Note on the derivation of the angular momentum and spin precessing equations in SpinTaylor codes.

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This is a technical note meant to accompany the SpinTaylor waveform review to explain how the derivative of the Newtonian angular momentum is computed. A future development would be to include in the code instantaneous spin^2 terms.

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I. TIME DERIVATIVE OF THE NEWTONIAN ANGULAR MOMENTUM UNIT VECTOR

The evolution equations for precessing spins and orbital angular momentum are obtained by imposing

$$\dot{\vec{L}} = -\dot{\vec{S}}_1 - \dot{\vec{S}}_2\,, \tag{1}$$

i.e. imposing total angular momentum conservation and neglecting angular momentum emission by radiation, which is given by, see e.g. (4.115) of [1]

$$\frac{dL}{dt} = \frac{32}{5}\eta^2 M v^7 = \frac{32}{5M}\eta v^8 L\,, \tag{2}$$

with $\eta \equiv m_1 m_2/M^2$ the symmetric mass ratio, $m_{1,2}$ individual binary constituent masses and $M \equiv m_1 + m_2$.

At leading order in the PN expansion parameter x ($x = v^2 = (M\omega)^{2/3}$ in LAL codes, with ω the orbital phase derivative) one has, see e.g. eqs. (8-10) of [2]:

$$\begin{aligned}\dot{\vec{S}}_1 &= \frac{x^{5/2}}{2M}\left(3 - 2\frac{m_1}{M} - \frac{m_1^2}{M^2}\right)\vec{S}_1 \times \hat{L}\,,\\ \dot{\hat{L}} &= \frac{x^3}{2M}\left(1 + 3\frac{M}{m_1}\right)\hat{L} \times \vec{S}_1 + 1 \leftrightarrow 2\,.\end{aligned} \tag{3}$$

At alternate PN orders spin derivatives receive contributions from spin^2 terms (x^n in the spin-dot equations) and from $L \times S$ terms ($x^{(2n+1)/2}$), with $n = 2$ for the leading order, and L precession equation can be inferred from (1).

At leading order (and up to v^2 order included with respect to the leading) we can assume that Newtonian angular momentum and $\vec{L}$ are parallel: $\hat{L}_N = \hat{L}$, however at higher order the relationship between $\vec{L}$ and $\vec{L}_N$ is actually, see eq. 4.7 of [3]

$$\begin{aligned}\vec{L} \;=\; & \hat{L}_N |\vec{L}_N^{(0)}| \left(1 + v^2 L_{1PN} + v^4 L_{2PN} + v^6 L_{3PN}\right) + \\ & v^2 \left[-\frac{5}{6}\left(1 + 3\frac{M}{m_1}\right)\vec{S}_{1l} - \frac{1}{2}\left(1 - \frac{M}{m_1}\right)\vec{S}_{1n} - \left(1 + \frac{M}{m_1}\right)\vec{S}_{1\lambda} + \right] \\ & + v^4 \left[\vec{S}_{1n}\left(-\frac{11}{8}\left(1 - \frac{M}{m_1}\right) + \frac{\eta}{24}\left(1 - 10\frac{M}{m_1}\right)\right) + \vec{S}_{1\lambda}\left(-\frac{5}{2}\left(1 + \frac{M}{5m_1}\right) + \frac{\eta}{3}\left(1 + 4\frac{M}{m_1}\right)\right) \right. \\ & \left. + \vec{S}_{1L}\left(-\frac{7}{8}\left(5 + 3\frac{M}{5m_1}\right) + 7\frac{\eta}{72}\left(1 + 30\frac{M}{m_1}\right)\right)\right] + 1 \leftrightarrow 2\,,\end{aligned} \tag{4}$$

where

$$\begin{aligned} |\vec{L}_N^{(0)}| &= \frac{m_1 m_2}{v}\,, \\ L_{1PN} &= \frac{3}{2} + \frac{\eta}{6}\,, \\ L_{2PN} &= \frac{27}{8} - \frac{19}{8}\eta + \frac{1}{24}\eta^2\,, \\ L_{3PN} &= \frac{135}{16} + \left[-\frac{6889}{144} + \frac{41}{24}\pi^2\right]\eta + \frac{31}{24}\eta^2 + \frac{7}{1296}\eta^3\,, \end{aligned} \tag{5}$$

and we remind that

$$\vec{L}_N^{(0)} \equiv \frac{m_1 m_2}{v}\hat{L}_N \neq \frac{m_1 m_2}{\omega r}\hat{L}_N\,, \tag{6}$$

and $\vec{S}_{1n} \equiv \hat{n}^i(\hat{n}\cdot\vec{S}_1)$, $\vec{S}_{1\lambda} \equiv \hat{\lambda}^i(\hat{\lambda}\cdot\vec{S}_1)$, and $\vec{S}_{1l} \equiv \hat{L}_N^i(\hat{L}_N\cdot\vec{S}_1)$, being $\hat{n}, \hat{\lambda}$ the unit vectors in the direction respectively of binary relative separation and velocity.

Note that we may want to average over one orbit, so that

$$\begin{aligned} \langle n^i \rangle = \langle \lambda^i \rangle &= 0\,, \\ \langle \hat{n}^i \hat{n}^j \rangle = \langle \hat{\lambda}^i \hat{\lambda}^j \rangle &= \frac{1}{2}\left(\delta_{ij} - \hat{L}_N^i \hat{L}_N^j\right)\,. \end{aligned} \tag{7}$$

The spin corrections to the orbital angular momentum are $x^{3/2}$ order with respect to the leading contribution to $\vec{L}$, that comes from the Newtonian angular momentum. Note also that the spins in this formulae are the physical ones related to the LAL convention $\vec{S}_{LAL}$ by $\vec{S}_{i\,LAL} \equiv \vec{S}_i/M^2$ (we use here units $G_N = c = 1$). The velocity symbol v in [4] (denoted below v_{Kidder}) differs from the one adopted in LAL (and also in this document):

$$v_{Kidder} \equiv \omega r \neq v_{LAL} \equiv (M\omega)^{1/3}\,. \tag{8}$$

We have thus a simple precession equation for $\vec{L}$, but $\vec{L}_N$, or $\hat{L}_N$ is needed to construct the waveform, since $\hat{L}_N$ is the unit vector perpendicular to the instantaneous orbital plane. We can construct the $\dot{\hat{L}}_N$ by short-circuiting eq. (1) and eq. (4), to first obtain

$$
\begin{aligned}
|L_N|\dot{\hat{L}}_N &= \frac{d}{dt}\left\{\vec{L} - v^2\left[-\frac{1}{4}\left(3+\frac{M}{m_1}\right)\vec{S}_1 - \frac{1}{12}\left(1+27\frac{M}{m_1}\right)(\hat{L}_N\cdot\vec{S}_1)\hat{L}_N + 1\leftrightarrow 2\right]\right\} = \\
&= \frac{v}{\eta M}\left(1 - L_{1PN}v^2 + \ldots\right)\left\{-\dot{\vec{S}}_1 - \dot{\vec{S}}_2 - v^2\left[-\frac{1}{4}\left(3+\frac{M}{m_1}\right)\dot{\vec{S}}_1\right.\right. \\
&\qquad\left.\left. -\frac{1}{12}\left(1+27\frac{M}{m_1}\right)\frac{d\left((\hat{L}_N\cdot\vec{S}_1)\hat{L}_N\right)}{dt}\right] + 1\leftrightarrow 2\right\},
\end{aligned} \tag{9}
$$

where we averaged over an orbit and the change in L due to GW emission has been neglected. We can then see various effect here at play:

- $\vec{S}\cdot\hat{L}_N$ interaction at v^{2n-1} order in $\dot{\vec{S}}$ equations starting from $n = 2$
- spin^2 terms appearing at v^{2n} order, coded only for n=2 (leading term) in the orbit averaged version
- terms due to S contamination to L, which affect $\dot{\hat{L}}_N$ equations starting from v^7 order can be turned on by the `LALDict` structure via `XLALSimInspiralWaveformParamsInsertLscorr()`.

This is summarized in tab.I.

On the right hand side for the computation of $\dot{\vec{L}}$ we have to consider spin derivatives up to next^4 leading order v^9 whereas in the rest of the terms we can use spin derivative at next-to-next leading order and angular momentum derivatives at next-to leading order.

To conclude the implementation of precessional equation we define a precession vector

$$
\vec{\Omega}_{\hat{L}_N} \equiv \hat{L}_N \times \frac{d\hat{L}_N}{dt}, \tag{10}
$$

such that one can define and implement in LAL

$$
\frac{d\hat{L}_N^{(LAL)}}{dt} \equiv \vec{\Omega}_{\hat{L}_N} \times \hat{L}_N = \frac{d\hat{L}_N}{dt} - \left(\frac{d\hat{L}_N}{dt}\cdot\hat{L}_N\right)\hat{L}_N, \tag{11}
$$

which can be derived by aid of the identity $(\vec{A}\times\vec{B})\times C = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$. The pseudo-vector $\vec{\Omega}_{\hat{L}_N}$ is orthogonal to $\hat{L}_N$ and so it takes into account of only the genuine precession of $\hat{L}_N$.

order	L	NL	N^{2}L	N^{3}L	N^{4}L
spinO	3	4	5	6	7
v order	v^5	v^6	v^7	v^8	v^9
$\vec{S}\times\hat{L}$	✓		✓		✓
$\vec{S}^2$		$\checkmark_{avg}$		×	
$\vec{S}^3$					×
J_S			$\checkmark_{flag}$	$\checkmark_{flag}$	$\checkmark_{flag}$

TABLE I: Summary of spin precession effects implemented in the *LALSimInspiralSpinTaylor.c* code in the $\dot{S}_{1,2}$ equations.

II. SPINTAYLORT5

The SpinTaylorT5 waveform construction is explained in [5], here we just recall the basic definitions of the main orbital phase:

$$\frac{d\omega}{dt} = \frac{1}{\frac{dE(\omega)/d\omega}{dE/dt}} = \frac{96 M_c^{5/3}\omega^{11/3}}{5\,(1+2E_{1PN}-F_{1PN}\dots)}\,, \tag{12}$$

with the usual identification $v \equiv (M\omega)^{1/3}$, being M the total mass of the binary system and M_c the chirp mass and we assumed $E = -1/2\eta M(M\omega)^{2/3}(1+E_{1PN}\dots)$ and $dE/dt = 32/5\eta^2(M\omega)^{10/3}(1+F_{1PN}+\dots)$.

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