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Dynamic Resonance of Light in Fabry-Perot Cavities

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Abstract

The dynamics of light in Fabry-Perot cavities with varying length and input laser frequency are analyzed and the exact condition for resonance is derived. The resonance strongly depends on the light transit time in the cavity and the Doppler effect due to the mirror motions. It is shown that the cavity response to changes of its length is maximized while the response to changes of the laser frequency is minimized if the frequency of these changes is equal to multiples of the cavity free spectral range. Implications of these results for the detection of gravitational waves using kilometer-scale Fabry-Perot cavities are discussed.

Fabry-Perot cavities, optical resonators, are commonly utilized for high-precision frequency and distance measurements [1]. Currently, kilometer-scale Fabry-Perot cavities with suspended mirrors are being employed in efforts to detect cosmic gravitational waves [2, 3]. This application has stimulated renewed interest in cavities with moving mirrors [4, 5, 6, 7] and motivated many efforts to model the dynamics of such cavities on the computer [8, 9, 10, 11]. Recently the dynamics of £elds in Fabry-Perot cavities were analyzed in the transient regime [12, 13]. In this letter, we address the dynamics of cavity £elds in the steady state regime which is the intended operational state of gravitational wave detectors. We derive analytical solutions to the cavity £eld equations, yielding formulas for the cavity response to periodic variations in both the cavity length and the incident laser frequency. It is in this state of dynamic resonance that the detectors will continuously operate.

We consider a Fabry-Perot cavity with a laser £eld incident from one side (Fig. 1). Variations in the cavity length are due to the mirror displacements $x_a(t)$ and $x_b(t)$ which are measured with respect to the reference planes a and b. The nominal light transit time in the cavity and the cavity free spectral range (FSR) are de£ned by

$$T = L/c, \qquad \omega_{\rm fsr} = \pi/T. \tag{1}$$

The £eld incident upon the cavity and the £eld circulating in the cavity are described by plane waves with nominal frequency ω and wavenumber k ($k = \omega/c$). Variations in the laser frequency are denoted by $\delta\omega(t)$. We assume that the mirror displacements are much less than the nominal cavity length and that the deviations of the laser frequency are much less than the nominal frequency.

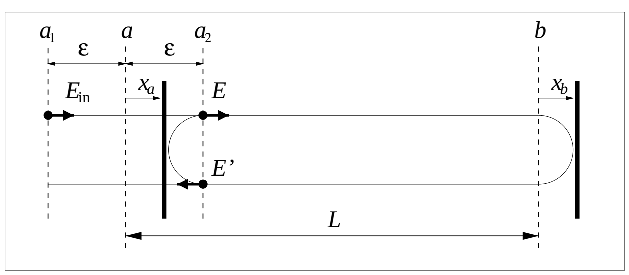

Figure 1: Mirror positions and £elds in Fabry-Perot cavity.

At any given place the electric £eld in the cavity $\mathcal{E}(t)$ oscillates at a very high frequency. For simplicity, we suppress this fast-oscillating factor, $\exp(i\omega t)$, and de£ne the slowly-varying £elds as $E(t) = \mathcal{E}(t)\exp(-i\omega t)$. To properly account for the phases of the propagating £elds, their complex amplitudes are de£ned at £xed locations, reference planes a_1 and a_2 shown in Fig. 1.

The equations for £elds in the cavity can be obtained by tracing a wavefront during its complete round-trip in the cavity (starting from the reference plane a_2). The propagation delays τ_1 and τ_2 depend on the mirror positions and are given by

$$c\,\tau_1 \;=\; L - \epsilon + x_b(t - \tau_1), \tag{2}$$

$$c\,\tau_2 \;=\; \epsilon - x_a(t - \tau_2), \tag{3}$$

where ϵ is de£ned as shown in Fig. 1. Then the £elds in the cavity satisfy the equations:

$$E'(t) \;=\; -r_b E(t - 2\tau_1) e^{-2i\omega\tau_1}, \tag{4}$$

$$E(t) \;=\; -r_a E'(t - 2\tau_2) e^{-2i\omega\tau_2} + t_a E_{\rm in}(t - 2\epsilon/c), \tag{5}$$

where r_a and r_b are the mirror re¤ectivities, and t_a is the transmissivity of the front mirror.

Because the £eld amplitudes E and E' do not change signi£cantly over times of order $x_{a,b}/c$, the small variations in these amplitudes during the changes in propagation times due to mirror displacements can be neglected. Furthermore, the reference planes a and b can be chosen so that the nominal length of the Fabry-Perot cavity becomes an integer multiple of the laser wavelength, making $\exp(-2ikL) = 1$. Finally, the small offset ϵ can be set to zero, and Eqs. (4)-(5), can be combined yielding the "iteration" equation for the cavity £eld

$$E(t) = t_a E_{\rm in}(t) + r_a r_b E(t - 2T)\exp[-2ik\delta L(t)]. \tag{6}$$

Here $\delta L(t)$ is the variation in the cavity length "seen" by the light circulating in the cavity,

$$\delta L(t) = x_b(t - T) - x_a(t). \tag{7}$$

Note that the time delay appears in the coordinate of the end mirror, but not the front mirror. This is simply a consequence of our placement of the laser source; the light that enters the cavity re¤ects from the end mirror £rst and then the front mirror. For $\delta L = 0$, Laplace transformation of both sides of Eq. (6) yields the basic cavity response function

$$H(s) \equiv \frac{\tilde{E}(s)}{\tilde{E}_{\rm in}(s)} = \frac{t_a}{1 - r_a r_b e^{-2sT}}, \tag{8}$$

where tildes denote Laplace transforms.

The static solution of Eq. (6) is found by considering a cavity with £xed length ($\delta L = \mathrm{const}$) and an input laser £eld with £xed amplitude and frequency, ($A, \delta\omega = \mathrm{const}$). In this case the input laser £eld and the cavity £eld are given by

$$\begin{aligned} E_{\mathrm{in}}(t) &= A\, e^{i\delta\omega t}, && (9) \\ E(t) &= E_0\, e^{i\delta\omega t}, && (10) \end{aligned}$$

where E_0 is the amplitude of the cavity £eld,

$$E_0 = \frac{t_a A}{1 - r_a r_b e^{-2i\psi}}, \tag{11}$$

and ψ is the phase offset,

$$\psi = T\delta\omega + k\ \delta L. \tag{12}$$

It can be seen from Eqs. (11) and (12) that the cavity £eld is maximized when the cavity length and the laser frequency are adjusted so that

$$\frac{\delta\omega}{\omega} = -\frac{\delta L}{L}. \tag{13}$$

This is the well-known static resonance condition. The maximum amplitude of the cavity £eld is given by

$$\bar{E} = \frac{t_a A}{1 - r_a r_b}. \tag{14}$$

Light can also resonate in a Fabry-Perot cavity when its length and the laser frequency are changing. For a £xed amplitude, the input laser £eld can be written as

$$E_{\mathrm{in}}(t) = A\, e^{i\phi(t)}, \tag{15}$$

where $\phi(t)$ is the variable phase,

$$\phi(t) = \int_0^t \delta\omega(t')dt'. \tag{16}$$

Then the steady-state solution of Eq. (6) is

$$E(t) = E_0\, e^{i\phi(t)}, \tag{17}$$

with the amplitude E_0 given by Eq. (11), under the condition that

$$\phi(t) - \phi(t - 2T) + 2k\ \delta L(t) = \mathrm{const} \equiv \psi. \tag{18}$$

The maximum amplitude of the cavity £eld occurs when $\psi = 0$.

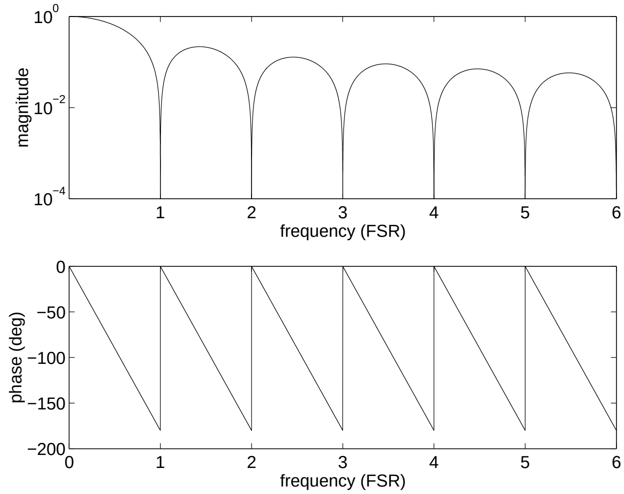

Figure 2: Bode plot of the frequency-to-length transfer function, $C(i\Omega)$.

Thus dynamic resonance occurs when the phase of the input laser £eld is corrected to compensate for the changes in the cavity length due to the mirror motions. The associated frequency correction is equal to the Doppler shift caused by re¤ection from the moving mirrors

$$\delta\omega(t) - \delta\omega(t - 2T) = -2\frac{v(t)}{c}\omega, \tag{19}$$

where $v(t)$ is the relative mirror velocity ($v = d\delta L/dt$). The equivalent formula in the Laplace domain is

$$C(s)\frac{\delta\tilde{\omega}(s)}{\omega} = -\frac{\delta\tilde{L}(s)}{L}, \tag{20}$$

where $C(s)$ is the normalized frequency-to-length transfer function which is given by

$$C(s) = \frac{1 - e^{-2sT}}{2sT}. \tag{21}$$

A Bode plot, which combines the magnitude and phase of $C(s)$ for $s = i\Omega$, is shown in Fig. 2. The magnitude is just the familiar sinc function, $|C(i\Omega)| = \sin\Omega T/\Omega T$, whereas the phase is a linear function of Ω. $C(s)$ has zeros at multiples of the cavity free spectral range,

$$z_n = i\omega_{\rm fsr}n, \tag{22}$$

where n is integer, and therefore can be written as an in£nite product,

$$C(s) = e^{-sT} \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{z_n^2}\right), \tag{23}$$

which is useful for control system design[1].

To maintain resonance, changes in the cavity length must be compensated by changes in the laser frequency according to Eq. (20). If the frequency of such changes is much less than the cavity free spectral range, then $C(s) \approx 1$. In this case, Eq. (20) reduces to the quasi-static approximation,

$$\frac{\delta\tilde{\omega}(s)}{\omega} \approx -\frac{\delta\tilde{L}(s)}{L}, \tag{24}$$

in which length and laser frequency variations are treated equally. At frequencies above the cavity free spectral range, $C(s) \sim 1/s$, and increasingly larger laser frequency changes are required to compensate for cavity length variations. Moreover, at multiples of the FSR no frequency-to-length compensation is possible.

In practice, Fabry-Perot cavities deviate from resonance, and a negative-feedback control system is employed to reduce the deviations. For small deviations from resonance, the cavity £eld can be described as

$$E(t) = [\bar{E} + \delta E(t)]e^{i\phi(t)}, \tag{25}$$

where $\bar{E}$ is the maximum £eld given by Eq.(14), and δE is a small perturbation ($|\delta E| \ll |\bar{E}|$). Substituting this equation into Eq. (6), we see that the perturbation evolves in time according to

$$\begin{aligned} \delta E(t) - r_a r_b \delta E(t - 2T) = & \\ & -i r_a r_b \bar{E}\left[\phi(t) - \phi(t-2T) + 2k\ \delta L(t)\right]. \end{aligned} \tag{26}$$

This equation is easily solved in the Laplace domain, yielding

$$\delta\tilde{E}(s) = -i r_a r_b \bar{E}\ \frac{\left(1 - e^{-2sT}\right)\tilde{\phi}(s) + 2k\ \delta\tilde{L}(s)}{1 - r_a r_b e^{-2sT}}. \tag{27}$$

Deviations of the cavity £eld from its maximum value can be measured by the Pound-Drever-Hall (PDH) error signal which is widely utilized for feedback control of Fabry-Perot cavities [14]. The PDH signal is obtained by coherent detection of phase-modulated light re¤ected by the cavity. With the appropriate

[1]This formula is derived using the in£nite-product: $\sin x = x\ \prod_{n=1}^{\infty}(1 - x^2/\pi^2 n^2)$.

choice of the demodulation phase, the PDH signal is proportional to the imaginary part of the cavity £eld and therefore can be written as

$$\delta\tilde{V}(s) = \mathrm{const} \times H(s) \left[\frac{\delta\tilde{L}(s)}{L} + C(s)\, \frac{\delta\tilde{\omega}(s)}{\omega} \right], \tag{28}$$

where $H(s)$ is given by Eq. (8). In the presence of length and frequency variations, feedback control will drive the error signal toward the null point, $\delta\tilde{V}(s) = 0$, thus maintaining dynamic resonance according to Eq. (20).

The response of the PDH signal to either length or laser frequency deviations can be found from Eq. (28). The normalized length-to-signal transfer function is given by

$$H_L(s) \equiv \frac{H(s)}{H(0)} = \frac{1 - r_a r_b}{1 - r_a r_b e^{-2sT}}. \tag{29}$$

A Bode plot of $H_L(s)$ is shown in Fig. 3 for the LIGO [2] Fabry-Perot cavities with $L = 4$ km, $r_a = 0.985$, and $r_b = 1$. The magnitude of the transfer function,

$$|H_L(i\Omega)| = \frac{1}{\sqrt{1 + F \sin^2 \Omega T}}, \tag{30}$$

is the square-root of the well-known Airy function with the coef£cient of £nesse, $F = 4 r_a r_b / (1 - r_a r_b)^2$. (The Airy function describes the intensity pro£le of a Fabry-Perot cavity [15].)

The poles of $H_L(s)$ are given by

$$p_n = i\omega_{\mathrm{fsr}} n - \frac{1}{\tau}, \tag{31}$$

where n is integer, and τ is the storage time of the cavity,

$$\tau = \frac{2T}{|\ln(r_a r_b)|}. \tag{32}$$

Therefore, H_L can be written as an in£nite product,

$$H_L(s) = e^{sT} \prod_{n=-\infty}^{\infty} \frac{p_n}{p_n - s}, \tag{33}$$

which can be truncated to a £nite number of terms for use in control system design.

The normalized frequency-to-signal transfer function is given by

$$H_\omega(s) \equiv C(s) H_L(s) \tag{34}$$

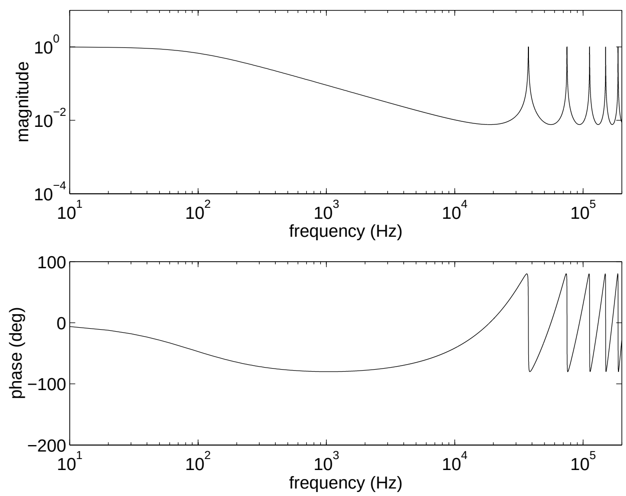

Figure 3: Bode plot of $H_L(i\Omega)$ for the LIGO 4-km-long Fabry-Perot cavities. The peaks occur at multiples of the FSR (37.5 kHz) and their half-width (91 Hz) is equal to the inverse of the cavity storage time.

$$= \left(\frac{1 - e^{-2sT}}{2sT}\right) \left(\frac{1 - r_a r_b}{1 - r_a r_b e^{-2sT}}\right). \tag{35}$$

A Bode plot of H_ω is shown in Fig. 4. H_ω has zeros given by Eq. (22) with $n \neq 0$, and poles given by Eq. (31). The poles and zeros come in pairs except for the lowest order pole ($n = 0$) which does not have a matching zero. Therefore $H_\omega(s)$ can be written as an in£nite product

$$H_\omega(s) = \frac{p_0}{p_0 - s} \prod_{n=-\infty}^{\infty}{}' \left(\frac{s - z_n}{s - p_n}\right), \tag{36}$$

where the prime indicates that $n = 0$ term is omitted from the product.

The zeros in the transfer function indicate that the cavity shows no response ($\delta E = 0$) to laser frequency deviations if these deviations occur at multiples of the cavity FSR. In this case, the amplitude of the circulating £eld is constant while the phase of the circulating £eld is changing in accordance with the phase of the input laser £eld.

In summary, we have shown that it is possible to maintain resonance of light in Fabry-Perot cavities even when the cavity length and laser frequency are changing. In this dynamic resonance state, changes in the laser frequency and changes in the cavity length play very different roles (Eq. (20)) in contrast to the quasi-static resonance state where they appear equally (Eq. (24)). Maintenance of dynamic resonance requires that the frequency-to-length transfer function, $C(s)$, be taken

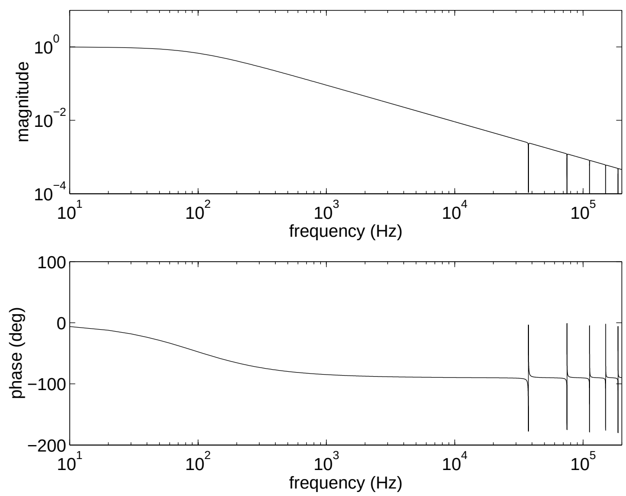

Figure 4: Bode plot of $H_\omega(i\Omega)$. The abrupt drops of magnitude are due to the zero-pole pairs at multiples of FSR.

into account when compensating for length variations by frequency changes and vice versa. Compensation for length variations by frequency changes becomes increasingly more dif£cult at frequencies above the FSR, and impossible at multiples of the FSR.

As can be seen in Fig. 4, the PDH error signal from frequency variations decreases as $1/\Omega$ above Ω_0 ($\Omega_0 = 1/\tau$) and is insensitive to frequency variations occurring at frequencies equal to multiples of the cavity FSR. For length variations, the PDH signal is a periodic function of frequency. It decreases as $1/\Omega$ above Ω_0, but only to the level of $1/\sqrt{F}$ and then returns to its maximum value (Fig. 3). Thus at multiples of the FSR, the sensitivity to length variations is maximum while the sensitivity to frequency variations is minimum.

The enhanced sensitivity to length variations and insensitivity to laser frequency variations might suggest that searches for high-frequency gravitational waves at multiples of FSR are feasible. However, because the gravitational wave interacts with the light as well as the cavity mirrors [16], the normalized response of the PDH error signal to gravitational waves is functionally equivalent to $H_\omega(s)$ rather than $H_L(s)$ [5]. Thus gravitational wave interferometers which utilize Fabry-Perot cavities are insensitive to gravitational waves at these higher frequencies where they are maximally sensitive to length ¤uctuations.

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