T > N_D$)

- The diffraction loss constraint dictates the *effective dimension* $M_T \sim N_D$ of our optimization problem (number of unknown coefficients in the PSWF modal expansion of the cavity field)

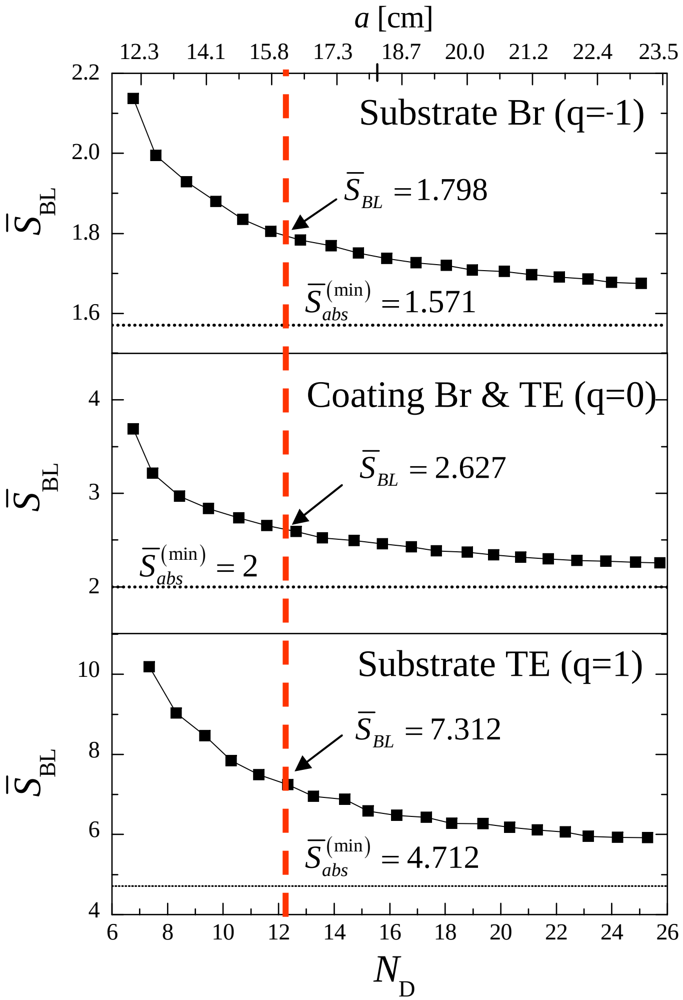
# Bandlimited approximants

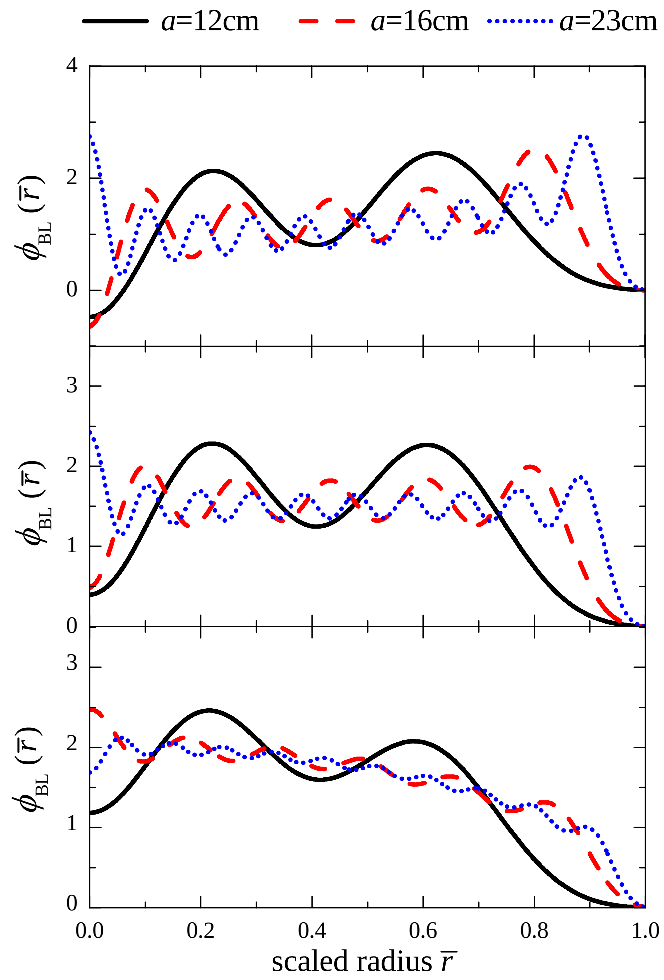
- Construct $L^2$ approximants of the (unphysical) fields obtained from minimum-noise variational-solutions
  - Suitable linear combinations of the lowest $N_D$ PSWFs

- Unlike compact-support variational solutions, these fields will satisfy *both* the diffraction-loss constraint *and* the spatial-bandlimitedness condition

- While there is NO guarantee that such fields may be supported by *some* mirror profile, the corresponding noise bound are expected to be tighter

# Bandlimited approximants (cont'd)




$$\boxed{\text{number of modes} = N_T \approx N_D = 2a^2 / \lambda L}$$

# How far did we reach?

[Pierro *et al.*, *Phys. Rev. D* **76,** 122003, 2007]

| $q$ | $\bar{S}^{(min)}_{abs}$ | $\bar{S}_{BL}/\bar{S}^{(min)}_{abs}$ | $\bar{S}_{GB}/\bar{S}^{(min)}_{abs}$ | $\bar{S}^{(min)}_{MB}/\bar{S}^{(min)}_{abs}$ | $\bar{S}_{GB}/\bar{S}_{BL}$ | $\bar{S}^{(min)}_{MB}/\bar{S}_{BL}$ |
|---|---|---|---|---|---|---|
| -1 | 1.5708 | 1.145 | 2.965 | 2.043 | 2.591 | 1.785 |
| 0 | 2 | 1.313 | 6.907 | 3.238 | 5.256 | 2.465 |
| 1 | 4.712 | 1.552 | 13.658 | 4.454 | 8.801 | 2.870 |

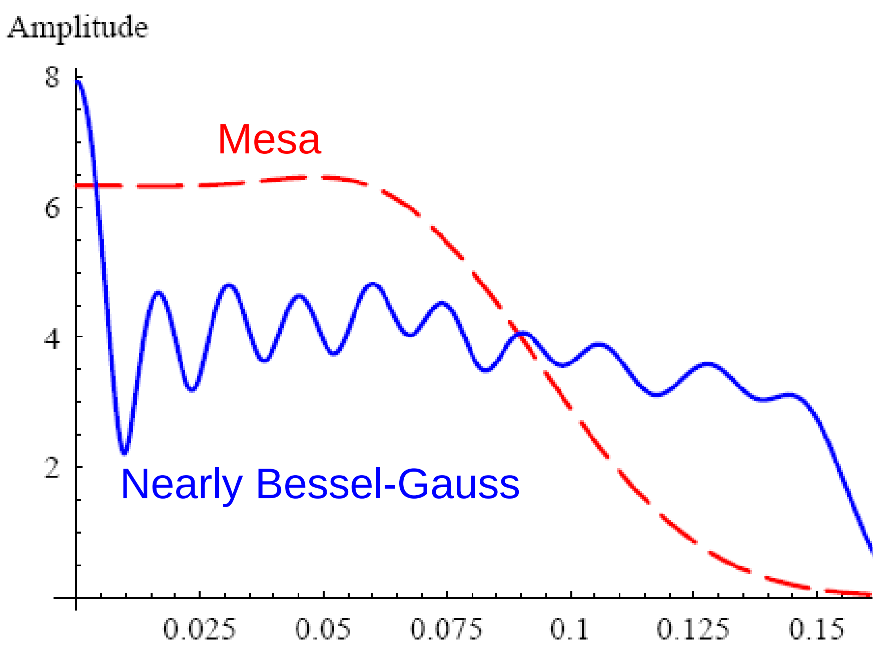
Parameters: *a*=16cm, diff. loss=1ppm

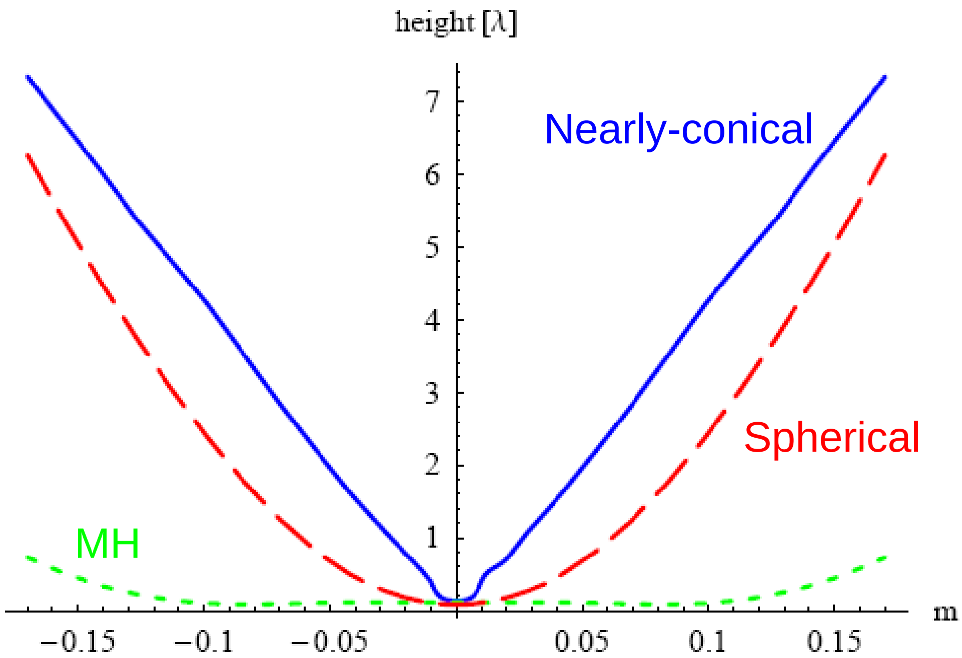
- Sensible potential improvement
  - Coating noises: Reduction of a factor ~2.5 w.r.t. MBs (~5.3 w.r.t. GBs)
- Yes, but *how close* to these lower bounds a *physically-feasible* profile can get?
  - Genetic-optimization results indicate reductions of a factor ~1.2

# A candidate (sub)optimal solution

[Bondarescu, PhD Thesis, 2007]

- "Brute-force" numerical optimization
  - Gauss-Laguerre expansion at the beam waist (physically-feasible by construction) [Galdi *et al.*, *Phys. Rev. D* **73,** 127101, 2006]
  - Constrained gradient-flow
  - Problem dimension consistent with our theoretical estimates
- Result: Nearly Bessel-Gauss beams, nearly conical mirrors



# A candidate (sub)optimal solution (cont'd)

[Bondarescu, PhD Thesis, 2007]

- *Global* or *local* optimum?
  - Impossible to assess self-consistently, **but**
  - Results *very close* to our estimated lower bounds
    - Coating noise reduction (w.r.t. MBs) of a factor ~2.3
    - If not optimum, certainly very good
- Open issues
  - Increased sensitivity (w.r.t. MH) to
    - Mirror fabrication tolerances (especially at large scales)
      - Figure error needs to be reduced by a factor ~10
    - Mirror tilt
      - Needs to be controlled at the level of ~ 3 nrad
    - Mirror translation
      - Needs to be controlled at the level of ~ 4μm
  - Non-Gaussian optics likely required
  - Parametric instabilities?

# Conclusions and outlook

- Absolute and realistic bounds for thermal noise reduction via beam-shaping derived
- Effective dimension of the problem related to the EM degrees of freedom $N_D = 2a^2/\lambda L$
- Large gap between the *best* currently available solutions (MB, HOGL) and the estimated lower bounds
  - Margins for further substantial noise reduction
    - Reduction factor ~2.5 (w.r.t. MBs) for coating noises
- (sub)optimal candidate solution [Bondarescu, PhD Thesis, 2007]
  - Nearly-Bessel-Gauss beams, nearly-conical mirrors
  - Reduction of a factor ~2.3 in coating noise PSD
  - Technologically challenging

# Conclusions and outlook (cont’d)

- Current studies
  - Validation against finite-test-mass numerical solution

- Future directions
  - Define a meaningful set of optimality criteria
    - Heterogeneous, competing constraints
  - Multiobjective optimization via robust (e.g., genetic) algorithms
    - Tradeoff (Pareto-type) curves