New Coordinates for the Amplitude Parameter Space of Continuous Gravitational Waves

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in collaboration with Reinhard Prix, Curt Cutler, Josh Willis
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Outline

1. Amplitude Parameters
   - Coordinates on Amplitude Parameter Space
   - $\mathcal{F}$- and $\mathcal{B}$-Statistics
2. New Coordinates
   - Definitions and Properties
   - Applications to $\mathcal{B}$-Statistic Integral

Preprint arXiv:1311.0065: JTW, Prix, Cutler & Willis: "New Coordinates for the Amplitude Parameter Space of Continuous Gravitational Waves", submitted to CQG

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Reminder: Amplitude Parameters & Signal Model

- CW signal model:

$$\overleftrightarrow{h}(\tau) = A_+ \cos[\phi(\tau) + \phi_0] \overleftrightarrow{e}_+ + A_\times \sin[\phi(\tau) + \phi_0] \overleftrightarrow{e}_\times$$

In a detector

$$h(t) = \overleftrightarrow{h}(\tau) : \overleftrightarrow{d} = A_+ F_+ \cos[\phi(\tau(t)) + \phi_0] + A_\times F_\times \sin[\phi(\tau(t)) + \phi_0]$$

- Sky position, f_0, spindowns, etc determine phase evolution
- Amplitude parameters are h_0, $\chi = \cos \iota$, ψ, ϕ_0
- $A_+ = \frac{h_0}{2}(1+\chi^2)$, $A_\times = h_0 \chi$
- JKS decomposition (*PRD* **58**, 063001 (1998))

$$\overleftrightarrow{h}(\tau) = \mathcal{A}^\mu(h_0, \chi, \psi, \phi_0) \overleftrightarrow{h}_\mu(\tau) \qquad \sum_{\mu=1}^{4} \text{ implied}$$

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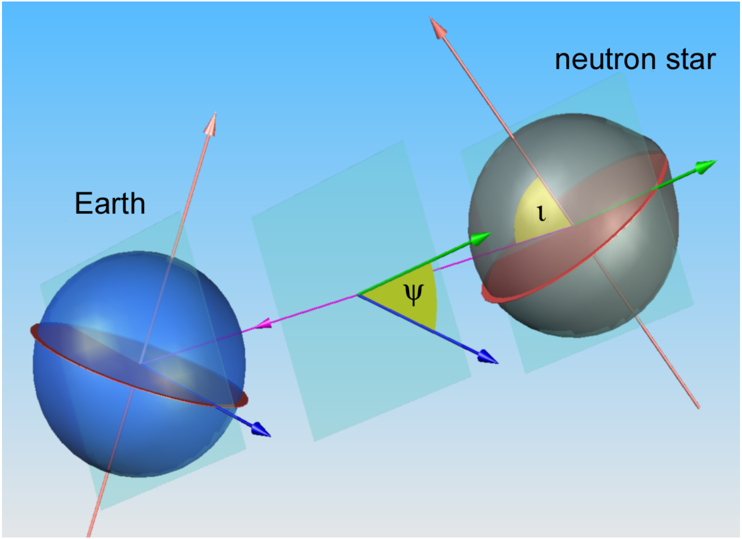
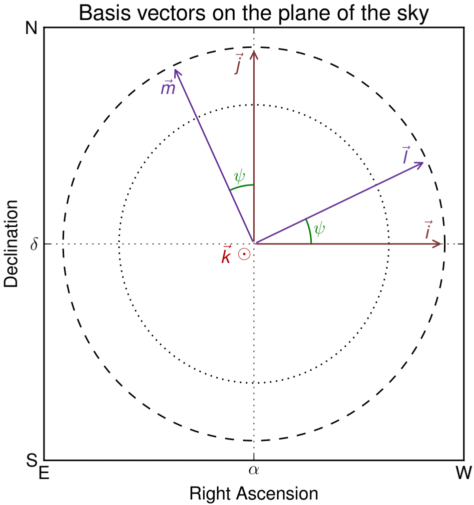
Inclination & Polarization Angles for Neutron Star



Geometric Definition of Polarization Bases



$$\overleftrightarrow{e}_+ = \vec{l}\otimes\vec{l} - \vec{m}\otimes\vec{m}$$
$$\overleftrightarrow{e}_\times = \vec{l}\otimes\vec{m} + \vec{m}\otimes\vec{l}$$
$$F_+ = \overleftrightarrow{d} : \overleftrightarrow{e}_+$$
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$$\overleftrightarrow{\varepsilon}_+ = \vec{i}\otimes\vec{i} - \vec{j}\otimes\vec{j}$$
$$\overleftrightarrow{\varepsilon}_\times = \vec{i}\otimes\vec{j} + \vec{j}\otimes\vec{i}$$
$$a = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_+$$
$$b = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_\times$$

Likelihood Ratio Statistics

- Log-likelihood ratio between noise model $\mathcal{H}_n$ & signal model $\mathcal{H}_s$ w/amplitude params $\mathcal{A}$, given data $\mathbf{x}$:

$$\Lambda(\mathcal{A};\mathbf{x}) := \ln \frac{\text{pdf}(\mathbf{x}|\mathcal{H}_s,\mathcal{A})}{\text{pdf}(\mathbf{x}|\mathcal{H}_n)} = \mathcal{A}^\mu x_\mu - \frac{1}{2}\mathcal{A}^\mu \mathcal{M}_{\mu\nu}\mathcal{A}^\nu$$

- $\mathcal{F}$-stat is this maximized over amplitude parameters:

$$\mathcal{F}(\mathbf{x}) = \max_{\mathcal{A}} \Lambda(\mathcal{A};\mathbf{x})$$

- Prix & Krishnan: optimal statistic is *marginalized* over $\mathcal{A}$:

$$\mathcal{B}(\mathbf{x}) = \int e^{\Lambda(\mathcal{A};\mathbf{x})}\,\text{pdf}(\mathcal{A}|\mathcal{H}_s)\,d^4\mathcal{A}$$

  (*CQG* **26**, 204013 (2009)) Depends on prior $\text{pdf}(\mathcal{A}|\mathcal{H}_s)$
- With uniform prior in $\{\mathcal{A}^\mu\}$, can show $\mathcal{B} \propto e^{\mathcal{F}}$
- Physically, prior should be uniform in χ, ψ, ϕ_0 & maybe h_0

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Physical Measure on Amplitude Parameter Space

- $\mathcal{F}$-statistic equivalent to $\mathcal{B}$-statistic w/uniform prior on $\{\mathcal{A}^\mu\}$
- Physically, prior should be uniform in χ, ψ, ϕ_0 & maybe h_0
- Related by Jacobian determinant

$$d\mathcal{A}^1\, d\mathcal{A}^2\, d\mathcal{A}^3\, d\mathcal{A}^4 = 16\left(h_0 \frac{1-\chi^2}{4}\right)^3 dh_0\, d\chi\, d\psi\, d\phi_0$$

  Shown using MAXIMA in Prix & Krishnan;
  New coordinates allow quick analytic computation
- Log-Likelihood ratio is quadratic in $\{\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3, \mathcal{A}^4\}$;
  Physical prior is uniform in $\{h_0, \chi = \cos\iota, \psi, \phi_0\}$

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- Jaranowski-Królak-Schutz decomposition:

$$\overleftrightarrow{h}(\tau) = \mathcal{A}^{\mu}(h_0, \chi, \psi, \phi_0)\overleftrightarrow{h}_{\mu}(\tau)$$

- Basis waveforms are

$$\overleftrightarrow{h}_1(\tau) = \overleftrightarrow{\varepsilon}_{+} \cos\phi(\tau) \qquad \overleftrightarrow{h}_2(\tau) = \overleftrightarrow{\varepsilon}_{\times} \cos\phi(\tau)$$
$$\overleftrightarrow{h}_3(\tau) = \overleftrightarrow{\varepsilon}_{+} \sin\phi(\tau) \qquad \overleftrightarrow{h}_4(\tau) = \overleftrightarrow{\varepsilon}_{\times} \sin\phi(\tau)$$

- $(\mathcal{A}^1, \mathcal{A}^3)$ are "plus"-pol amplitudes, $(\mathcal{A}^2, \mathcal{A}^4)$ "cross". Straightforward but not simple functions of $\{h_0, \chi, \psi, \phi_0\}$

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CPF Coordinates

- JKS-like decomposition in $\{\mathcal{A}^{\breve{\mu}}\} \equiv \{\mathcal{A}^{\breve{1}}, \mathcal{A}^{\breve{2}}, \mathcal{A}^{\breve{3}}, \mathcal{A}^{\breve{4}}\}$:

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- Define CPF-polar coordinates ($\mathcal{A}^{\breve{1}} = A_{\mathrm{R}} \cos\phi_{\mathrm{R}}$, etc):

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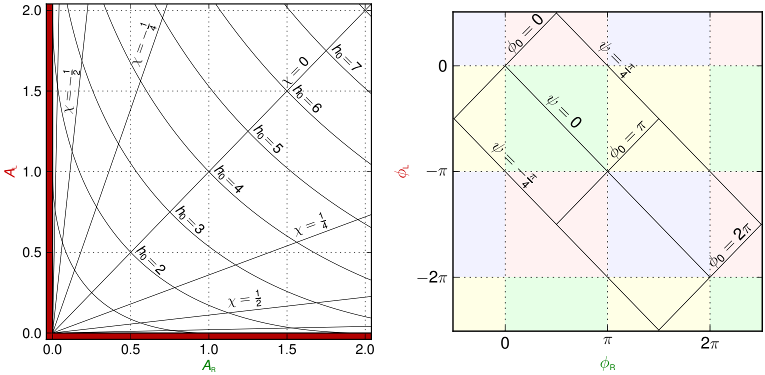
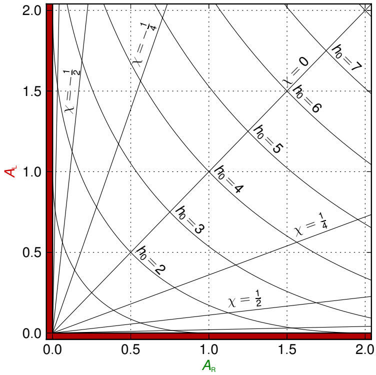
CPF-Polar Coordinates

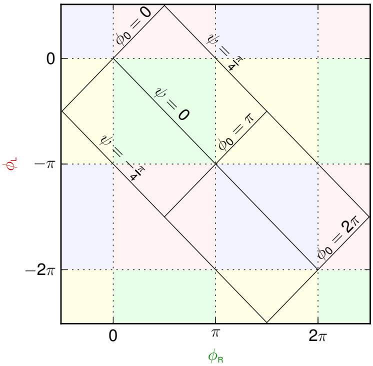
- $0 \leq A_R < \infty$ & $0 \leq A_L < \infty$ cover allowed $h_0, \chi = \cos\iota$
- $0 \leq \phi_R < 2\pi$ & $0 \leq \phi_L < 2\pi$ cover unique ψ, ϕ_0



CPF-Polar Coordinates

- $0 \leq A_{\text{R}} < \infty$ & $0 \leq A_{\text{L}} < \infty$ cover allowed $h_0, \chi = \cos\iota$
- $0 \leq \phi_{\text{R}} < 2\pi$ & $0 \leq \phi_{\text{L}} < 2\pi$ cover unique ψ, ϕ_0





Circular Polarization Coordinate Singularities

- When $\iota = 0$, $\chi = 1$ (right circular polarization), $A_{\mathrm{L}} = 0$ and $\phi_{\mathrm{L}} = \phi_0 - 2\psi$ arbitrary
- When $\iota = \pi$, $\chi = -1$ (left circular polarization), $A_{\mathrm{R}} = 0$ and $\phi_{\mathrm{R}} = \phi_0 + 2\psi$ arbitrary
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Coordinate Transformations and Jacobians

- Factor appearing in Jacobian between JKS[1] coordinates and physical amplitude params is $\left(h_0\frac{1-\chi^2}{4}\right)^3 = (A_R A_L)^{3/2}$ so coordinate singularity at circ pol ($A_R = 0$ or $A_L = 0$) makes measure of $\mathcal{B}$-stat integral singular in JKS[1] coords
- Physical measure is

$$dh_0\,d\chi\,d\psi\,d\phi_0 = \frac{dA_R\,dA_L\,d\phi_R\,d\phi_L}{4\sqrt{A_R A_L}} = \frac{A_R\,dA_R\,d\phi_R\,A_L\,dA_L\,d\phi_L}{4(A_R A_L)^{3/2}}$$

$$= 4\,r_R\,dr_R\,d\phi_R\,r_L\,dr_L\,d\phi_L = 4\,dx_R\,dy_R\,dx_L\,dy_L$$

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- Form of likelihood ratio manageable via

$$\mathcal{A}^{\check{1}} = {r_R}^3\,x_R \qquad \mathcal{A}^{\check{2}} = {r_R}^3\,y_R \qquad \mathcal{A}^{\check{3}} = {r_L}^3\,x_L \qquad \mathcal{A}^{\check{4}} = {r_L}^3\,y_L$$

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Approximate Calculation of $\mathcal{B}$-Statistic

- Assume uniform prior in $\{h_0, \chi = \cos\iota, \psi, \phi_0\}$ so

$$\mathcal{B}(\mathbf{x}) \propto \int e^{\Lambda(\mathcal{A};\mathbf{x})} dh_0\, d\chi\, d\psi\, d\phi_0$$

- In root-radius coordinates, measure is constant. $\Lambda(x_R, y_R, x_L, y_L; \mathbf{x})$ is not quadratic, but we can Taylor expand about maximum-likelihood point and find

$$\ln \mathcal{B}(\mathbf{x}) \approx \mathcal{F}(\mathbf{x}) - \frac{3}{2} \ln(\widehat{A}_R(\mathbf{x}) \widehat{A}_L(\mathbf{x}))$$

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Explicit Evaluation for Simple Metric

- Log-likelihood ratio is

$$\Lambda(\mathcal{A};\mathbf{x}) = \mathcal{F}(\mathbf{x}) - \frac{1}{2}\mathcal{M}_{\breve{\mu}\breve{\nu}}(\mathcal{A}^{\breve{\mu}} - \widehat{\mathcal{A}}^{\breve{\mu}}(\mathbf{x}))(\mathcal{A}^{\breve{\nu}} - \widehat{\mathcal{A}}^{\breve{\nu}}(\mathbf{x}))$$

where amp param metric $\{\mathcal{M}_{\breve{\mu}\breve{\nu}}\}$ determined by geometry and $\mathcal{F}(\mathbf{x}) = \frac{1}{2}\mathcal{M}_{\breve{\mu}\breve{\nu}}\widehat{\mathcal{A}}^{\breve{\mu}}(\mathbf{x})\widehat{\mathcal{A}}^{\breve{\nu}}(\mathbf{x})$

- Can get simpler form if $\langle a^2\rangle = \langle b^2\rangle \gg \langle ab\rangle$; then $\mathcal{M}_{\breve{\mu}\breve{\nu}} = h_{\text{det}}^{-2}\delta_{\breve{\mu}\breve{\nu}}$ where h_{det} is a sensitivity scale, and $\Lambda(\mathcal{A};\mathbf{x}) = \Lambda_{\text{R}}(A_{\text{R}},\phi_{\text{R}};\widehat{A}_{\text{R}},\widehat{\phi}_{\text{R}}) + \Lambda_{\text{L}}(A_{\text{L}},\phi_{\text{L}};\widehat{A}_{\text{L}},\widehat{\phi}_{\text{L}})$ with

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$\mathcal{B}$-Statistic for Simple Metric

- In this special case, can analytically integrate in CPF-polar coordinates to get solution in terms of confluent hypergeometric functions ${}_1F_1(a,b,z) = M(a,b,z)$:

$$\frac{\mathcal{B}(\mathbf{x})}{\mathcal{B}(\mathbf{0})} = \frac{\mathcal{B}_{\mathrm{R}}(\widehat{A}_{\mathrm{R}})}{\mathcal{B}_{\mathrm{R}}(0)} \frac{\mathcal{B}_{\mathrm{L}}(\widehat{A}_{\mathrm{L}})}{\mathcal{B}_{\mathrm{L}}(0)} = {}_1F_1\left(\frac{1}{4}, 1, \frac{\widehat{A}_{\mathrm{R}}{}^2}{2h_{\mathrm{det}}^2}\right) {}_1F_1\left(\frac{1}{4}, 1, \frac{\widehat{A}_{\mathrm{L}}{}^2}{2h_{\mathrm{det}}^2}\right)$$

- Can compare this to $\mathcal{F}$-statistic

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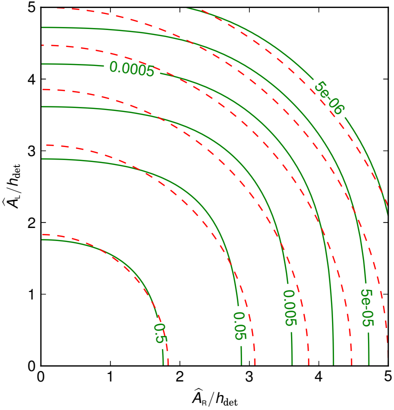
Contours of constant $\mathcal{B}$-Stat & $\mathcal{F}$-Stat

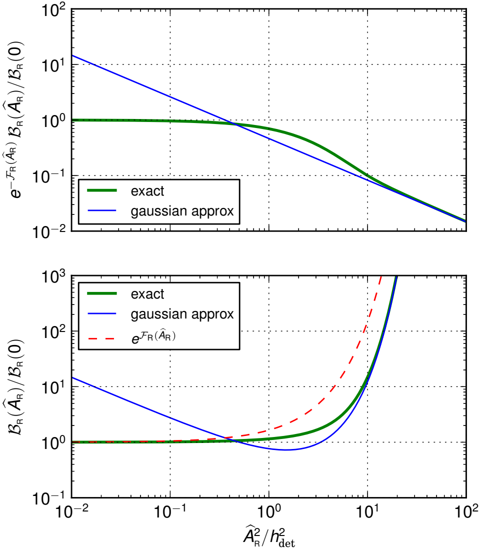


$\mathcal{B}$ & $\mathcal{F}$ contours are drawn at same false alarm rates

$\mathcal{F}$-stat prior overweights linear polarization for signal hypothesis

$\mathcal{B}$-stat “corrects” this; circ pol more likely to imply $\mathcal{H}_s$ @ given $\mathcal{F}$

Comparison of exact & approximate $\mathcal{B}$-Stat



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  $\mathcal{A}^{\breve{3}} = A_{\mathrm{L}} \cos\phi_{\mathrm{L}}$, $\mathcal{A}^{\breve{4}} = A_{\mathrm{L}} \sin\phi_{\mathrm{L}}$ form alternate JKS decomp
- Jacobian btwn phys $\{h_0, \chi = \cos\iota, \psi, \phi_0\}$ coordinates

& CPF or JKS is $(A_{\mathrm{R}} A_{\mathrm{L}})^{-3/2}$; coord singularity @ circ polarization

- Coordinate change allows approx evaluation of $\mathcal{B}$ statistic (marginalized likelihood ratio assuming uniform prior in physical params), $\mathcal{B} \approx (\text{const}) \frac{e^{\mathcal{F}}}{(\widehat{A}_{\mathrm{R}} \widehat{A}_{\mathrm{L}})^{3/2}}$
- If amplitude param metric is $\mathcal{M}_{\breve{\mu}\breve{\nu}} = h_{\mathrm{det}}^{-2} \delta_{\breve{\mu}\breve{\nu}}$, exact soln is
  $\frac{\mathcal{B}(\mathbf{x})}{\mathcal{B}(\mathbf{0})} = {}_1F_1\left(\frac{1}{4}, 1, \frac{\widehat{A}_{\mathrm{R}}{}^2}{2h_{\mathrm{det}}^2}\right) \; {}_1F_1\left(\frac{1}{4}, 1, \frac{\widehat{A}_{\mathrm{L}}{}^2}{2h_{\mathrm{det}}^2}\right)$
- Preprint arXiv:1311.0065; send comments!