# Treatment of Calibration Uncertainty in Multi-Baseline Cross-Correlation Searches for Gravitational Waves

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## Abstract

*Residual uncertainty in the calibration of gravitational wave (GW) detector data leads to systematic errors which must be accounted for in setting limits on the strength of GW signals. When cross-correlation measurements are made using data from a pair of instruments, as in searches for a stochastic GW background, the calibration uncertainties associated with the two instruments can be combined into an uncertainty associated with the pair. With the advent of multi-baseline GW observation (e.g., networks consisting of multiple detectors such as the LIGO observatories and Virgo), a more sophisticated treatment is called for. We describe how the correlations between calibration factors associated with different pairs can be taken into account by marginalizing over the uncertainty associated with each instrument.*

## Calibration Uncertainty with One Baseline

Consider an experiment to measure a physical quantity $\mu$ (e.g., the stochastic GW background strength $\Omega$). An optimal combination $x$ of cross-correlation measurements provides a point estimate of $\mu$ with error bar $\sigma$. Given the likelihood function $P(x|\mu)$, we can use Bayes's Theorem to construct the posterior

$$P(\mu|x) = \frac{P(x|\mu)P(\mu)}{P(x)} = \frac{P(x|\mu)P(\mu)}{\int d\mu\, P(x|\mu)\, P(\mu)}$$

so we need the $\mu$ dependence of the likelihood $P(x|\mu)$. Due to calibration uncertainties in each of the instruments which make up the baseline for the cross-correlation, $x$ is an estimator not of $\mu$, but of $\lambda\mu$, where $\lambda$ is an unknown calibration factor described by an uncertainty $\varepsilon$. Thus likelihood depends on calibration factor $\lambda$

$$P(x|\mu,\lambda) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\lambda\mu)^2}{2\sigma^2}\right)$$

and we need the marginalized likelihood

$$P(x|\mu) = \int d\lambda\, P(x|\mu,\lambda)P(\lambda)\ .$$

Assuming a Gaussian distribution for calibration factor

$$P(\lambda) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(\lambda-1)^2}{2\varepsilon^2}\right)$$

allows us to marginalize analytically:

$$P(x|\mu) = \sqrt{\frac{M(\mu,\sigma_\alpha,\varepsilon)}{2\pi}} \exp\left(-\frac{1}{2}M(\mu,\sigma_\alpha,\varepsilon)\,(x-\mu)^2\right)$$

where

$$M(\mu,\sigma_\alpha,\varepsilon) = \frac{1}{\sigma^2+\varepsilon^2\mu^2}\ .$$

This is the method used in stochastic GW searches with two LIGO sites, e.g., [1, 2].
However, the cal. factor $\lambda$ is multiplicative & always positive, so it makes more sense to use a log-normal distribution

$$P(\lambda) = \frac{1}{\lambda\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(\ln\lambda)^2}{2\varepsilon^2}\right)$$

or, in terms of $\Lambda = \ln\lambda$,

$$P(\Lambda) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{\Lambda^2}{2\varepsilon^2}\right)$$

This was the approach taken in the stochastic GW search using LIGO & ALLEGRO[3], but has the drawback of requiring numerical integration over $\Lambda$ because

$$P(x|\mu,\Lambda) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-e^{\Lambda}\mu)^2}{2\sigma^2}\right)$$

gives a factor which is not Gaussian in $\Lambda$.

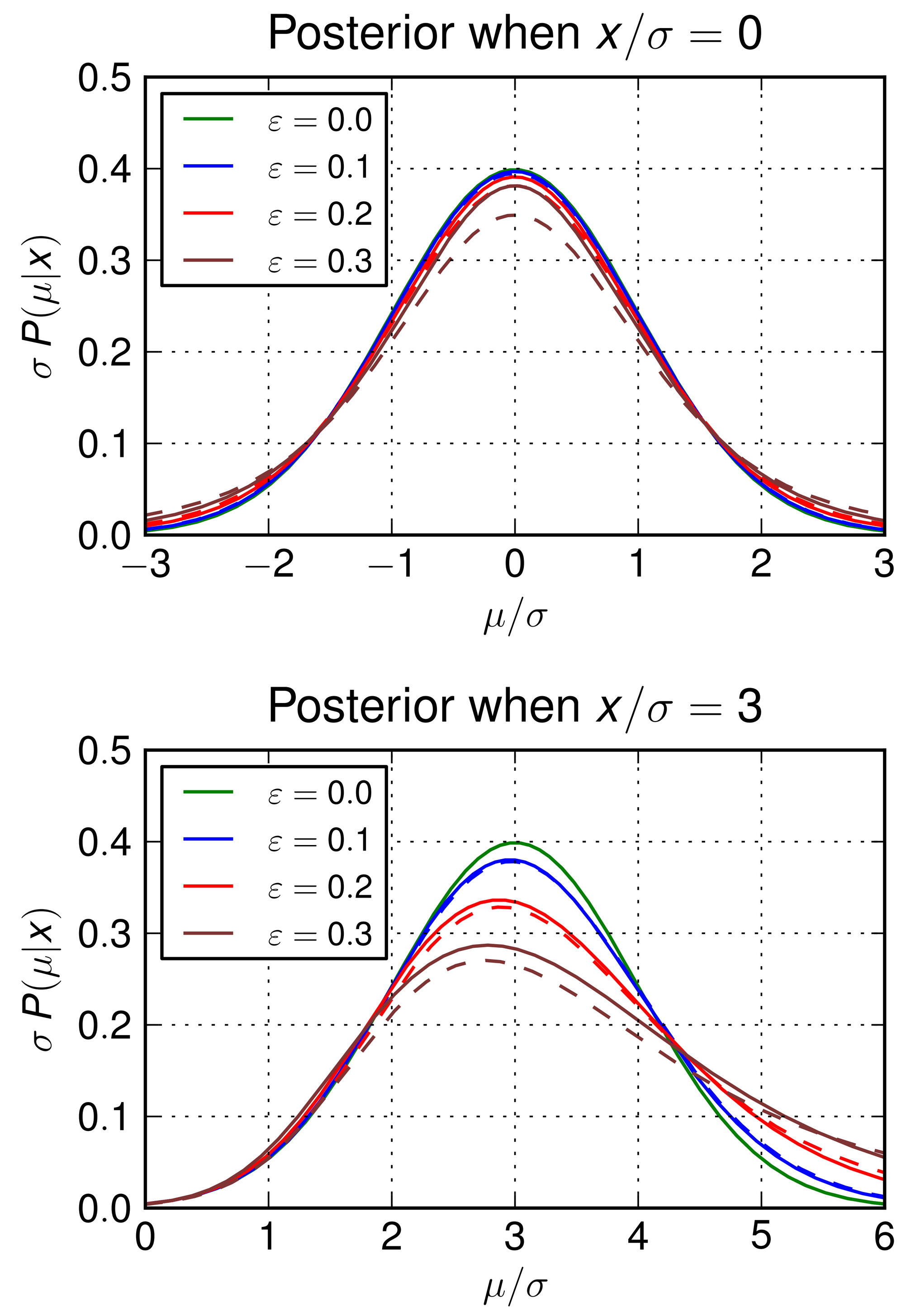


**Figure 1:** *Effects of marginalizing analytically over a single calibration factor in the absence of a signal (i.e., $x = 0$) and in the presence of a signal $x = 3\sigma$. The solid line is a numerical marginalization with a log-normal prior on $\lambda$; the dashed line is an analytic marginalization with a Gaussian prior on $\lambda$.*

## Calibration Uncertainty with Multiple Baselines

With more than two instruments, we have multiple baselines & multiple calibration uncertainties to marginalize over. For instance, the LIGO-Virgo stochastic search[4, 5] involves

- Instruments $I \in \{H1, H2, L1, V1\}$
- Baselines $\alpha \in \{H1L1, H1V1, H2L1, H2V1, L1V1\}$

Since the cross-correlation measurements for different baselines involve different calibration factors, we can't optimally combine all of them before marginalizing over calibration. Instead, optimally combine all measurements for baseline $\alpha$ into point estimate $x_\alpha$ with error bar $\sigma_\alpha$. Each baseline has unknown cal. factor $\lambda_\alpha$, so the likelihood is

$$P(\{x_\alpha\}|\mu,\{\lambda_\alpha\}) = \left(\prod_\alpha \frac{1}{\sigma_\alpha\sqrt{2\pi}}\right) \exp\left(-\sum_\alpha \frac{(x_\alpha-\lambda_\alpha\mu)^2}{2{\sigma_\alpha}^2}\right)\ .$$

Underlying calibration uncertainty $\varepsilon_I$ for each instrument. The cal. factor $\lambda_\alpha$ for each baseline is $\lambda_{IJ} = \lambda_I\lambda_J$ determined by per-instrument cal. factors $\{\lambda_I\}$. The per-baseline cal. factors $\{\lambda_\alpha\}$ have means, variances & covariances:

$$\langle\lambda_{IJ}\rangle = 1 + \mathcal{O}(\varepsilon^2)$$
$$\langle\lambda_{IJ}\lambda_{IJ}\rangle = 1 + {\varepsilon_I}^2 + {\varepsilon_J}^2 + \mathcal{O}(\varepsilon^4)$$
$$\langle\lambda_{IJ}\lambda_{JK}\rangle = 1 + {\varepsilon_J}^2 + \mathcal{O}(\varepsilon^4) \qquad \text{if } I \neq K\ .$$

Could use these to construct a multivariate Gaussian prior $P(\{\lambda_\alpha\})$ & marginalize per-baseline:

$$P(\{x_\alpha\}|\mu) = \left(\prod_\alpha \int d\lambda_\alpha\right) P(\{x_\alpha\}|\mu,\{\lambda_\alpha\})\, P(\{\lambda_\alpha\})\ .$$

This integral would be analytic. But $\lambda_{IJ} = \lambda_I\lambda_J$ implies e.g.,

$$\lambda_{IJ}\lambda_{KL} - \lambda_{IK}\lambda_{JL} = 0\ .$$

For a multivariate Gaussian prior on $\{\lambda_\alpha\}$, this is only true if the correlation matrix is degenerate.

## Per-Instrument Calibration Marginalization

We can instead set a prior on each per-instrument cal. factor $\lambda_I$ or equivalently on $\Lambda_I = \ln\lambda_I$. Per-pair $\Lambda_\alpha = \ln\lambda_\alpha$ so

$$\Lambda_{IJ} = \ln\lambda_{IJ} = \ln(\lambda_I\lambda_J) = \Lambda_I + \lambda_J\ .$$

The likelihood is

$$P(\{x_\alpha\}|\mu,\{\Lambda_I\}) = \left(\prod_\alpha \frac{1}{\sigma_\alpha\sqrt{2\pi}}\right) \exp\left(-\sum_\alpha \frac{(x_\alpha - e^{\Lambda_\alpha}\mu)^2}{2{\sigma_\alpha}^2}\right)$$

and marginalized likelihood is

$$P(\{x_\alpha\}|\mu) = \left(\prod_I \int d\Lambda_I\right) P(\{x_\alpha\}|\mu,\{\Lambda_I\})\, P(\{\Lambda_I\})\ .$$

An obvious prior is log-normal on $\lambda_I$, Gaussian on $\Lambda_I$:

$$P(\{\Lambda_I\}) = \left(\prod_I \frac{1}{\varepsilon_I\sqrt{2\pi}}\right) \exp\left(-\sum_I \frac{{\Lambda_I}^2}{2{\varepsilon_I}^2}\right)\ .$$

The exact integral over $\{\Lambda_I\}$ would need to be done numerically for each $\mu$, but if $\{\varepsilon_I\}$ are small, we can approximate

$$e^{\Lambda_{IJ}} \approx 1 + \Lambda_{IJ} = 1 + \Lambda_I + \Lambda_J$$

in likelihood to get a Gaussian integral over $\{\Lambda_I\}$ which can be done analytically. We end up with a likelihood of the form

$$P(\{x_\alpha\}|\mu) = \sqrt{\det\left\{\frac{M_{\alpha\beta}(\mu,\{\sigma_\alpha\},\{\varepsilon_I\})}{2\pi}\right\}}$$
$$\times \exp\left(-\frac{1}{2}\sum_\alpha\sum_\beta (x_\alpha-\mu)\, M_{\alpha\beta}(\mu,\{\sigma_\alpha\},\{\varepsilon_I\})\,(x_\beta-\mu)\right)\ .$$

For one baseline, the matrix $\{M_{\alpha\beta}\}$ becomes a number

$$M_{12,12} = \sigma_{12}^2 + \mu^2({\varepsilon_1}^2 + {\varepsilon_2}^2)$$

and this approximation gives the same result as assuming a Gaussian prior in $\lambda_{12}$ with

$${\varepsilon_{12}}^2 = {\varepsilon_1}^2 + {\varepsilon_2}^2\ .$$

## Ongoing Work

More generally, we may be using $\{x_\alpha\}$ to estimate multiple physical quantities $\mu_a$, such as different spherical harmonic modes of a non-isotropic stochastic GW background [6, 7]. These methods of analytic marginalization with either a multivariate Gaussian prior or an approximate likelihood function can be applied to the effects of calibration uncertainty in that search as well. Additionally, these calibration effects may also be considered in other cross-correlation searches, such as the modelled cross-correlation search for periodic GW signals.[8]

## References

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