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B. ELECTROMAGNETICALLY COUPLED BROADBAND GRAVITATIONAL ANTENNA

1. Introduction

The prediction of gravitational radiation that travels at the speed of light has been an essential part of every gravitational theory since the discovery of special relativity. In 1918, Einstein,[1] using a weak-field approximation in his very successful geometrical theory of gravity (the general theory of relativity), indicated the form that gravitational waves would take in this theory and demonstrated that systems with time-variant mass quadrupole moments would lose energy by gravitational radiation. It was evident to Einstein that since gravitational radiation is extremely weak, the most likely measurable radiation would come from astronomical sources. For many years the subject of gravitational radiation remained the province of a few dedicated theorists; however, the recent discovery of the pulsars and the pioneering and controversial experiments of Weber[2,3] at the University of Maryland have engendered a new interest in the field.

Weber has reported coincident excitations in two gravitational antennas separated 1000 km. These antennas are high-Q resonant bars tuned to 1.6 kHz. He attributes these excitations to pulses of gravitational radiation emitted by broadband sources concentrated near the center of our galaxy. If Weber's interpretation of these events is correct, there is an enormous flux of gravitational radiation incident on the Earth.

Several research groups throughout the world are attempting to confirm these results with resonant structure gravitational antennas similar to those of Weber. A broadband antenna of the type proposed in this report would give independent confirmation of the existence of these events, as well as furnish new information about the pulse shapes.

The discovery of the pulsars may have uncovered sources of gravitational radiation which have extremely well-known frequencies and angular positions. The fastest known pulsar is NP 0532, in the Crab Nebula, which rotates at 30.2 Hz. The gravitational flux incident on the Earth from NP 0532 at multiples of 30.2 Hz can be 10^{-6} erg/cm^2/s at most. This is much smaller than the intensity of the events measured by Weber. The detection of pulsar signals, however, can be benefited by use of correlation techniques and long integration times.

The proposed antenna design can serve as a pulsar antenna and offers some distinct advantages over high-Q acoustically coupled structures.

2. Description of a Gravitational Wave in the General Theory of Relativity

In his paper on gravitational waves (1918), Einstein showed by a perturbation argument that a weak gravitational plane wave has an irreducible metric tensor in an almost Euclidean space. The total metric tensor is $g_{ij} = \eta_{ij} + h_{ij}$, where

$$\eta_{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

is the Minkowski background metric tensor, h_{ij} is the perturbation metric tensor resulting from the gravitational wave, and it is assumed that all components of this tensor are much smaller than 1. If the plane wave propagates in the x_1 direction, it is always possible to find a coordinate system in which h_{ij} takes the irreducible form

$$h_{ij} = \left(\begin{array}{c|cc} 0 & & 0 \\ \hline & h_{22} & h_{23} \\ 0 & h_{32} & h_{33} \end{array}\right)$$

with $h_{22} = -h_{33}$, and $h_{23} = h_{32}$. The tensor components have the usual functional dependence $f(x_1 - ct)$.

To gain some insight into the meaning of a plane gravitational wave, assume that the wave is in the single polarization state $h_{23} = h_{32} = 0$, and furthermore let $h_{22} = -h_{33} = h \sin(kx_1 - \omega t)$. The interval between two neighboring events is then given by

$$ds^2 = g_{ij}dx^i dx^j = c^2dt^2 - \left[dx_1^2 + (1 + h \sin(kx_1 - \omega t))\, dx_2^2 + (1 - h \sin(kx_1 - \omega t))\, dx_3^2\right].$$

The metric relates coordinate distances to proper lengths. In this metric coordinate time is proper time; however, the spatial coordinates are not proper lengths. Some reality can be given to the coordinates by placing free noninteracting masses at various points in space which then label the coordinates. The proper distance between two coordinate points may then be defined by the travel time of light between the masses. Assume a light source at $x_2 = -a/2$ and a receiver at $x_2 = a/2$. For light, the total interval is always zero so that

$$ds^2 = 0 = c^2dt^2 - (1 + h \sin(kx_1 - \omega t))\, dx_2^2.$$

Since $h \ll 1$,

$$cdt = \left[1 + \frac{h}{2} \sin(kx_1 - \omega t)\right] dx_2.$$

If the travel time of light, Δt, is much less than the period of the wave, the integral for

Δt becomes simple and we get

$$\Delta t = \left(1 - \frac{h}{2} \sin \omega t\right) \frac{a}{c}.$$

In the absence of the gravitational wave $\Delta t = \ell_o/c = a/c$, the coordinate distance becomes the proper length. The variation in Δt because of the gravitational wave is given by

$$\delta\Delta t = \left(\frac{h}{2} \sin \omega t\right) \frac{\ell_o}{c}.$$

This can be interpreted as though the gravitational wave produces a strain in space in the x_2 direction of

$$\frac{\Delta \ell}{\ell_o} = \frac{h}{2} \sin \omega T = \frac{h_{22}}{2}.$$

There is a comparable strain in the x_3 direction, however, inverted in phase.

This geometric description of the effects of a gravitational wave is useful for showing the interaction of the wave with free stationary particles. It becomes cumbersome when the particles have coordinate velocities or interact with each other. Weber[4] has developed a dynamic description of the effect of a gravitational wave on interacting matter which has negligible velocity. For the case of two masses m separated by a proper distance ℓ along the x_2 direction that are coupled by a lossy spring, the equation for the differential motion of the masses in the gravitational wave of the previous example becomes

$$\frac{d^2x_{2R}}{dt^2} + \frac{\omega_o}{Q} \frac{dx_{2R}}{dt} + \omega_o^2 x_{2R} = c^2 R_{2020} \ell,$$

where x_{2R} is the proper relative displacement of the two masses, and R_{2020} is that component of the Riemannian curvature tensor which interacts with the masses to give relative displacements in the x_2 direction; it can be interpreted as a gravitational gradient force.

For the plane wave,

$$R_{2020} = \frac{1}{2c^2} \frac{d^2h_{22}}{dt^2}$$

If the masses are free, the equation of differential motion becomes

$$\frac{d^2x_{2R}}{dr^2} = \frac{1}{2} \frac{d^2h_{22}}{dt^2} \ell$$

and, for zero-velocity initial conditions, the strain becomes $\frac{x_{2R}}{\ell} = \frac{1}{2} h_{22}$, which is the same result as that arrived at by the geometric approach.

The intensity of the gravitational wave in terms of the plane-wave metric tensor is given by Landau and Lifshitz[5] as

$$I_g = \frac{c^3}{16\pi G}\left[\left(\frac{dh_{23}}{dt}\right)^2 + \frac{1}{4}\left(\frac{dh_{22}}{dt} - \frac{dh_{33}}{dt}\right)^2\right]. \tag{1}$$

3. Gravitational Radiation Sources – Weber Events and Limits on Pulsar Radiation

The strain that Weber observes in his bars is of the order of $\Delta\ell/\ell \sim 10^{-16}$. If the strain is caused by impulsive events that can excite a 1.6 kHz oscillation in the bars, the events must have a rise time of 10^{-3} second or less – the fact that the bars have a high Q does not enter into these considerations. The peak incident gravitational flux of these events is truly staggering. Using Eq. 1, we calculate $I_g \geq 5 \times 10^9$ erg/s/cm^2.

If the sources of this radiation, which are alleged to be at the center of the galaxy, radiate isotropically, each pulse carries at least 5×10^{52} ergs out of the galaxy, the equivalent of the complete conversion to gravitational energy of 1/40 of the sun's rest mass. Weber observes on the average one of these events per day. At this rate the entire known rest mass of the galaxy would be converted into gravitational radiation in 10^{10} years. Gravitational radiation would then become the dominant energy loss mechanism for the galaxy.

Gravitational radiation by pulsar NP 0532, even at best, is not expected to be as spectacular as the Weber pulses. Gold[6] and Pacini[7] have proposed that pulsars are rotating neutron stars with off-axis magnetic fields. In a neutron star the surface magnetic field can be so large ($\sim 10^{12}$-10^{13} G) that the magnetic stresses perceptibly distort the star into an ellipsoid with a principal axis along the magnetic moment of the star. The star, as viewed in an inertial coordinate system, has a time-dependent mass quadrupole moment that could be a source of gravitational radiation at twice the rotation frequency of the star. Gunn and Ostriker[8] have made a study of this pulsar model and conclude from the known lifetime and present decay of the rotation frequency of NP 0532 that no more than 1/6 of the rotational energy loss of the pulsar could be attributed to gravitational radiation. The measured and assumed parameters for NP 0532 are listed below.

Rotation Frequency	$\nu = 30.2155 \ldots \ (\pm 3.4 \times 10^{-9})$ Hz
Slowdown Rate	$d\nu/dt = -3.859294 \pm .000053 \times 10^{-10}$ Hz/s
Distance	$d = 1.8$ kpc
Mass	$m = 1.4\ m_\odot$
Radius	$r = 10$ km.

The gravitational radiation intensity at 60.4 Hz incident on the Earth must be less than $I_g \leq 1 \times 10^{-6}$ erg/cm^2/s. The strain amplitude corresponding to this intensity is $\Delta\ell/\ell \leq 10^{-24}$.

4. Proposed Antenna Design

The principal idea of the antenna is to place free masses at several locations and measure their separations interferometrically. The notion is not new; it has appeared as a gedanken experiment in F. A. E. Pirani's[9] studies of the measurable properties of the Riemann tensor. However, the realization that with the advent of lasers it is feasible to detect gravitational waves by using this technique grew out of an undergraduate seminar that I ran at M.I.T. several years ago, and has been independently discovered by Dr. Philip Chapman of the National Aeronautics and Space Administration, Houston.

A schematic diagram of an electromagnetically coupled gravitational antenna is shown in Fig. V-20. It is fundamentally a Michelson interferometer operating in vacuum with the mirrors and beam splitter mounted on horizontal seismometer suspensions. The suspensions must have resonant frequencies far below the frequencies in the gravitational wave, a high Q, and negligible mechanical mode cross coupling. The laser beam makes multiple passes in each arm of the interferometer. After passing through the beam splitter, the laser beam enters either interferometer arm through a hole in the reflective coating of the spherical mirror nearest the beam splitter. The beam is reflected and refocused by the far mirror, which is made slightly astigmatic. The beam continues to bounce back and forth, hitting different parts of the mirrors, until eventually it emerges through another hole in the reflective coating of the near mirror. The beams from both arms are recombined at the beam splitter and illuminate a photodetector. Optical delay lines of the type used in the interferometer arms have been described by Herriott.[10] An experimental study of the rotational and transverse translational stability of this kind of optical delay line has been made by M. Wagner.[11]

The interferometer is held on a fixed fringe by a servo system which controls the optical delay in one of the interferometer arms. In such a mode of operation, the servo output signal is proportional to the differential strain induced in the arms. The servo signal is derived by modulating the optical phase in one arm with a Pockel-effect phase shifter driven at a frequency at which the laser output power fluctuations are small, typically frequencies greater than 10 kHz. The photo signal at the modulation frequency is synchronously detected, filtered, and applied to two controllers: a fast controller which is another Pockel cell optical phase shifter that holds the fringe at high frequencies, and a slow large-amplitude controller that drives one of the suspended masses to compensate for thermal drifts and large-amplitude low-frequency ground noise.

The antenna arms can be made as large as is consistent with the condition that the travel time of light in the arm is less than one-half the period of the gravitational wave

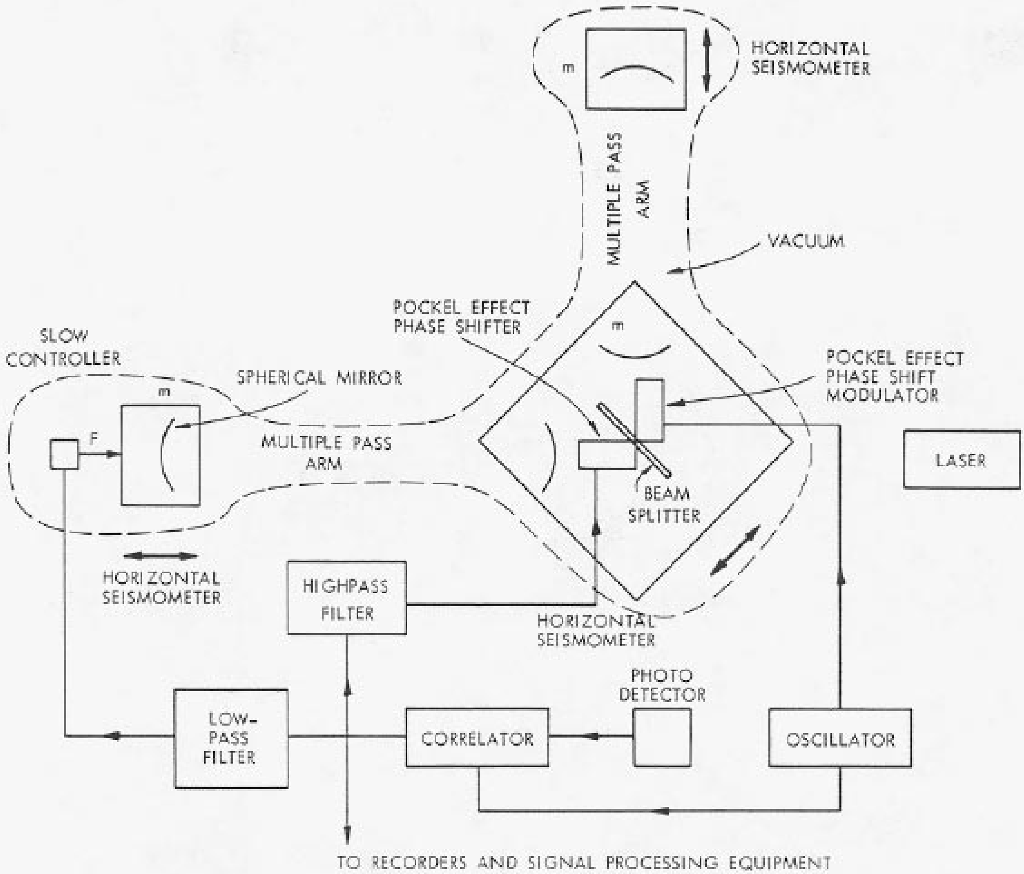

Fig. V-20. Proposed antenna.

that is to be detected. This points out the principal feature of electromagnetically coupled antennas relative to acoustically coupled ones such as bars; that an electromagnetic antenna can be longer than its acoustic counterpart in the ratio of the speed of light to the speed of sound in materials, a factor of 10^5. Since it is not the strain but rather the differential displacement that is measured in these gravitational antennas, the proposed antenna can offer a distinct advantage in sensivity relative to bars in detecting both broadband and single-frequency gravitational radiation. A significant improvement in thermal noise can also be realized.

5. Noise Sources in the Antenna

The power spectrum of noise from various sources in an antenna of the design shown in Fig. V-20 is estimated below. The power spectra are given in equivalent displacements squared per unit frequency interval.

a. Amplitude Noise in the Laser Output Power

The ability to measure the motion of an interferometer fringe is limited by the fluctuations in amplitude of the photo current. A fundamental limit to the amplitude noise in a laser output is the shot noise in the arrival rate of the photons, as well as the noise generated in the stochastic process of detection. At best, a laser can exhibit Poisson amplitude noise. This limit has been approached in single-mode gas lasers that are free of plasma oscillations and in which the gain in the amplifying medium at the frequency of the oscillating optical line is saturated.[12, 13]

The equivalent spectral-noise displacement squared per unit frequency interval in an interferometer of the design illustrated by Fig. V-20, illuminated by a Poisson noise-limited laser and using optimal signal processing, is given by

$$\frac{\Delta x^2(f)}{\Delta f} \geq \frac{hc\lambda}{8\pi^2 \epsilon P b^2 e^{-b(1-R)}},$$

where h is Planck's constant, c the velocity of light, λ the wavelength of the laser light, ϵ the quantum efficiency of the photodetector, P the total laser output power, b the number of passes in each interferometer arm, and R the reflectivity of the spherical mirrors. The expression has a minimum value for $b = 2/(1-R)$.

As an example, for a 0.5 W laser at 5000 Å and a mirror reflectivity of 99.5% using a photodetector with 50% quantum efficiency, the minimum value of the spectral noise power is

$$\frac{\Delta x^2(f)}{\Delta f} \geq 10^{-33}\ \text{cm}^2/\text{Hz}.$$

b. Laser Phase Noise or Frequency Instability

Phase instability of the laser is transformed into displacement noise in an interferometer with unequal path lengths. In an ideal laser the phase noise is produced by spontaneous emission which adds photons of random phase to the coherent laser radiation field. The laser phase performs a random walk in angle around the noise-free phase angle given by $\phi_o = \omega_o t$. The variance in the phase grows as $\overline{(\Delta\phi)}^2 = t/st_c$, where s is the number of photons in the laser mode, t_c the laser cavity storage time, and t the observation time. This phase fluctuation translates into an oscillating frequency width of the laser given by $\delta = 1/4\pi t_c s$.

Armstrong[14] has made an analysis of the spectral power distribution in the output of a two-beam interferometer illuminated by a light source in which the phase noise has a Gaussian distribution in time. By use of his results, the equivalent power spectrum of displacement squared per unit frequency in the interferometer is given by

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{4}{3}\lambda^2\delta^2\tau^3$$

for the case $f\tau \ll 1$ and $\delta\tau \ll 1$, where τ is the difference in travel time of light between the two paths in the interferometer.

The main reason for using a Michelson interferometer in the gravity antenna is that τ can be made small (equal to zero, if necessary), so that excessive demands need not be made on the laser frequency stability. In most lasers δ is much larger than that because of spontaneous emission, especially for long-term measurements (large τ). For small τ, however, δ does approach the theoretical limit. In a typical case δ might be of the order of 10 Hz and τ approximately 10^{-9} second, which gives

$$\frac{\Delta x^2(f)}{\Delta f} \leq 10^{-34}\ \text{cm}^2/\text{Hz}.$$

c. Mechanical Thermal Noise in the Antenna

Mechanical thermal noise enters the antenna in two ways. First, there is thermal motion of the center of mass of the masses on the horizontal suspensions and second, there is thermal excitation of the internal normal modes of the masses about the center of mass. Both types of thermal excitation can be handled by means of the same technique. The thermal noise is modeled by assuming that the mechanical system is driven by a stochastic driving force with a spectral power density given by

$$\frac{\Delta F^2(f)}{\Delta f} = 4kT\alpha \qquad \text{dyn}^2/\text{Hz},$$

where k is Boltzmann's constant, T the absolute temperature of the damping medium, and α the damping coefficient. We can express α in terms of Q, the resonant frequency, ω_o, of the mechanical system, and the mass. Thus $\alpha = m\omega_o/Q$. The spectral power density of the displacement squared, because of the stochastic driving force on a harmonic oscillator, is

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{1}{m^2\omega_o^4}\,\frac{1}{(1-z^2)^2 + z^2/Q^2}\,\frac{4kT\omega_o m}{Q},$$

where $z = \omega/\omega_o$. The seismometer suspension should have a resonant frequency much lower than the frequency of the gravitational wave that is to be detected; in this case $z \gg 1$ and $Q \gg 1$, to give

$$\frac{\Delta x^2(f)}{\Delta f} = 4\frac{\omega_o}{\omega^4}\frac{kT}{mQ}.$$

On the other hand, the lowest normal-mode frequencies of the internal motions of the masses, including the mirrors and the other suspended optical components, should be higher than the gravitational wave frequency. Some care must be taken to make the entire suspended optical system on each seismometer mount as rigid as possible. For the internal motions $z \ll 1$ and $Q \gg 1$, so that

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{4kT}{\omega_o^3 mQ}.$$

It is clear that, aside from reducing the temperature, the thermal noise can be minimized by using high-Q materials and a high-Q suspension, as long as the gravitational wave frequency does <u>not</u> fall near one of the mechanical resonances. The range of Q for internal motions is limited by available materials: quartz has an internal Q of approximately 10^6, while for aluminum it is of the order of 10^5. The Q of the suspension can be considerably higher than the intrinsic Q of materials. The relevant quantity is the ratio of the potential energy stored in the materials to that stored in the Earth's gravitational field in the restoring mechanism.

The suspensions are critical components in the antenna, and there is no obvious optimal design. The specific geometry of the optics in the interferometer can make the interferometer output insensitive to motions along some of the degrees of freedom of the suspension. For example, the interferometer shown in Fig. V-20 is first-order insensitive to motions of the suspended masses transverse to the direction of propagation of light in the arms. It is also first-order insensitive to rotations of the mirrors. Motions of the beam splitter assembly along the 45° bisecting line of the interferometer produce common phase shifts in both arms and therefore do not appear in the interferometer output. Nevertheless, the success of the antenna rests heavily on the mechanical design of the suspensions because the thermal noise couples in through them, and they also have to provide isolation from ground noise.

The general problem with suspensions is that in the real world they do not have only one degree of freedom but many, and these modes of motion tend to cross-couple nonlinearly with each other, so that, by parametric conversion, noise from one mode appears in another. A rule of thumb, to minimize this problem in suspensions, is to have as few modes as possible, and to make the resonance frequencies of the unwanted modes high relative to the operating mode.[15]

It is still worthwhile to look at an example of the theoretical thermal noise limit of a single-degree-of-freedom suspension. If the internal Q is 10^5, the mass 10 kG, and the lowest frequency resonance in the mass 10 kHz, the thermal noise from internal motions at room temperature for frequencies less than 10 kHz is

$$\frac{\Delta x^2(f)}{\Delta f} \sim 10^{-35}\ \text{cm}^2/\text{Hz}.$$

The thermal noise from center-of-mass motion on the suspension for a $Q \sim 10^4$ and a resonant frequency of 5×10^{-2} Hz becomes

$$\frac{\Delta x^2(f)}{\Delta f} \sim \frac{10^{-24}}{f^4}\ \text{cm}^2/\text{Hz}$$

for frequencies greater than the resonant frequency of the suspension. With the chosen sample parameters, the Poisson noise in the laser amplitude is larger than the thermal noise at frequencies greater than 200 Hz. An antenna that might be used in the pulsar radiation search would require, at room temperature, an mQ product 10^2 larger than the example given, to match the Poisson noise of the laser.

d. Radiation-Pressure Noise from the Laser Light

Fluctuations in the output power of the laser can drive the suspended masses through the radiation pressure of light. In principle, if the two arms of the interferometer are completely symmetric, both mechanically and optically, the interferometer output is insensitive to these fluctuations. Since complete symmetry is hard to achieve, this noise source must still be considered. An interesting point is that although one might find a high modulation frequency for the servo system where the laser displays Poisson noise, it is the spectral power density of the fluctuations in the laser output at the lower frequency of the gravitational wave which excites the antenna. In other words, if this is a serious noise source, the laser has to have amplitude stability over a wide range of frequencies.

Radiation-pressure noise can be treated in the same manner as thermal noise. If the laser displays Poisson noise, the spectral power density of a stochastic radiation-pressure force on one mirror is

$$\frac{\Delta F_{rad}^2(f)}{\Delta f} = \frac{4b^2 hP}{\lambda c} \qquad \text{dyn}^2/\text{Hz},$$

where b is the number of times the light beam hits the mirror, and P is the average total laser power. Using the same sample parameters for the suspension as we used in calculating the thermal noise, and those for the laser in the discussion of the amplitude noise, the ratio of stochastic radiation pressure forces relative to stochastic thermal forces is

$$\frac{\Delta F^2_{rad}(f)}{\Delta F^2_{thermal}(f)} \sim 10^{-6}.$$

e. Seismic Noise

If the antenna masses were firmly attached to the ground, the seismic noise, both through horizontal and tilt motions of the ground, would be larger than any of the other noise sources considered thus far. The seismic noise on the earth at frequencies higher than 5 Hz has been studied by several investigators[16-18] at various locations both on the surface and at different depths. In areas far from human industrial activity and traffic, the high-frequency noise can be characterized by a stationary random process. The noise at the surface appears higher than at depths of 1 km or more, but an unambiguous determination of whether the high-frequency noise is due to Rayleigh or to body waves has not been carried out. Measurements made in a zinc mine at Ogdensburg, New Jersey,[16] at a depth of approximately 0.5 km have yielded the smallest published values of seismic noise. In the region 10-100 Hz, the power spectrum is approximated by

$$\frac{\Delta x^2_{\ell}(f)}{\Delta f} \sim \frac{3 \times 10^{-14}}{f^4}\ \mathrm{cm}^2/\mathrm{Hz}.$$

Although the spectrum has not been measured at frequencies higher than 100 Hz, it is not expected to decrease more slowly with frequency at higher frequencies. Surface measurements are typically larger by an order of magnitude.

By mounting the antenna masses on horizontal seismometer suspensions, we can substantially reduce the seismic noise entering the interferometer. The isolation provided by a single-degree-of-freedom suspension is given by

$$\left|\frac{\Delta x_m(f)}{\Delta x_{\ell}(f)}\right|^2 = \frac{\left[(1-z^2) + (2/Q)^2\right]^2 + (z^3/Q)^2}{\left[(1-z^2)^2 + (z/Q)^2\right]^2},$$

where $z = f/f_o$, and f_o is the resonant frequency of the suspension. $\Delta x_m(f)$ is the displacement of an antenna mass at frequency f relative to an inertial frame, and $\Delta x_{\ell}(f)$ is the motion of the Earth measured in the same reference frame.

At frequencies for which $z \gg 1$, the isolation ratio is

$$\left|\frac{\Delta x_m(f)}{\Delta x_{\ell}(f)}\right|^2 \sim \left(\frac{f_o}{f}\right)^4 + \left(\frac{f_o}{f}\right)^2 \frac{1}{Q^2}.$$

For the sample suspension parameters given, the estimated seismic noise entering the antenna is

$$\frac{\Delta x^2(f)}{\Delta f} > \frac{2 \times 10^{-18}}{f^8}\ \text{cm}^2/\text{Hz}; \qquad 10 < f < 10\ \text{kHz}$$

with the average seismic driving noise at the Earth's surface assumed. For frequencies higher than 100 Hz, the effect of seismic noise is smaller than the noise from the laser amplitude fluctuations.

Although the isolation is adequate for detecting Weber-type events, an antenna to detect pulsar radiation would require better rejection of the ground noise. Several approaches are possible. Clearly, the suspension period can be increased to be longer than 20 s, but suspensions of very long periods are difficult to work with. Several shorter period suspensions may be used in series, since their isolation factors multiply. The disadvantage of this is that by increasing the number of moving members, the mode cross-coupling problem is bound to be aggravated.

An interesting possibility of reducing the seismic noise is to use a long-baseline antenna for which the period of the gravitational wave is much shorter than the acoustic travel time through the ground between the antenna end points. In this situation, the sections of ground at the end points are uncoupled from each other and the gravitational wave moves the suspended mass in the same way as the ground around it. In other words, there is little differential motion between the suspended mass and the neighboring ground because of the gravitational wave. Differential motion would result primarily from seismic noise. The differential motion can be measured by using the suspended mass as an inertial reference in a conventional seismometer. This information can be applied to the interferometer output to remove the seismic-noise component.

f. Thermal-Gradient Noise

Thermal gradients in the chamber housing the suspension produce differential pressures on the suspended mass through the residual gas molecules. The largest unbalanced heat input into the system occurs at the interferometer mirror where, after multiple reflections, approximately 1/10 of the laser power will be absorbed.

The excess pressure on the mirror surface is approximately $p \sim nk\Delta T$, where n is the number of gas molecules/cm^3, k is Boltzmann's constant, and ΔT is the difference in temperature between the mirror surface and the rest of the chamber. The fluctuations in ΔT can be calculated adequately by solving the one-dimensional problem of thermal diffusion from the surface into the interior of the mirror and the associated antenna mass, which are assumed to be at a constant temperature.

The mirror surface temperature fluctuations, $\Delta T(f)$, driven by incident intensity fluctuations $\Delta I(f)$, is given by

$$\Delta T(f) = \frac{\Delta I(f)}{4\epsilon\sigma T_o^3 + (\pi c_v \rho k_t)^{1/2} f^{1/2}}.$$

The first term in the denominator is the radiation from the surface, with ϵ the emissivity, σ the Stefan-Boltzmann constant, and T_o the ambient temperature. The second term is due to thermal diffusion from the surface into the interior, with c_v the specific heat, ρ the density, and k_t the thermal conductivity of the mirror.

If the laser exhibits Poisson noise, the spectral force density on the antenna mass becomes

$$\frac{\Delta F^2(f)}{\Delta f} = \frac{2(nk)^2}{f(\pi c_v \rho k_t)} \frac{hc}{\lambda} \overline{P} \qquad \text{dyn}^2/\text{Hz}.$$

Radiation is neglected because it is much smaller than the thermal diffusion. Using the following parameters for glass, $c_v \sim 10^6$ erg/gm °K, $\rho \sim 4$, $k_t \sim 10^3$ erg/s cm °K, an average laser power of 0.5 W and a vacuum of 1×10^{-8} mm Hg, the ratio of the thermal-gradient noise to the thermal noise forces in the sample suspension is

$$\frac{\Delta F^2_{T,G}(f)}{\Delta F^2_{th}(f)} \sim \frac{10^{-15}}{f}.$$

g. Cosmic-Ray Noise

The principal component of the high-energy particle background both below and on the Earth's surface is muons with kinetic energies[19] greater than 0.1 BeV. A muon that passes through or stops in one of the antenna masses imparts momentum to the mass, thereby causing a displacement that is given by

$$\Delta x = \frac{\Delta E \cos\theta}{m\omega_o c},$$

where ΔE is the energy loss of the muon in the antenna mass, θ the angle between the displacement and the incident muon momentum, m the antenna mass, and ω_o the suspension resonant frequency.

The energy loss of muons in matter is almost entirely through electromagnetic interactions so that the energy loss per column density, k(E), is virtually constant with energy for relativistic muons. A 10^{-1} BeV muon loses 3 MeV/gm/cm^2, while a 10^4 BeV muon loses ~30 MeV/gm/cm^2.

The vertical flux of muons at sea level with an energy greater than 10^{-1} BeV is approximately 10^{-2} particles/cm^2 sec sr. For energies larger than 10 BeV, the integrated flux varies as $\sim 10^{-1}/E^2$(BeV).

Since the flux falls off steeply with energy and the energy loss is almost independent of energy, the bulk of the muon events will impart the same momentum to the suspension. If we use the following sample suspension parameters, $m \sim 10^4$ g, $f_o \sim 5 \times 10^{-2}$ Hz, $\rho \sim 3$, and typical linear dimensions ~10 cm, the average energy loss per muon is $\sim 10^{-1}$ BeV. At sea level the antenna mass might experience impulsive displacements of $\sim 10^{-18}$ cm occurring at an average rate of once a second. An event arising from the passage of a 10^4 BeV muon results in a displacement of 10^{-17} cm at a rate of once a year.

Although the shape of the antenna mass can be designed to reduce somewhat the effect and frequency of muon interactions especially if we take advantage of the anisotropy of the muon flux, the best way of reducing the noise is to place the antenna masses underground. The pulse rate at depths of 20 m, 200 m, and 2 km is approximately 3×10^{-2}, 10^{-4}, 10^{-9} pulses/second.

If the antenna output is measured over times that include many muon pulses, as it would be in a search for pulsar radiation, the noise can be treated as a stationary distribution. Under the assumption that the muon events are random and, for ease of calculation, that the magnitude of the momentum impacts is the same for all muons, the spectral power density of displacement squared of the antenna mass is

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{4N(\Delta E/c)^2}{(2\pi)^4 m^2 f^4} \qquad \text{cm}^2/\text{Hz}$$

for $f \gg f_o$, where N is the average number of pulses per second, $\Delta E/c$ the momentum imparted to the mass per pulse, and m the antenna mass. For the sample suspension parameters at sea level

$$\frac{\Delta x^2(f)}{\Delta f} \sim 10^{-40}/f^4 \qquad \text{cm}^2/\text{Hz}.$$

h. Gravitational-Gradient Noise

The antenna is sensitive to gravitational field gradients, that is, differential gravitational forces exerted on the masses defining the ends of the interferometer arms. No data are available concerning high-frequency gravitational gradients that occur naturally on or near the surface of the earth. Two effects can bring about gravitational-gradient noise: first, time-dependent density variations in both the atmosphere and the ground, and second, motions of existing inhomogeneities in the mass distribution around the antenna.

An estimate of these two effects can be made with a crude model. Assume that one of the antenna masses is at the boundary of a volume that has a fluctuating density. The amount of mass that can partake in a coherent density fluctuation at frequency f and exert a force on the mass is roughly that included in a sphere with a radius equal to half the acoustic wavelength, λ, in the ground. The fluctuating gravitational force on the mass is

$$\frac{F_g(f)}{m} \sim \frac{2}{3}\pi\lambda\Delta\rho(f)\,G,$$

where $\Delta\rho(f)$ is the density fluctuation at frequency f, and G the Newtonian gravitational constant. The density fluctuations driven by ground noise in the sphere are

$$\Delta\rho(f) = 3\langle\rho\rangle\frac{\Delta x_e(f)}{\lambda},$$

where $\langle\rho\rangle$ is the average density of the ground, and $\Delta x_e(f)$ is the ground noise displacement. If f is larger than the resonant frequency of the suspension, the ratio of the displacement squared of the mass to that of the ground motion is given by

$$\frac{\Delta x_m^2(f)}{\Delta x_e^2(f)} = \left[\frac{\langle\rho\rangle G}{2\pi f^2}\right]^2.$$

For the earth, this isolation factor is

$$\frac{\Delta x_m^2(f)}{\Delta x_e^2(f)} \sim \frac{10^{-14}}{f^4},$$

which is much smaller than the isolation factor for the attenuation of direct ground motion by the sample suspension.

A comparable approach can be used in estimating the effect of motions of inhomogeneities in the distribution of matter around the antenna which are driven by ground noise. If we assume an extreme case of a complete inhomogeneity, for example, an atmosphere-ground interface, the mass that partakes in a coherent motion, $\Delta x(f)$, could be $m \sim \lambda^3\langle\rho\rangle$. The fluctuating force on the nearest antenna mass is

$$\frac{F_g(f)}{m} = \frac{2}{3}\pi G\langle\rho\rangle\,\Delta x(f).$$

The isolation factor is

$$\frac{\Delta x_m^2(f)}{\Delta x_e^2(f)} \sim \left[\frac{G\langle\rho\rangle}{6\pi f^2}\right]^2,$$

which is comparable to the isolation factor attributable to density fluctuations. These factors become smaller if the distance between the masses is less than λ.

i. Electric Field and Magnetic Field Noise

Electric fields in dielectric-free conducting vacuum chambers are typically 10^{-3} V/cm. These fields result from variations in the work function of surfaces and occur even when all surfaces in a system are constructed of the same material, since the work function of one crystal face is different from that of another. Temporal fluctuations in these fields are caused by impurity migrations and variations in adsorbed gas layers. Little is known about the correlation time of these fluctuations, except that at room temperature it seems to be longer than a few seconds and at cryogenic temperatures it is possible to keep the fields constant to better than 10^{-12} V/cm for several hours.[20]

The electric force on a suspended antenna mass is

$$F_e \sim \frac{1}{4\pi}\mathcal{E}^2 A,$$

where A is the exposed antenna surface, and $\mathcal{E}$ is the fluctuating electric field at the surface. Under the assumption that the power spectrum of the field fluctuations is similar to that of the flicker effect in vacuum tubes or to the surface effects in semiconductors, both of which come from large-scale, but slow, changes in the surface properties of materials, the electric force power spectrum might be represented by

$$\frac{\Delta F_e^2(f)}{\Delta f} \sim \frac{\frac{2}{\pi}\langle F_e^2\rangle\, 1/\tau_o}{(1/\tau_o)^2 + (2\pi f)^2} \qquad \text{dyn}^2/\text{Hz},$$

where τ_o is the correlation time of the fluctuations, and $\langle F_e^2\rangle$ is the average electric force squared.

If the gravitational wave frequency is much greater than $1/\tau_o$ and also higher than the resonant frequency of the suspension, the power spectrum of the displacements squared becomes

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{\langle \mathcal{E}^4\rangle A^2}{32\pi^6 m^2 \tau_o f^4} \qquad \text{cm}^2/\text{Hz}.$$

For $m \sim 10^4$ gm, $A \sim 10^2$ cm^2, $\mathcal{E} \sim 10^{-5}$ stat V/cm and $\tau_o \sim 1$ s,

$$\frac{\Delta x^2(f)}{\Delta f} \sim 10^{-38}/f^4 \qquad \text{cm}^2/\text{Hz}.$$

This noise is considerably less than that from the Poisson noise of the laser. Nevertheless, it is necessary to take care to shield, electrostatically, the deflection mirror surfaces.

Geomagnetic storms caused by ionospheric currents driven by the solar wind and cosmic rays create fluctuating magnetic fields at the surface of the Earth. The smoothed power spectrum of the magnetic field fluctuations in mid-latitude regions at frequencies greater than 10^{-3} Hz is approximately[21]

$$B^2(f) \sim B_o^2/f^2 \qquad G^2/\text{Hz},$$

with $B_o \sim 3 \times 10^{-8}$ G. Large pulses with amplitudes $\sim 5 \times 10^{-3}$ G are observed occasionally; the rise time of these pulses is of the order of minutes.[22]

Fluctuating magnetic fields interact with the antenna mass primarily through eddy currents induced in it or, if it is constructed of insulating material, in the conducting coating around the antenna that is required to prevent charge buildup. The interaction, especially at low frequencies, can also take place through ferromagnetic impurities in nonmagnetic materials. Magnetic field gradients cause center-of-mass motions of the suspended mass. Internal motions are excited by magnetic pressures if the skin depth is smaller than the dimensions of the antenna mass.

In an extreme model it would be assumed that the fluctuating magnetic fields are completely excluded by the antenna mass and that the field changes over the dimensions of the mass are equal to the fields. The magnetic forces are $F_m = \frac{1}{4\pi} B^2 A$.

The power spectrum for center-of-mass motions, with $f \gg f_o$, becomes

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{A^2 B_o^4}{16\pi^3 m^2 f^4} \qquad \text{cm}^2/\text{Hz}.$$

For the sample suspension, using the smoothed power spectrum of magnetic field fluctuations, we have

$$\Delta x^2(f) \sim 10^{-36}/f^4 \qquad \text{cm}^2/\text{Hz}.$$

The displacements arising from internal motions driven by magnetic pressures at frequencies lower than the internal resonant frequency, $f_{o_{int}}$, are given by

$$\frac{\Delta x^2(f)}{\Delta f} = \frac{A^2 B_o^4}{16\pi^3 m^2 f_{o_{int}}^2 f^2} \qquad \text{cm}^2/\text{Hz}.$$

Although the disturbances caused by the smoothed power spectrum do not appear troublesome in comparison with the other noise sources, the occasional large magnetic pulses will necessitate placing both conducting and high-μ magnetic shields around the antenna masses. (It is not inconceivable that Weber's coincident events may be caused by pulses in geomagnetic storms, if his conducting shielding is inadequate. It would require a pulse of 10^{-2} G with a rise time $\sim 10^{-3}$ s to distort his bars by $\Delta\ell/\ell \sim 10^{-16}$.)

6. Detection of Gravitational Waves in the Antenna Output Signal

The interferometer (servo) output signal is filtered after detection. The gravitational wave displacements in the filtered output signal are given by

$$\Delta x_g^2 = \frac{1}{4} \int_0^\infty |F(f)|^2 \, h^2(f) \, \ell^2 \, df,$$

where F(f) is the filter spectral response, $h^2(f)$ is the spectral power density of the gravitational wave metric components, and ℓ is the arm length of the antenna interferometer. The noise displacements in the filtered output signal are given by

$$\Delta x_n^2 = \int_0^\infty |F(f)|^2 \frac{\Delta x_n^2(f)}{\Delta f} \, df,$$

where $\Delta x_n^2(f)/\Delta f$ is the spectral power density of the displacement noise. In order to observe a gravitational wave, the signal-to-noise ratio has to be greater than 1. That is, $\Delta x_g^2/\Delta x_n^2 > 1$.

The dominant noise source for the antenna appears to be the amplitude fluctuations in the laser output power. When translated into equivalent displacement of the masses, the noise has been shown to have a flat spectrum given by $\Delta x_n^2(f)/\Delta f \sim 10^{-33}$ cm^2/Hz.

If we assume this noise and an idealized unity gain bandpass filter with cutoff frequencies f_2 and f_1, then the signal-to-noise ratio becomes

$$\frac{\Delta x_g^2}{\Delta x_n^2} = \frac{1/4 \int_{f_1}^{f_2} h_2(f) \, \ell^2 \, df}{\frac{\Delta x_n^2(f)}{\Delta f} (f_2 - f_1)}.$$

For continuous gravitational waves, the minimum detectable gravitational wave metric spectral density is then

$$h^2(f) > \frac{4}{\ell^2} \frac{\Delta x_n^2(f)}{\Delta f} \approx \frac{4 \times 10^{-33}}{\ell^2(\text{cm})} \qquad \text{Hz}^{-1}.$$

Detectability criteria for pulses cannot be so well defined; a reasonable assumption is that the pulse "energy" be equal to the noise "energy." The optimum filter should have a bandwidth comparable to the pulse bandwidth. The spectral density of a pulse of duration τ is roughly distributed throughout a $1/\tau$ bandwidth. A possible signal-to-noise criterion for pulses is then

$$\Delta x_g^2 \tau > \frac{\Delta x_n^2(f)}{\Delta f},$$

or in terms of h,

$$h^2 \tau > \frac{4}{\ell^2} \frac{\Delta x_n^2(f)}{\Delta f}.$$

As an example, the Weber pulses induce impulsive strains of $h \sim 2 \times 10^{-16}$ for a duration of approximately 10^{-3} s, so that $h^2\tau \sim 4 \times 10^{-35}$. A 1-m interferometer arm antenna of the proposed design would have a noise "energy" of 4×10^{-37}, so that the signal-to-noise ratio for Weber events would approach 100/1.

A meaningful search for the pulsar radiation requires a more elaborate and considerably more expensive installation. The spectral density of the pulsar gravitational wave metric is

$$h^2(f) = h_o^2 \delta(f-f_p),$$

where f_p is a multiple of the pulsar rotation frequency. The signal-to-noise ratio is

$$\frac{\Delta x_g^2}{\Delta x_n^2} = \frac{1/4\, h_o^2 \ell^2}{\frac{\Delta x_n^2}{\Delta f}(f_2-f_1)}.$$

By coherent amplitude detection, using a reference signal at multiples of the pulsar rotation frequency, we can reduce the filter bandwidth by increasing the postdetection integration time. The integration time, t_{int}, required to observe the pulsar radiation with a signal-to-noise ratio greater than 1 is given by

$$t_{int} > \frac{4 \frac{\Delta x_n^2(f_p)}{\Delta f}}{h_o^2 \ell^2}.$$

Assuming the Gunn-Ostriker upper limit for the gravitational radiation of the Crab Nebula pulsar, $h_o \sim 2 \times 10^{-24}$, and an antenna with a 1-km interferometer arm, we find that the integration time is around one day.

An interesting point, suggested by D. J. Muehlner, is that the Weber events, if they are gravitational radiation pulses, could constitute the dominant noise in a pulsar radiation search. Under the assumption that the Weber pulses cause steplike strains, h_o, at an average rate of n per second, and that the integration time includes many pulses, the power spectrum of displacement squared is given roughly by

$$\frac{\Delta x^2(f)}{\Delta f} \sim \frac{N\ell^2 h_o^2}{16\pi^2 f^2} \qquad cm^2/Hz.$$

With $f \sim 60$ Hz, $h_o \sim 10^{-16}$, $\ell \sim 10^5$ cm, and $N \sim 10^{-5}$/s, the noise is $\sim 10^{-32}$ cm^2/Hz, which is greater than the Poisson noise of the laser. Large pulses can be observed directly in the broadband output of the antenna and can therefore be removed in the data analysis of the pulsar signal. If the energy spectrum of gravitational radiation pulses is, however, such that there is a higher rate for lower energy pulses, in particular, if Nh_o^2 is constant as h_o gets smaller, gravitational radiation may prove to be the dominant noise source in the pulsar radiation measurements.

Appendix

Comparison of Interferometric Broadband and Resonant Bar Antennas for Detection of Gravitational Wave Pulses

Aside from their greater possible length, interferometric broadband antennas have a further advantage over bars, in that the thermal noise in the detection bandwidth for the gravitational wave pulse is smaller than for the bar. In the following calculation it is assumed that the thermal noise is the dominant noise in both types of antennas.

Let the gravitational radiation signal be a pulse given by

$$h(t) = \begin{cases} 0 & t < 0 \\ h & 0 \leq t \leq t_o \\ 0 & t > t_o \end{cases}$$

The spectral energy density of the pulse is

$$h^2(\omega) = \frac{2h^2t_o^2}{(2\pi)^2} \frac{\sin^2 \omega t_o/2}{(\omega t_o/2)^2} \qquad 0 \leq \omega < \infty$$

$$\cong \begin{cases} \dfrac{2h^2t_o^2}{(2\pi)^2} & 0 < \omega \leq \pi/t_o \\ 0 & \omega > \pi/t_o \end{cases} \qquad \text{the equivalent energy box spectrum}$$

The gravitational force spectral density is

$$F_g^2(\omega) = \frac{1}{4}\,\omega^4 h^2(\omega)\,\ell^2 m^2.$$

Using the dynamic interpretation for the interaction of the bar with the gravitational-wave pulse, the "energy" in the bar after the pulse excitation is given by

$$\int_0^\infty x_g^2(t)\,dt = E_g = \frac{h^2 t_o^2 \ell^2 \omega_o Q}{4\pi} \qquad Q \gg 1, \quad \frac{\pi}{t_o} > \omega_o,$$

where ω_o is the resonant frequency of the bar.

The pulse "energy" is distributed throughout the ringing time of the bar so that

$$E_g \sim x_g^2(t)\; 2Q/\omega_o,$$

and the average displacement of the ends of the bar becomes

$$x_g^2(t) \sim \frac{h^2 t_o^2 \omega_o^2 \ell^2}{8\pi}.$$

The average thermal-noise displacement is

$$\langle x_{TH}^2 \rangle \sim \frac{4kT}{m\omega_o^2}.$$

The thermal noise also rings on the average for a period $\tau \sim 2Q/\omega_o$.

The signal-to-noise ratio for the bar is given by

$$\frac{x_g^2}{\langle x_{TH}^2 \rangle} \sim \frac{h^2 \ell^2 (t_o \omega_o)^2\, m\omega_o^2}{32\pi kT}.$$

Now make the same calculation for the broadband antenna with a filter matched to the pulse spectrum. The displacement spectrum is

$$x^2(\omega) = h^2(\omega)\,\ell^2 = \frac{2h^2 t_o^2 \ell^2}{(2\pi)^2}.$$

The pulse "energy" in a filter with matched bandwidth and a low-frequency cutoff ω_L is

$$E_g = 2\pi \int_{\omega_L}^{\pi/t_o} x^2(\omega)\, d\omega \cong \ell^2 h^2 t_o \qquad \omega_L \ll \pi/t_o.$$

If the resonant frequency, ω_o, of the suspension is smaller than ω_L, and the suspension has a high Q, the thermal "energy" in the same bandwidth is given by

$$E_{TH} = \langle x_{TH}^2 \rangle t_o = t_o \int_{\omega_L}^{\pi/t_o} \frac{1}{m^2\omega^4} \frac{4kT\omega_o m}{Q}\, d\omega.$$

Generally $\omega_L \ll \frac{\pi}{t_o}$, the thermal "energy" becomes

$$E_{TH} \sim \frac{4kT\omega_o t_o}{3Qm\omega_L^3}.$$

The signal-to-noise ratio for the broadband antenna is

$$\frac{E_g}{E_{TH}} = \frac{x_g^2}{\langle x_{TH}^2 \rangle} = \frac{3h^2\ell^2 Qm\omega_L^3}{4kT\omega_o}.$$

The signal-to-noise ratio for the broadband antenna relative to the equivalent-length resonant bar antenna at the same temperature is

$$R = \frac{(S/N)_{BB}}{(S/N)_B} = \frac{24\pi Q_{BB} m_{BB} \omega_L^3/\omega_{oBB}}{(t_o\omega_{oB})^2\, m_B \omega_{oB}^2}.$$

The best case for the bar is a pulse with $t_o \sim \frac{\pi}{\omega_o}$. If we assume Weber bar parameters $m_B \sim 10^6$ g, $\omega_{oB} \sim 10^4$ and the sample suspension parameters previously given, $Q_{BB} \sim 10^4$, $m_{BB} \sim 10^4$ g, $\omega_L \sim 10^3$, $\omega_{oBB} \sim 3 \times 10^{-1}$, the signal-to-noise ratio approaches $\sim 10^4$. This entire factor cannot be realized because the laser amplitude noise dominates in the interferometric antenna.

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