Visualization of Antenna Pattern Factors via Projected Detector Tensors*

John T. Whelan†

Sat Jan 28 12:07:07 2012 -0500
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Abstract

This note shows how the response of an interferometric gravitational-wave detector to waves coming from a particular direction can be visualized using a projection of the detector's response tensor onto a transverse, traceless subspace.

1 Transverse Traceless Basis Tensors

A plane gravitational wave propagating along a unit vector $\vec{k}$, written in the transverse traceless gauge, will be described by a metric perturbation with components

$$h_{ij} = h_+ e_{+\,ij} + h_\times e_{\times\,ij} \tag{1.1}$$

where h_+ and $h_\times$ are functions of $t - \vec{k} \cdot \vec{r}/c$. In coördinate-free tensor notation, this can be written as

$$\overleftrightarrow{h} = h_+ \overleftrightarrow{e}_+ + h_\times \overleftrightarrow{e}_\times \tag{1.2}$$

The transverse, traceless *polarization basis tensors* $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ are constructed from a pair of unit vectors $\vec{\ell}$ and $\vec{m}$ which form a right-handed triple with the propagation direction $\vec{k}$. In terms of the tensor (dyad) product, they are written as

$$\overleftrightarrow{e}_+ = \vec{\ell} \otimes \vec{\ell} - \vec{m} \otimes \vec{m} \tag{1.3a}$$

$$\overleftrightarrow{e}_\times = \vec{\ell} \otimes \vec{m} + \vec{m} \otimes \vec{\ell} \tag{1.3b}$$

which is equivalent, in component notation, to

$$e_{+\,ij} = \ell_i \ell_j - m_i m_j \tag{1.4a}$$

$$e_{\times\,ij} = \ell_i m_j + m_i \ell_j \tag{1.4b}$$

*LIGO-T1100431-v2

†Center for Computational Relativity and Gravitation and School of Mathematical Sciences, Rochester Institute of Technology, 85 Lomb Memorial Drive, Rochester, NY 14623, USA

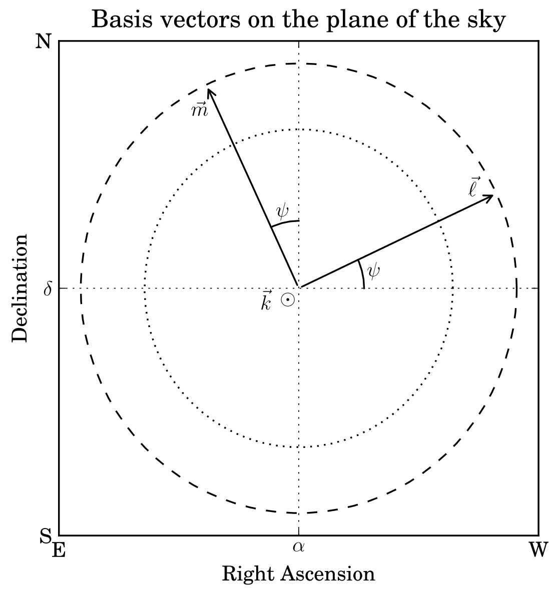

Figure 1: Source-specific polarization basis illustrated on the plane of the sky. The unit vector $\vec{k}$ pointing from the source to the observer is pointing directly out of the page. The convention for polarization angle ψ is the one used in e.g., [4].

To define the polarization basis, we need the propagation direction, (equivalently the sky position described by right ascension α and declination δ) and also a *polarization angle* ψ which describes the orientation of the basis relative to some reference direction.[1, 2, 3, 4] This is most easily illustrated via a sketch "on the plane of the sky," as shown in figure 1.

In this view the propagation unit vector $\vec{k}$ is pointed directly towards us so that right ascension increases from right to left and declination increases from bottom to top. (The labels N≡North, S≡South, E≡East, and W≡West reinforce this.) The dashed circle has unit radius and the dotted one has a radius of $1/\sqrt{2}$. We can represent the basis tensors $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ on the same diagram in figure 2. We draw $\overleftrightarrow{e}_+ = \vec{\ell}\otimes\vec{\ell} - \vec{m}\otimes\vec{m}$ by drawing outward pointing arrows along $\vec{\ell}$ and $-\vec{\ell}$ to represent $\vec{\ell}\otimes\vec{\ell}$ and inward pointing arrows along $\vec{m}$ and $-\vec{m}$ to represent $-\vec{m}\otimes\vec{m}$. To represent the basis tensor $\overleftrightarrow{e}_\times = \vec{\ell}\otimes\vec{m} + \vec{m}\otimes\vec{\ell}$ we resolve it along its eigenvectors $\frac{\vec{\ell}\pm\vec{m}}{\sqrt{2}}$ as

$$\overleftrightarrow{e}_\times = \frac{\vec{\ell}+\vec{m}}{\sqrt{2}}\otimes\frac{\vec{\ell}+\vec{m}}{\sqrt{2}} - \frac{\vec{\ell}-\vec{m}}{\sqrt{2}}\otimes\frac{\vec{\ell}-\vec{m}}{\sqrt{2}} \tag{1.5}$$

and draw outward- and inward-pointing arrows along those directions.

While many sources have a preferred polarization basis which makes the form of h_+ and $h_\times$ simple[1], it is also convenient to define a reference basis which depends only on the sky position. This is done by choosing a vector $\vec{\imath}$ pointing due West on the sky (in

[1]For instance, in a rotating neutron star or slowly-inspiralling non-precessing binary, choosing $\vec{m}$ along the projection of the system's angular momentum onto the plane of the sky, so that $\vec{\ell}$ lies in the equatorial or orbital plane of the source produces h_+ and $h_\times$ which oscillate 1/4 cycle out of phase.

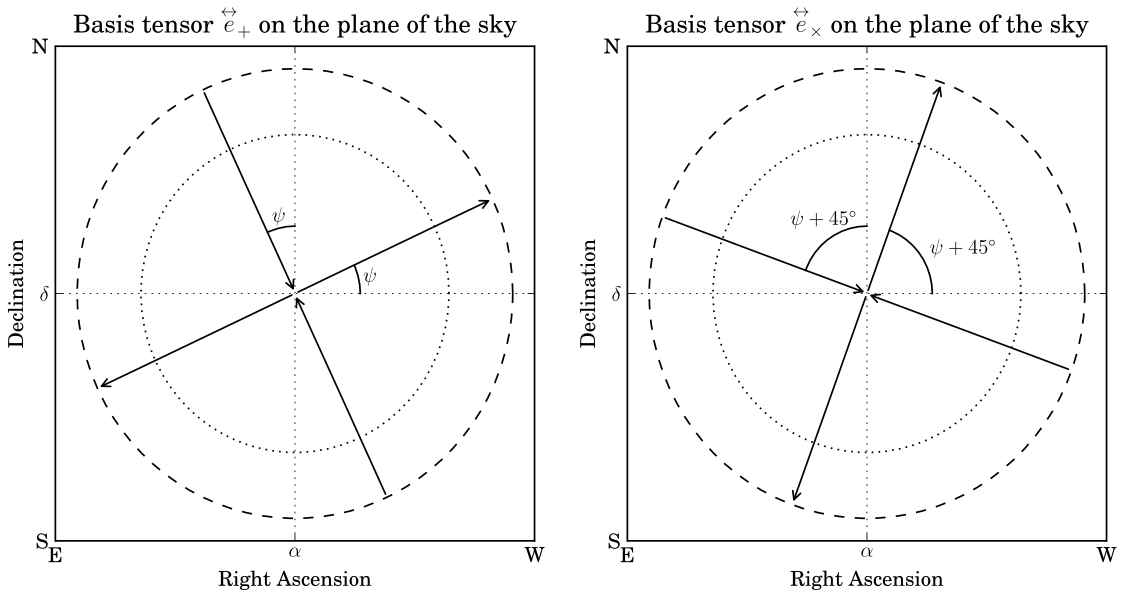

Figure 2: Source-specific polarization basis tensors $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ illustrated in the same plane as in figure 1.

the direction of decreasing right ascension) and $\vec{\jmath}$ pointing due North (in the direction of increasing right ascension), as shown in figure 3 Those vectors can be combined into plus and cross polarization basis tensors as before:

$$\overleftrightarrow{\varepsilon}_+ = \vec{\imath} \otimes \vec{\imath} - \vec{\jmath} \otimes \vec{\jmath} \tag{1.6a}$$

$$\overleftrightarrow{\varepsilon}_\times = \vec{\imath} \otimes \vec{\jmath} + \vec{\jmath} \otimes \vec{\imath} \tag{1.6b}$$

These are illustrated in figure 4. The relationship between the source-specific basis tensors and the reference ones is simply a polarization rotation:

$$\overleftrightarrow{e}_+ = \overleftrightarrow{\varepsilon}_+ \cos 2\psi + \overleftrightarrow{\varepsilon}_\times \sin 2\psi \tag{1.7a}$$

$$\overleftrightarrow{e}_\times = -\overleftrightarrow{\varepsilon}_+ \sin 2\psi + \overleftrightarrow{\varepsilon}_\times \cos 2\psi \tag{1.7b}$$

2 Detector Tensors

In the long-wavelength limit, the signal measured by an interferometer with arms along unit vectors $\vec{u}$ and $\vec{v}$ will be

$$h = \frac{h_{ij}}{2}(u^i u^j - v^i v^j) = \frac{\vec{u} \cdot \overleftrightarrow{h} \cdot \vec{u} - \vec{v} \cdot \overleftrightarrow{h} \cdot \vec{v}}{2} \tag{2.1}$$

where the first expression uses the Einstein summation convention and the second defines the dot product between a tensor and a vector in the usual way. If we define a *detector tensor*

$$\overleftrightarrow{d} = \frac{\vec{u} \otimes \vec{u} - \vec{v} \otimes \vec{v}}{2} \tag{2.2}$$

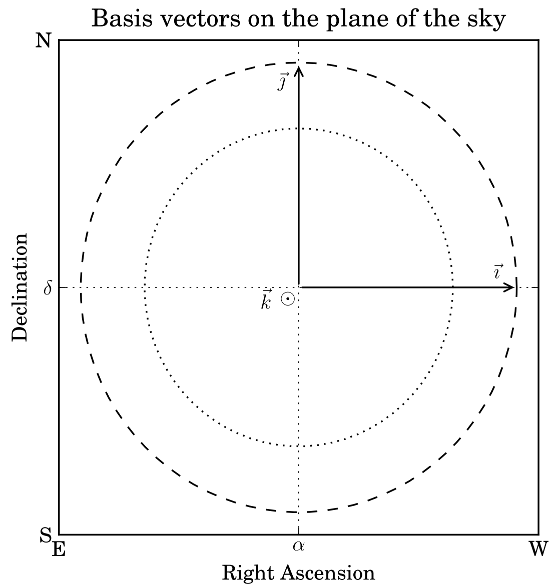

Figure 3: Canonical polarization basis vectors illustrated on the plane of the sky. These are defined only by the sky position, and have the unit vector $\vec{\imath}$ pointing in the direction of decreasing right ascension (West) and $\vec{\jmath}$ in the direction of increasing declination (North).

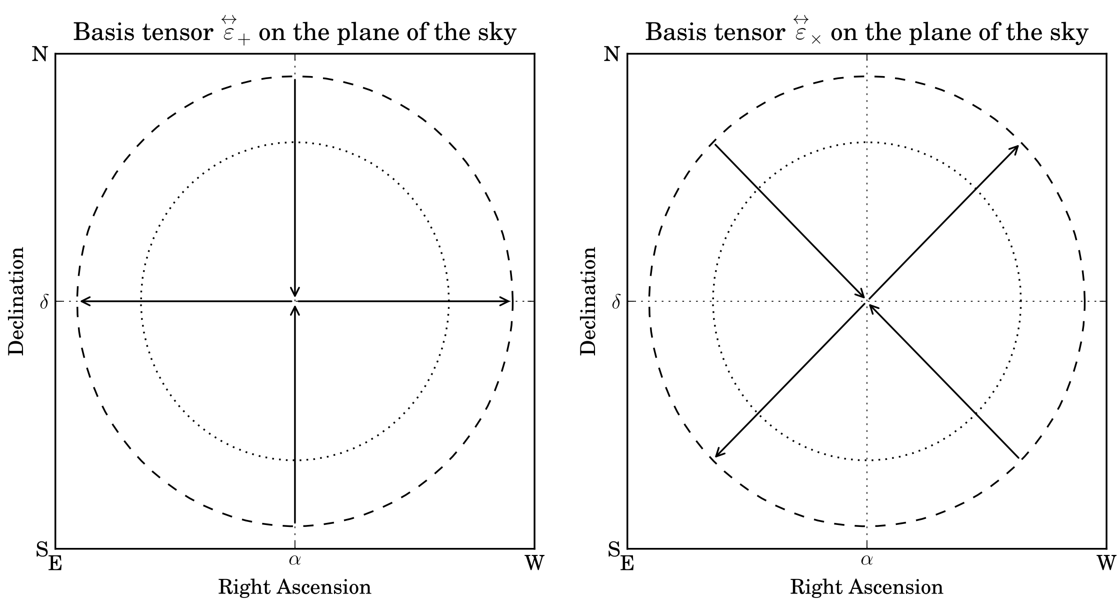

Figure 4: Canonical polarization basis tensors $\overleftrightarrow{\varepsilon}_+$ and $\overleftrightarrow{\varepsilon}_\times$ illustrated in the same plane as in figure 3.

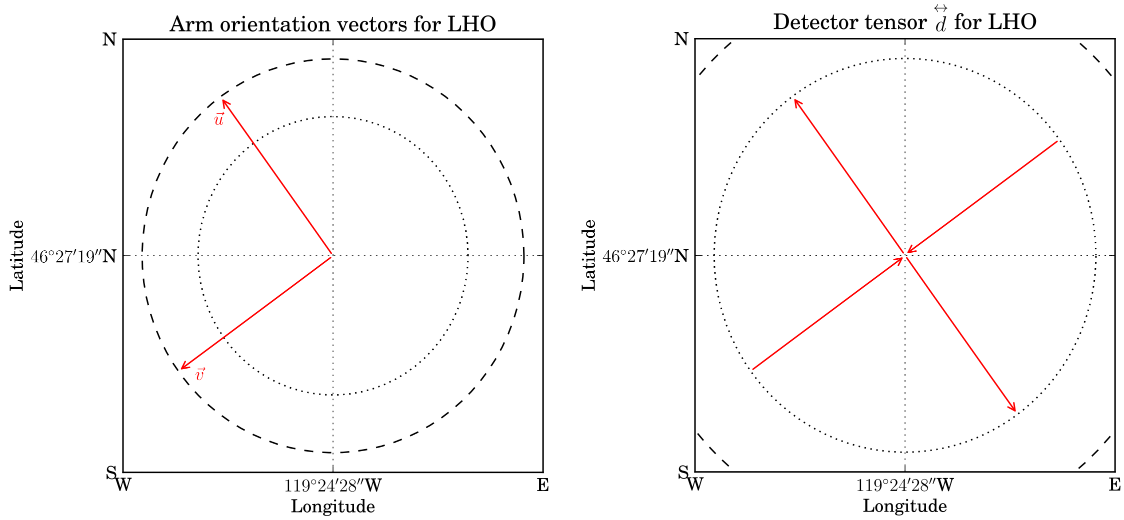

Figure 5: Definition of the detector tensor for LIGO Hanford Observatory, shown in a plane tangent to the Earth, seen from above. The small tilt angles between the detector arm directions and this plane are ignored.

the response is (again using the Einstein summation convention)

$$h = h_{ij}d^{ij} = \overleftrightarrow{h} : \overleftrightarrow{d} \ . \tag{2.3}$$

The last expression use the double dot product of two tensors, defined as the scalar[2]

$$\overleftrightarrow{S} : \overleftrightarrow{T} = S_{ij}T^{ji} \tag{2.4}$$

We illustrate this detector tensor for the specific case of LIGO Hanford Observatory (LHO) in figure 5 The geometry is most naturally viewed on the surface of the Earth, i.e., in a plane tangent to the Earth at LHO's latitude and longitude.[3] Note that in these figures, West is to the left and East is to the right, as in the usual map view of the Earth. Because the detector tensor is

$$\overleftrightarrow{d} = \frac{\vec{u}\otimes\vec{u} - \vec{v}\otimes\vec{v}}{2} = \frac{\vec{u}}{\sqrt{2}}\otimes\frac{\vec{u}}{\sqrt{2}} - \frac{\vec{v}}{\sqrt{2}}\otimes\frac{\vec{v}}{\sqrt{2}} \tag{2.5}$$

the outward and inward pointing arrows only reach the inner, dashed circle, with a radius of $1/\sqrt{2}$.

We can also represent this tensor "on the plane of the sky" for the point directly overhead at LHO, as shown in figure 6 Since the right ascension and declination of this point vary with the rotation of the Earth, we also need to specify the Greenwich sidereal time (GST); for example, at GST 00:00:00, the right ascension of the relevant point is equal to the longitude of the detector:

[2]This is a more elegant choice than $S_{ij}T^{ij}$, but it doesn't matter in this application since all of the tensors we consider are symmetric.

[3]For this illustration, we neglect the small tilt angles between the detector arm directions and the local horizontal plane.

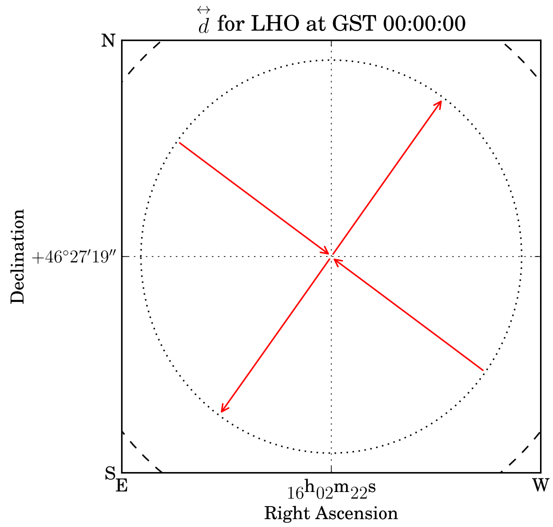

Figure 6: The tensor in figure 5, shown on the plane of the sky, directly overhead at the observatory, as seen from below.

3 Antenna Pattern Factors

Rather than working with the tensors, it is convenient and customary to combine (1.2) and (2.3) and write the signal h in a detector in terms of h_+ and $h_\times$ as

$$h = h_+ F_+ + h_\times F_\times \tag{3.1}$$

in terms of the *antenna pattern factors*

$$F_+ = \overleftrightarrow{d} : \overleftrightarrow{e}_+ \qquad \text{and} \qquad F_\times = \overleftrightarrow{d} : \overleftrightarrow{e}_\times \tag{3.2}$$

The tensor $\overleftrightarrow{d}$ depends on the location and orientation of the detector on the Earth, and also on the sidereal time; the tensors $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ depend on the location of the source on the sky and also on the polarization angle ψ. For a given detector at a given time, F_+ and $F_\times$ thus depend on the source's sky position (α, δ) and ψ. It is useful to separate out the dependence on the polarization angle ψ and define *amplitude modulation factors*[2]

$$a = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_+ \qquad \text{and} \qquad b = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_\times \tag{3.3}$$

(which are just the antenna pattern factors in the reference basis for that sky position), which are related to the antenna pattern factors by a basis rotation:

$$F_+(\alpha, \delta, \psi) = \quad a(\alpha, \delta) \cos 2\psi + b(\alpha, \delta) \sin 2\psi \tag{3.4a}$$

$$F_\times(\alpha, \delta, \psi) = -a(\alpha, \delta) \sin 2\psi + b(\alpha, \delta) \cos 2\psi \tag{3.4b}$$

The problem with visualizing the dot products in (3.2) or (3.3) is that the detector tensor $\overleftrightarrow{d}$ lies in a different plane than the polarization basis tensors. However, for gravitational

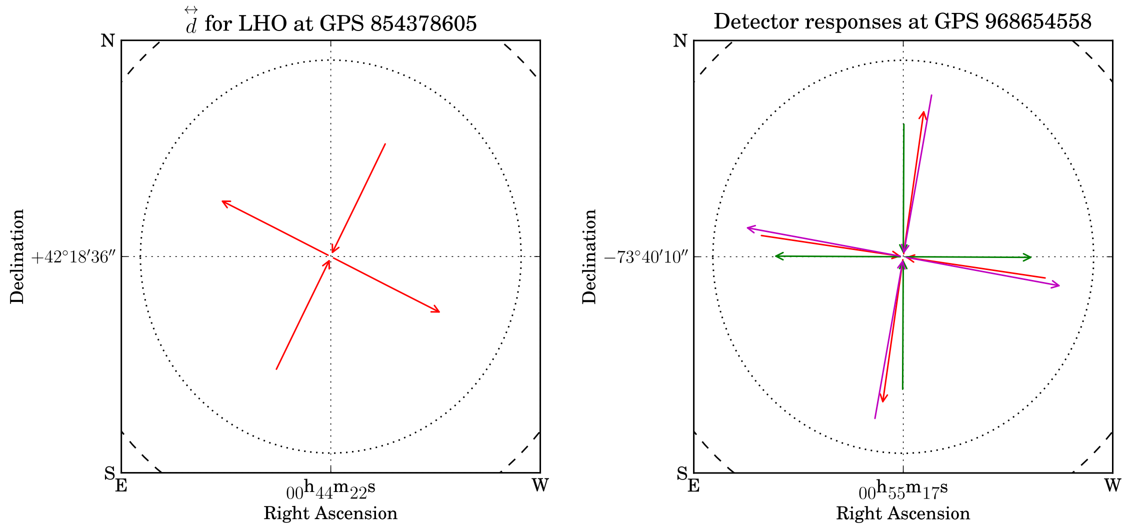

Figure 7: Examples of projected detector tensors for specific times and sky positions. The figure on the left shows the LHO detector geometry at the time and location of GRB070201. The figure on the right shows LLO, LHO and Virgo at the time and location of the Big Dog blind injection. The lengths of the arrows indicate the overall response of each detector, and their orientation captures the preferred polarization basis associated with the detector for that point on the sky.

waves propagating in a particular direction, we can replace $\overleftrightarrow{d}$ with its projection onto the transverse, traceless basis:

$$F_+(\vec{k},\psi) = \overleftrightarrow{d}^{\mathrm{TT}\vec{k}} : \overleftrightarrow{e}_+(\vec{k},\psi) \qquad \text{and} \qquad F_\times(\vec{k},\psi) = \overleftrightarrow{d}^{\mathrm{TT}\vec{k}} : \overleftrightarrow{e}_\times(\vec{k},\psi) \tag{3.5}$$

where the *projected detector tensor* is given by

$$\overleftrightarrow{d}^{\mathrm{TT}\vec{k}} = \frac{a}{2}\overleftrightarrow{\varepsilon}_+ + \frac{b}{2}\overleftrightarrow{\varepsilon}_\times \ . \tag{3.6}$$

The factor of $\frac{1}{2}$ in (3.6) is necessary because $\overleftrightarrow{\varepsilon}_+ : \overleftrightarrow{\varepsilon}_+ = 2 = \overleftrightarrow{\varepsilon}_\times : \overleftrightarrow{\varepsilon}_\times$. (Of course $\overleftrightarrow{\varepsilon}_+ : \overleftrightarrow{\varepsilon}_\times = 0$).

To represent the projected detector tensor $\overleftrightarrow{d}^{\mathrm{TT}\vec{k}}$ on the plane of the sky at the sky position corresponding to propagation direction $\vec{k}$, we just need to find the polarization angle ψ_d for which $F_\times = 0$ and $F_+ = \sqrt{a^2+b^2}$, and then

$$\overleftrightarrow{d}^{\mathrm{TT}\vec{k}} = \frac{\sqrt{a^2+b^2}}{2}\,\overleftrightarrow{e}_+(\vec{k},\psi_d) = \left(r\vec{\ell}_d\right)\otimes\left(r\vec{\ell}_d\right) - \left(r\vec{m}_d\right)\otimes\left(r\vec{m}_d\right) \tag{3.7}$$

It is straightforward to find

$$r = \frac{\sqrt[4]{a^2+b^2}}{\sqrt{2}} \qquad \text{and} \qquad \psi_d = \frac{\mathtt{atan2}(b,a)}{2} \tag{3.8}$$

If the source is directly overhead at the detector, so that $\overleftrightarrow{d}^{\mathrm{TT}\vec{k}} = \overleftrightarrow{d}$, then $r = 1/\sqrt{2}$ and ψ_d is (modulo 180°) the angle, measured clockwise on the Earth, from due West to the detector's X arm. For a general sky position, the projected tensor shows how much the detector responds to each polarization.

As an example, consider the response of LHO at the sky position and time of GRB070201.[5] This GRB occurred at right ascension $00^{\mathrm{h}}44^{\mathrm{m}}22^{\mathrm{s}}$ and declination $+42°18'36''$ at a GPS time of 854378605, which is a Greenwich sidereal time of 00:09:00. The amplitude modulation factors were $a = 0.2588$ and $b = -0.3434$. Since $\sqrt{a^2 + b^2} = 0.4300$, we draw our projected detector tensor with $r = 0.4637$. The polarization angle for which this response would be entirely in the plus polarization is $\psi_d = -26.5°$ so we draw the tensor in the diagram on the left (the dashed circle, with radius $1/\sqrt{2}$, is where the arrows would reach for an optimally-located detector). This is shown on the left in figure 7. The example on the right is for the Big Dog blind injection,[6] which was performed at right ascension $00^{\mathrm{h}}55^{\mathrm{m}}17^{\mathrm{s}}$ and declination $-73°40'10''$ at a GPS time of 968654558, which is a Greenwich sidereal time of 06:22:49. The projected detector tensors are shown for LHO (in red), LLO (in green) and Virgo (in magenta).

References

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