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Note on the derivation of the angular momentum and spin precessing equations in SpinTaylor codes.

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This is a technical note meant to accompany the SpinTaylor waveform review to explain how the derivative of the Newtonian angular momentum is computed. A future development would be to include in the code instantaneous spin^2 terms.

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I. TIME DERIVATIVE OF THE NEWTONIAN ANGULAR MOMENTUM UNIT VECTOR

The evolution equations for precessing spins and orbital angular momentum are obtained by imposing

$$\dot{\vec{L}} = -\dot{\vec{S}}_1 - \dot{\vec{S}}_2 \,, \tag{1}$$

i.e. imposing total angular momentum conservation and neglecting angular momentum emission by radiation, which is given by, see e.g. (4.115) of [1]

$$\frac{dL}{dt} = \frac{32}{5}\eta^2 M v^7 = \frac{32}{5M}\eta v^8 L \,, \tag{2}$$

with $\eta \equiv m_1 m_2/M^2$ the symmetric mass ratio, $m_{1,2}$ individual binary constituent masses and $M \equiv m_1 + m_2$.

At leading order $(|_L)$ in the PN expansion parameter x ($x = v^2 = (M\omega)^{2/3}$ in LAL codes, with ω the orbital phase derivative) one has, see e.g. eqs. (8-10) of [2]:

$$\begin{aligned}
\dot{\vec{S}}_1\Big|_L &= \frac{x^{5/2}}{2M}\left(3-2\frac{m_1}{M}-\frac{m_1^2}{M^2}\right)\hat{L}\times\vec{S}_1\,,\\
\dot{\hat{L}}\Big|_L &= \frac{x^3}{2M^3}\left(1+3\frac{M}{m_1}\right)\vec{S}_1\times\hat{L}+1\leftrightarrow 2\\
&= \frac{x^3 m_1^2}{2M^3}\left(1+3\frac{M}{m_1}\right)\vec{\chi}_1\times\hat{L}+1\leftrightarrow 2\\
&= \frac{x^3}{2M}\left(1+3\frac{M}{m_1}\right)\vec{\chi}_{1LAL}\times\hat{L}+1\leftrightarrow 2\,,
\end{aligned} \tag{3}$$

where $\vec{\chi}_1 \equiv \vec{S}_a/m_a^2$, and $\vec{\chi}_{aLAL} \equiv \vec{S}_a/M^2$, for $a = 1, 2$, to reflect that dimension-less spins in LAL are obtained by dividing physical ones by the total mass M squared.

At alternate PN orders spin derivatives receive contributions from spin^2 terms (x^n in the spin-dot equations) and from $L \times S$ terms ($x^{(2n+1)/2}$), with $n = 2$ for the leading order, and L precession equation can be inferred from (1).

At leading order (and up to v^2 order included with respect to the leading) we can assume that Newtonian angular momentum and $\vec{L}$ are parallel: $\hat{L}_N = \hat{L}$, however at higher order the relationship between $\vec{L}$ and $\vec{L}_N$ is actually, see eq. (4.7) of [3]

$$\begin{aligned}
\vec{L} &= \hat{L}_N|\vec{L}_N^{(0)}|\left(1+v^2L_{1PN}+v^4L_{2PN}+v^6L_{3PN}\right)+\\
&\quad \eta v^2\left\{\left[-\frac{1}{2}\left(1-\frac{M}{m_1}\right)\vec{S}_{1n}-\left(1+\frac{M}{m_1}\right)\vec{S}_{1\lambda}-\frac{5}{6}\left(1+3\frac{M}{m_1}\right)\vec{S}_{1l}\right]\right.\\
&\qquad +v^2\left[\vec{S}_{1n}\left(-\frac{11}{8}\left(1-\frac{M}{m_1}\right)+\frac{\eta}{24}\left(1-10\frac{M}{m_1}\right)\right)+\vec{S}_{1\lambda}\left(-\frac{5}{2}\left(1+\frac{M}{5m_1}\right)+\frac{\eta}{3}\left(1+4\frac{M}{m_1}\right)\right)\right.\\
&\qquad\quad \left.+\vec{S}_{1l}\left(-\frac{7}{8}\left(5+3\frac{M}{m_1}\right)+7\frac{\eta}{72}\left(1+30\frac{M}{m_1}\right)\right)\right]\\
&\qquad\quad +v^2\left[\vec{S}_{1n}\left(-\frac{61}{16}\left(1-\frac{M}{m_1}\right)+\frac{\eta}{24}\left(\frac{137}{2}-367\frac{M}{m_1}\right)+\frac{\eta^2}{48}\left(1+\frac{5}{48}\frac{M}{m_1}\right)\right)\right.\\
&\qquad\qquad +\vec{S}_{1\lambda}\left(-\frac{1}{2}\left(\frac{25}{2}+\frac{M}{m_1}\right)+\frac{\eta}{3}\left(20-\frac{79}{8}\frac{M}{m_1}\right)-\frac{2}{3}\eta^2\frac{M}{m_1}\right)\\
&\qquad\qquad \left.\left.+\vec{S}_{1l}\left(-\frac{81}{16}\left(3+\frac{M}{m_1}\right)+\frac{3}{4}\eta\left(\frac{55}{4}+39\frac{M}{m_1}\right)+\frac{\eta^2}{16}\left(1-15\frac{M}{m_1}\right)\right)\right]\right\}+1\leftrightarrow 2\,,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
|\vec{L}_N^{(0)}| &= \frac{m_1m_2}{v}\,,\\
L_{1PN} &= \frac{3}{2}+\frac{\eta}{6}\,,\\
L_{2PN} &= \frac{27}{8}-\frac{19}{8}\eta+\frac{1}{24}\eta^2\,,\\
L_{3PN} &= \frac{135}{16}+\left[-\frac{6889}{144}+\frac{41}{24}\pi^2\right]\eta+\frac{31}{24}\eta^2+\frac{7}{1296}\eta^3\,,
\end{aligned} \tag{5}$$

and we remind that

$$\vec{L}_N^{(0)} \equiv \frac{m_1 m_2}{v}\hat{L}_N \neq \frac{m_1 m_2}{\omega r}\hat{L}_N\,, \tag{6}$$

and $\vec{S}_{1n} \equiv \hat{n}(\hat{n}\cdot\vec{S}_1)$, $\vec{S}_{1\lambda} \equiv \hat{\lambda}(\hat{\lambda}\cdot\vec{S}_1)$, and $\vec{S}_{1l} \equiv \hat{L}_N(\hat{L}_N\cdot\vec{S}_1)$. Eqs. (4) is based on the following convention:

$$\begin{aligned}\dot{\hat{L}}_N &= \vec{\Omega}\times\hat{L}_N\,,\\ \dot{\hat{e}}_1 &= \vec{\Omega}\times\hat{e}_1\,,\\ \dot{\hat{e}}_2 &= \vec{\Omega}\times\hat{e}_2\,,\end{aligned} \tag{7}$$

with

$$\vec{\Omega} \equiv \hat{L}_N\times\dot{\hat{L}}_N \simeq \frac{x^3}{2M}\left(1+3\frac{M}{m_1}\right)\left[\vec{\chi}_{1LAL}-\left(\vec{\chi}_{1LAL}\cdot\hat{L}_N\right)\hat{L}_N\right]+1\leftrightarrow 2+\text{higher order terms}\,. \tag{8}$$

One can rewrite (4) as

$$\vec{L} = L_N\hat{L}_N+\eta v^2\left[c_1\vec{S}_1+c_{1n}\vec{S}_{1n}+c_{1l}\vec{S}_{1l}+1\leftrightarrow 2\right]+\ldots\,, \tag{9}$$

with $L_N \equiv m_1 m_2/v\left(1+v^2 L_{1PN}+v^4 L_{2PN}+v^6 L_{3PN}\ldots\right)$ using the identity

$$\delta^{ij} = \hat{n}^i\hat{n}^j+\hat{\lambda}^i\hat{\lambda}^j+\hat{L}^i\hat{L}^j, \tag{10}$$

i.e. express the projections of $\vec{S}_a$ in terms of $\vec{S}_a$, $\vec{S}_{an}$, and $\vec{S}_{al}$ (i.e. substitute $\vec{S}_{a\lambda}$ in terms of $\vec{S}_a, \vec{S}_{an}$ and $\vec{S}_{al}$), having defined

$$\begin{aligned}c_1 &\equiv -\left(1+\frac{M}{m_1}\right)\,,\\ c_{1n} &\equiv \frac{1}{2}\left(1+3\frac{M}{m_1}\right)\,,\\ c_{1l} &\equiv \frac{1}{2}\left(\frac{1}{3}-3\frac{M}{m_1}\right)\,.\end{aligned} \tag{11}$$

The spin corrections to the orbital angular momentum are $x^{3/2}$ order with respect to the leading contribution to $\vec{L}$, that is the Newtonian angular momentum. The velocity symbol v in [4] (denoted below v_{Kidder}) differs from the one adopted in LAL (and also in this document):

$$v_{Kidder} \equiv \omega r \neq v_{LAL} \equiv (M\omega)^{1/3}\,. \tag{12}$$

We have thus a simple precession equation for $\vec{L}$, but $\hat{L}_N$, which is the unit vector perpendicular to the instantaneous orbital plane, is needed to construct the waveform.

We can construct the $\dot{\hat{L}}_N$ by short-circuiting eq. (1) and eq. (9), to first obtain

$$\begin{aligned} \dot{L}_N \hat{L}_N + L_N \dot{\hat{L}}_N &= -\dot{\vec{S}}_1 - \eta v^2 \left[c_1 \dot{\vec{S}}_1 + c_{1n} \dot{\vec{S}}_{1n} + c_{1l} \dot{\vec{S}}_{1l} \right. \\ &\quad \left. + v^2 \left(d_1 \dot{\vec{S}}_1 + d_{1n} \dot{\vec{S}}_{1n} + d_{2l} \dot{\vec{S}}_{1l} \right) + \ldots \right] + 1 \leftrightarrow 2 \,, \end{aligned} \tag{13}$$

where we have defined

$$\begin{aligned} d_1 &\equiv -\frac{1}{2}\left(5 + \frac{M}{m_1}\right) + \frac{\eta}{3}\left(1 + 4\frac{M}{m_1}\right) , \\ d_{1n} &\equiv \frac{3}{8}\left(3 + 5\frac{M}{m_1}\right) - \frac{7\eta}{4}\left(\frac{1}{6} + \frac{M}{m_1}\right) , \\ d_{1l} &\equiv -\frac{1}{8}\left(15 + 17\frac{M}{m_1}\right) + \frac{\eta}{4}\left(-\frac{17}{18} + \frac{19}{3}\frac{M}{m_1}\right) . \end{aligned}$$

Few features worth noticing:

- $\vec{S}_a \times \hat{L}_N$ terms appear at v^{2n+1} order in $\dot{\vec{S}}_a$ equations starting from $n = 2$
- spin^2 terms in $\dot{\vec{S}}_a$ equation appear at v^{2n+2} order, starting from $n = 2$
- terms linear in S_a contaminate L, affect $\dot{\hat{L}}_N$ equations starting from v^5 order (in the instantaneous treatment), as

$$\dot{\vec{S}}_{an} = \left[\frac{v^3}{M} \left(\hat{\lambda} \cdot \vec{S}_a \right) + \hat{n} \cdot \dot{\vec{S}}_a \right] \hat{n} + \frac{v^3}{M} \left(\hat{n} \cdot \vec{S}_a \right) \hat{\lambda} \,. \tag{14}$$

  Note that $M\omega = v^3$ is an exact equation (it does not have PN corrections). The spin-depedent terms in the orbital angular momentum $\vec{L}$ start at NNNL order. They can be turned on by the `LALDict` structure via `XLALSimInspiralWaveformParamsInsertLscorr()`.

- one can decide to use instantaneous values of spin or their orbit-average values, see subsec. II and summary tab.I.

Since $\dot{\hat{L}}_N \cdot \hat{L}_N = 0$, we can split eq.(13) into a part perpendiculat to $\hat{L}_N$ and a part along it. The part parallel to $\hat{L}_N$ gives for the spin-induced variation of the modulus of L_N:

$$\dot{L}_N = -\dot{\vec{S}}_1 \cdot \hat{L}_N - \eta v^2 \left[c_1 \dot{\vec{S}}_1 \cdot \hat{L}_N + c_{1l} \left(\dot{\hat{L}}_N \cdot \vec{S}_1 + \hat{L}_N \cdot \dot{\vec{S}}_1 \right) + \ldots \right] + 1 \leftrightarrow 2 \,. \tag{15}$$

Plugging back into eq. (13) one has

$$\begin{aligned} L_N \dot{\hat{L}}_N &= -\left[\dot{\vec{S}}_1 - \left(\dot{\vec{S}}_1 \cdot \hat{L}_N \right) \hat{L}_N \right] \\ &\quad -\eta v^2 \left\{ c_1 \left[\dot{\vec{S}}_1 - \left(\dot{S}_1 \cdot \hat{L}_N \right) \hat{L}_N \right] + c_{1n} \dot{\vec{S}}_{1n} + c_{1l} \left(\hat{L}_N \cdot \vec{S}_1 \right) \dot{\hat{L}}_N + \ldots \right\} + 1 \leftrightarrow 2 \,. \end{aligned} \tag{16}$$

II. ORBITAL AVERAGING AND SPIN CORRECTIONS TO ORBITAL ANGULAR MOMENTUM

The precession equations at leading order were given in eq. (3). One may want to average over one orbit to drop terms in $\hat{n}$, $\hat{\lambda}$, and keep only $\hat{L}_N$ according to:

$$\begin{aligned} \langle n^i \rangle = \langle \lambda^i \rangle &= 0 = \langle \hat{n}^i \hat{\lambda}^j \rangle , \\ \langle \hat{n}^i \hat{n}^j \rangle = \langle \hat{\lambda}^i \hat{\lambda}^j \rangle &= \frac{1}{2}\left(\delta_{ij} - \hat{L}^i_N \hat{L}^j_N\right) + O(v^3) , \end{aligned} \tag{17}$$

where the v^3 error made in averaging over an orbit is due to the precession of the Newtonian angular momentum. Since $\dot{\hat{L}}_N \sim v^6/M$ and $T \sim M/v^3$, one has

$$\Delta \hat{L}_N = \int_0^T \dot{\hat{L}}_N dt \sim \frac{M}{v^3}\frac{v^6}{M}\langle \vec{S}_{1LAL} \times \hat{L}\rangle = v^3 \langle \vec{S}_{1LAL} \times \hat{L}\rangle, \tag{18}$$

hence bringing an error at N^{3}L order in the spins and angular momentum. This implies that if an average is taken, N^{3}L spin terms receives un-controlled contributions from the averaging, so **precessing equations in the averaged case can only be extended up to to NNL order after averaging is first taken, i.e. at NNNL order (included).**

To summarize one has at various v order for the instantaneous case (understanding summation under $1 \leftrightarrow 2$):

$$\begin{aligned} L_N \dot{\hat{L}}_N|_{v^5} &= -\dot{\vec{S}}_1|_{v^5} - \eta v^2 c_{1n}\left[\left(\dot{\hat{n}}|_{v_3} \cdot \vec{S}_1\right)\hat{n} + \left(\hat{n}\cdot \vec{S}_1\right)\dot{\hat{n}}|_{v^3}\right] , \\ L_N \dot{\hat{L}}_N|_{v^6} &= -\left[\dot{\vec{S}}_1|_{v^6} - \left(\dot{\vec{S}}_1|_{v^6}\cdot \hat{L}_N\right)\hat{L}_N\right] , \\ L_N \dot{\hat{L}}_N|_{v^7} &= -\dot{\vec{S}}_1|_{v^7} - \eta v^2\left[c_1 \dot{\vec{S}}_1|_{v^5} + c_{1n}\left(\hat{n}\cdot\dot{\vec{S}}_1|_{v^5}\right)\hat{n}\right] \\ &\qquad -\eta v^4\left\{d_{1n}\left[\left(\dot{\hat{n}}|_{v^3}\cdot \vec{S}_1\right)\hat{n} + \left(\hat{n}\cdot\vec{S}_1\right)\dot{\hat{n}}|_{v^3}\right]\right\} , \\ L_N \dot{\hat{L}}_N|_{v^8} &= -\left[\dot{\vec{S}}_1|_{v^8} - \left(\dot{\vec{S}}_1|_{v^8}\cdot\hat{L}_N\right)\hat{L}_N\right] - \eta v^2\left\{c_1\left[\dot{\vec{S}}_1|_{v^6} - \left(\dot{\vec{S}}_1|_{v^6}\cdot\hat{L}_N\right)\hat{L}_N\right]\right. \\ &\qquad \left. + c_{1n}\left[\left(\hat{n}\cdot\dot{\vec{S}}_1|_{v^6}\right)\hat{n}\right] + c_{1l}\left[\left(\hat{L}_N\cdot\vec{S}_1\right)\dot{\hat{L}}_N|_{v^6}\right]\right\} \\ L_N \dot{\hat{L}}_N|_{v^9} &= -\dot{\vec{S}}_1|_{v^9} - \eta v^2\left[c_1\dot{\vec{S}}_1|_{v^7} + c_{1n}\left(\hat{n}\cdot\dot{\vec{S}}_1|_{v^7}\right)\hat{n} + c_{1l}\left(\hat{L}_N\cdot\vec{S}_1\right)\dot{\hat{L}}_N|_{v^7}\right] \\ &\qquad -\eta v^4\left[d_1\dot{\vec{S}}_1|_{v^5} + d_{1n}\left(\hat{n}\cdot\dot{\vec{S}}_1|_{v^5}\right)\hat{n}\right] - \eta v^6 e_{1n}\left[\left(\dot{\hat{n}}|_{v_3}\cdot\vec{S}_1\right)\hat{n} + \left(\hat{n}\cdot\vec{S}_1\right)\dot{\hat{n}}|_{v^3}\right] , \end{aligned} \tag{19}$$

where it has been used that $\dot{\vec{S}}_1 \perp \hat{L}_N$ at v^5, v^7, and v^9 order and

$$e_{1n} \equiv -\frac{3}{16}\left(13 + 23\frac{M}{m_1}\right) - \eta\left(\frac{61}{16} + 12\frac{M}{m_1}\right) + \frac{\eta^2}{48}\left(1 + 37\frac{M}{m_1}\right) . \tag{20}$$

For the orbit-averaged case just drop the terms involving $\dot{\hat{n}}^i\hat{n}^j$, set c_{1n}, d_{1n}, and e_{1n} to 0, and substitute in (19) $c_1 \to c_1 + c_{1n}/2$ and $c_{1l} \to c_{1l} - c_{1n}/2$, and analogously $d_1 \to d_1 + d_{1n}/2$ and $d_{1l} \to d_{1l} - d_{1n}/2$.

order	L	NL	N^2L	N^3L	N^4L
spinO	3	4	5	6	7
v order	v^5	v^6	v^7	v^8	v^9
$\vec{S}\times\hat{L}$	✓		✓		$\checkmark_{flag-phenom}$
$\vec{S}^2$		$\checkmark_{avg}$		$\checkmark_{avg}$	
$\vec{S}^3$					×
J_S			$\checkmark_{flagls}$	$\checkmark_{flagls}$	×

TABLE I: Summary of spin precession effects implemented in the *LALSimInspiralSpinTaylor.c* code in the $\dot{S}_{1,2}$ equations.

III. SPIN EVOLUTION EQUATIONS

The spin evolution equation at NL order involve scalar product with $\hat{n},\hat{\lambda}$, see (A2) of [5]:

$$\dot{\vec{S}}_1\Big|_{NL}=\frac{v^6}{M}\left[-\vec{S}_{2LAL}+3(\hat{n}\cdot\vec{S}_{2LAL})\hat{n}+3\kappa_1\left(\frac{M}{m_1}-1\right)(\hat{n}\cdot\vec{S}_{1LAL})\hat{n}\right]\times\vec{S}_1\,, \tag{21}$$

which after orbital averaging turns into

$$\langle\dot{\vec{S}}_1\rangle\Big|_{NL}=\frac{v^6}{2M}\left[\vec{S}_{2LAL}-3\left(\hat{L}_N\cdot\vec{S}_{2LAL}\right)\hat{L}_N-3\kappa_1\left(\frac{M}{m_1}-1\right)\left(\hat{L}_N\cdot\vec{S}_{1LAL}\right)\hat{L}_N\right]\times\vec{S}_1\,.$$

A. NNL order

At NNL order we have linear in spin effects, see (7.8) of [6] and the notebook attached to this dcc:

$$\dot{S}_1\big|_{NNL}=\frac{v^7}{M}\left(\frac{9}{8}-\frac{m_1}{2M}+\frac{7m_1^2}{12M^2}-\frac{7m_1^3}{6M^3}-\frac{m_1^4}{24M^4}\right)\hat{L}_N\times\vec{S}_1\,, \tag{22}$$

and since these terms are linear in the spin, no averaging is necessary.

B. NNNL order

At N^{3}L order one has the 1PN corrections to the spin^2 corrections appearing at NL order and the instantaneous precessing equation, see (A2) of [5] is

$$\begin{aligned}\dot{\vec{S}}_1\Big|_{N^3L} = \frac{v^8}{M}\Bigg\{ & \left[-\frac{m_1^2}{2M^2} - 3\frac{M}{m_1} + \frac{7}{2} + \kappa_1\left(-3\frac{m_1^2}{2M^2} - 3\frac{m_1}{M} + 3\frac{M}{2m_1} + 3\right)\right](\hat{n}\cdot\vec{S}_1)\hat{n} \\ & + \left[-3\frac{M}{2m_1} - 3\frac{m_1}{2M} + 3 + \kappa_1\left(3\frac{M}{m_1} - 3\right)\right](\hat{\lambda}\cdot\vec{S}_1)\hat{\lambda} + \\ & + \left[-\frac{m_1}{M} - \frac{3}{2}\right]\vec{S}_2 + \left[\frac{m_1^2}{M^2} + 2\frac{m_1}{M} + \frac{3}{2}\right](\hat{n}\cdot\vec{S}_2)\hat{n} \\ & + \left[\frac{m_1}{M} + \frac{3}{2}\right](\hat{\lambda}\cdot\vec{S}_2)\hat{\lambda}\Bigg\} \times \vec{S}_1 \,. \end{aligned} \tag{23}$$

The orbit averaged version of (23) reads

$$\begin{aligned}\langle\dot{\vec{S}}_1\rangle\Big|_{N^3L} = \frac{v^8}{M}\Bigg\{ & \left[\frac{9}{4}\frac{M}{m_1} - \frac{13}{4} + \frac{3}{4}\frac{m_1}{M} + \frac{m_1^2}{4M^2} + \kappa_1\left(-\frac{9}{4}\frac{M}{m_1} + \frac{3}{2}\frac{m_1}{M} + \frac{3}{4}\frac{m_1^2}{M^2}\right)\right]\left(\hat{L}_N\cdot\vec{S}_{1LAL}\right)\hat{L}_N \\ & \left[\frac{m_1}{2M} + \frac{m_1^2}{2M^2}\right]\vec{S}_{2LAL} + \left[-\frac{3}{2} - \frac{3}{2}\frac{m_1}{M} - \frac{m_1^2}{2M^2}\right]\left(\hat{L}_N\cdot S_{2LAL}\right)\hat{L}_N\Bigg\} \times \vec{S}_1 \,. \end{aligned} \tag{24}$$

After this order on, one has to remember eq. (18) introduces an error because of the averaging process, hence only instantaneous contributions will be used.

C. N^{4}L order

The instantaneous dynamical equations at N^{4}L order can be read from eq. (4.5) of [3]

$$\dot{\vec{S}}_1\Big|_{N^4L} = \frac{v^9}{M}\left(\frac{27}{16} - \frac{51m_1}{8M} + \frac{181m_1^2}{16M^2} - \frac{23m_1^3}{6M^3} - \frac{39m_1^4}{16M^4} - \frac{3m_1^5}{8M^5} - \frac{m_1^6}{48M^6}\right)\hat{L}_N \times \vec{S}_1 \,. \tag{25}$$

IV. ACCELERATION

Now following [7] we investigate how the precessing terms modify the relationship between ω and $\dot{\phi}$, where ω is defined by the relationship

$$\vec{a}\cdot\hat{n} = -\omega^2 r \qquad \text{(definition of } \omega) \tag{26}$$

and $\dot{\phi}$ is the time derivative of the orbital phase. Let us consider a triad $(\hat{e_1}, \hat{e_2}, \hat{L}_N)$ co-rotating with $\hat{L}_N$ (but not with the binaries), that is

$$\begin{aligned}
\dot{\hat{e}}_1 &= \vec{\Omega} \times \hat{e}_1 = \left(\vec{\Omega} \cdot \hat{e}_2\right) \hat{e}_2 \times \hat{e}_1 = -\Omega_{e_2} \hat{L}_N \,, \\
\dot{\hat{e}}_2 &= \vec{\Omega} \times \hat{e}_2 = \left(\vec{\Omega} \cdot \hat{e}_1\right) \hat{e}_1 \times \hat{e}_2 = \Omega_{e_1} \hat{L}_N \,, \\
\dot{\hat{L}}_N &= \vec{\Omega} \times \hat{L}_N = \Omega_{e_2} \hat{e}_2 \times \hat{L}_N + \Omega_{e_1} \hat{e}_1 \times \hat{L}_N = \Omega_{e_2} \hat{e}_1 - \Omega_{e_1} \hat{e}_2 \,,
\end{aligned} \tag{27}$$

the notation $\Omega_X \equiv \vec{\Omega} \cdot \hat{X}$ has been adopted. For the binary system we have

$$\begin{aligned}
\vec{r} &= r\hat{n} \equiv r\left(\cos\phi\, \hat{e}_1 + \sin\phi\, \hat{e}_2\right) \,, \\
\hat{\lambda} &\equiv \hat{L}_N \times \hat{n} = \left(-\sin\phi\, \hat{e}_1 + \cos\phi\, \hat{e}_2\right) \,.
\end{aligned} \tag{28}$$

Note the useful equalities ($\dot{\phi} \neq \omega$, as we will show)

$$\begin{aligned}
e_1 &= \cos\phi\hat{n} - \sin\phi\hat{\lambda} \\
e_2 &= \sin\phi\hat{n} + \cos\phi\hat{\lambda}
\end{aligned} \tag{29}$$

$$\begin{aligned}
\dot{\hat{n}} &= \dot{\phi}\hat{\lambda} + \left[-\cos\phi\Omega_{e_2} + \sin\phi\Omega_{e_1}\right]\hat{L}_N = \dot{\phi}\hat{\lambda} - \Omega_\lambda \hat{L}_N \,, \\
\dot{\hat{\lambda}} &= -\dot{\phi}\hat{n} + \left[\sin\phi\Omega_{e_2} + \cos\phi\Omega_{e_1}\right]\hat{L}_N = -\dot{\phi}\hat{n} + \Omega_n \hat{L}_N \,, \\
\dot{\hat{L}}_N &= \Omega_\lambda \hat{n} - \Omega_n \hat{\lambda} \,.
\end{aligned} \tag{30}$$

Taking one derivative we get

$$\vec{v} \equiv \dot{\vec{r}} = \dot{r}\hat{n} + r\dot{\phi}\hat{\lambda} - r\Omega_\lambda \hat{L}_N \,, \tag{31}$$

taking another derivative

$$\begin{aligned}
\vec{a} \equiv \dot{\vec{v}} &= \ddot{r}\hat{n} + \dot{r}\dot{\phi}\hat{\lambda} - \dot{r}\Omega_\lambda \hat{L}_N + \\
&\quad (\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\lambda} - r\dot{\phi}^2\hat{n} + r\dot{\phi}\Omega_n \hat{L}_N + \\
&\quad (-\dot{r}\Omega_\lambda - r\dot{\Omega}_\lambda)\hat{L}_N - r\Omega_\lambda(\Omega_\lambda \hat{n} - \Omega_n \hat{\lambda}) \\
&= (\ddot{r} - r\dot{\phi}^2 - r\Omega_\lambda^2)\hat{n} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi} + r\Omega_n\Omega_\lambda\right)\hat{\lambda} + \\
&\quad \left[-2\dot{r}\Omega_\lambda + 2r\dot{\phi}\Omega_n - r\dot{\vec{\Omega}} \cdot \hat{\lambda}\right]\hat{L}_N \,.
\end{aligned} \tag{32}$$

Projecting the acceleration onto $\hat{n}$ and dividing by $-r$, we find ω^2 of the harmonic motion, that is (using $\hat{n} \cdot \hat{L}_N = 0 = \hat{n} \cdot \hat{\lambda}$)

$$-\frac{1}{r}\vec{a} \cdot \hat{n} = \left(\dot{\phi}^2 - \ddot{r}/r + \Omega_\lambda^2\right) \,, \tag{33}$$

which shows that $\omega \neq \dot{\phi}$ for terms of the order v^4 (because of $\delta\ddot{r}/r$, see below). Note at leading order $\dot{\phi}^2 \sim v^6/M^2$, $\Omega \sim v^6/M$ implying that the Ω_λ^2 terms represents a 3PN corrections to the phasing formula, as first pointed out in [7], i.e.

$$\dot{\phi} \simeq \omega - \frac{1}{2}\frac{\Omega_\lambda^2}{\omega} \stackrel{\text{orbit average}}{\to} \omega - \frac{1}{4}\frac{\Omega^2}{\omega}\,, \tag{34}$$

where Ω can be read in eq. (8).

To estimate $\ddot{r}$ let us re-write the equation of motion, neglecting dissipation, as

$$\begin{aligned} \vec{a} \;=\; & a_N\left(1 + a_{1PN} + a_{\hat{n}1.5PN} + a_{\hat{n}2PN} + \ldots\right)\hat{n} \\ & + a_N\left(a_{\hat{\lambda}2PN} + \ldots\right)\hat{\lambda} + a_N\left(a_{\hat{L}_N 1.5PN} + a_{\hat{L}_N 2PN} + \ldots\right)\hat{L}_N\,, \end{aligned} \tag{35}$$

where $a_N \equiv -G_N m/r^3$. Then one finds, following App. B of [8], defining

$$\vec{S}_0 \equiv \left(1 + \frac{m_2}{m_1}\right)\vec{S}_1 + \left(1 + \frac{m_1}{m_2}\right)\vec{S}_2\,, \tag{36}$$

and using the $\hat{\lambda}$ component of eq. (32)

$$\begin{aligned} 2\dot{r}\dot{\phi} + r\ddot{\phi} \;&=\; -\frac{3G_N}{Mr^4}\left(\hat{n}\cdot\vec{S}_0\right)\left(\hat{\lambda}\cdot\vec{S}_0\right)\,, \\ \Longrightarrow\quad \frac{d}{dt}\left(r^2\dot{\phi}\right) \;&=\; -\frac{3G_N}{2Mr^3\dot{\phi}}\frac{d}{dt}\left(\hat{n}\cdot\vec{S}_0\right)^2\,, \end{aligned} \tag{37}$$

which shows that in-plane components of the spin induce deviation from cicular orbits, i.e. deviations from $\ddot{\phi} = 0 = \dot{r}$, giving:

$$2\dot{\phi}\delta r + r\delta\dot{\phi} = -\frac{3G_N}{2Mr^4\dot{\phi}}\left(n\cdot S_0\right)^2 + C\,, \tag{38}$$

where C a (dimensionless) integration constant and r and $\dot{\phi}$ can be considered constant, resulting that, barring cancellation between $2\dot{\phi}\delta r$ and $r\delta\dot{\phi}$,

$$\frac{\delta\dot{\phi}}{\dot{\phi}} \sim \frac{\delta r}{r} \sim v^4\,. \tag{39}$$

Assuming that $\delta\dot{\phi}$ are fluctuations around mean value of respective quantity which average to zero over an orbit, one has

$$2\dot{\phi}\delta r + r\delta\dot{\phi} = \frac{3G_N}{4Mr^4\dot{\phi}}\left[\vec{S}_0^2 - \left(\hat{L}_N\cdot\vec{S}_0\right)^2 - 2\left(\hat{n}\cdot S_0\right)^2\right]\,. \tag{40}$$

To solve for both δr and $\delta\dot{\phi}$ one needs another equation: one can use the one from $\vec{a}\cdot\hat{n}$ which gives:

$$\delta\ddot{r} - \delta r\dot{\phi}^2 - 2r\dot{\phi}\delta\dot{\phi} = \bar{r}\bar{\dot{\phi}}^2 - \frac{G_N M}{r^2}\left(1 - \frac{2\delta r}{r}\right)\left[1 + \ldots + \frac{3G_N}{2M^2r^2}\left(\vec{S}_0^2 - 3\left(\hat{n}\cdot\vec{S}_0\right)^2\right)\right]\,, \tag{41}$$

which leads to

$$\delta\ddot{r} - 3\dot{\phi}^2\delta r - 2r\dot{\phi}\delta\dot{\phi} = -\frac{9G_N}{4Mr^4}\left[\vec{S}_0^2 - \left(\hat{L}_N\cdot S_0\right)^2 - 2\left(\hat{n}\cdot\vec{S}_0\right)^2\right]. \tag{42}$$

Now short-circuiting eqs.(40) and (42) one finds

$$\delta\ddot{r} + \dot{\phi}^2\delta r = -\frac{3G_N}{4Mr^4}\left[\vec{S}_0^2 - \left(\hat{L}_N\cdot S_0\right)^2 - 2\left(\hat{n}\cdot\vec{S}_0\right)^2\right]. \tag{43}$$

Find a particular solution of (43) and solving also for $\delta\dot{\phi}$ one obtains

$$\begin{aligned}\delta r &= \frac{G_N}{4M^2r}\left[\left(\hat{\lambda}\cdot\vec{S}_0\right) - \left(\hat{n}\cdot\vec{S}_0\right)\right],\\ \delta\dot{\phi} &= \frac{G_N\dot{\phi}}{4M^2r^2}\left[\left(\hat{\lambda}\cdot\vec{S}_0\right) - \left(\hat{n}\cdot\vec{S}_0\right)\right].\end{aligned} \tag{44}$$

This shows than in-plane components of the spins induce radial oscillations and modulation in the orbital frequency, hence circular orbit exist only on average. Solving for the orbital frequency evolution via the energy-flux balance equation

$$\dot{\omega} = \frac{dE/dt}{dE/d\omega} \tag{45}$$

is possible only averaging thos radial oscillations, which allows

$$\langle\omega^2\rangle = \langle\dot{\phi}^2 + \Omega_\lambda^2 - \frac{\ddot{r}}{r}\rangle = \bar{\dot{\phi}}^2 + \Omega_\lambda^2, \tag{46}$$

where the Ω_λ^2 term is a v^6 correction, and its presence in eq.(33) agrees with [7]. One could consider instantaneous orbital variables, but one should then solve for the complete set of equation of motions, with radiation reaction, to find the actual dynamics.

V. SPINTAYLORTX PHASING

The orbital phase differential equation can be PN expandend in different ways, according to the T1, T4 [9] or T5 [10] version. Here we just recall the basic definitions of the main orbital phase:

$$\begin{aligned}\frac{d\omega}{dt} &= \frac{dE/dt}{dE(\omega)/d\omega} = \\ &\stackrel{T1}{=} \frac{96M_c^{5/3}\omega^{11/3}}{5}\left(\frac{1+F_{1PN}+\dots}{1+2E_{1PN}+\dots}\right)\\ &\stackrel{T4}{=} \frac{96M_c^{5/3}\omega^{11/3}}{5}\left(1+F_{1PN}-2E_{1PN}\dots\right)\\ &\stackrel{T5}{=} \frac{96M_c^{5/3}\omega^{11/3}}{5}\left(\frac{1}{1+2E_{1PN}-F_{1PN}\dots}\right),\end{aligned} \tag{47}$$

with the usual identification $v \equiv (M\omega)^{1/3}$, being M the total mass of the binary system and M_c the chirp mass and we assumed $E = -(1/2)\eta M x(1 + xE_{1PN}\dots)$ and $dE/dt = (32/5)\eta^2 x^5(1 + xF_{1PN} + \dots)$ and $\dot\omega = \frac{5}{32\eta}x^{11/2}\left(1 + \dots + x^2\dot\omega_{2PN} + x^3\dot\omega_{3PNSS} + \dots\right)$. We report here for reference the the Spin^2 at 2PN order in their averaged

$$
\begin{aligned}
E_{2PNS^2avg} &= \frac{1}{2\eta}\left(\vec{S}_1\cdot S_2 - 3\hat{L}\cdot\vec{S}_1\hat{L}\cdot S_2\right) + \kappa_1\left(-\frac{3}{2}(\hat{L}_N\cdot\vec{S}_1)^2 + \frac{1}{2}\vec{S}_1^2\right) + 1\leftrightarrow 2\,,\\
F_{2PNS^2avg} &= \frac{1}{96\eta}\left(-103\vec{S}_1\cdot\vec{S}_2 + 289\hat{L}\cdot\vec{S}_1\hat{L}\cdot S_2\right)\\
&\quad + \frac{M^2}{96m_1^2}\left(7\vec{S}_1^2 - (\hat{L}\cdot\vec{S}_1)^2\right) + \kappa_1\left(\frac{M}{m_1}\right)^2\left(3(\hat{L}\cdot\vec{S}_1)^2 - \vec{S}_1^2\right) + 1\leftrightarrow 2\,,\\
\dot\omega_{2PNS^2-T4avg} &= \frac{1}{96\eta}\left(-247\vec{S}_1\cdot\vec{S}_2 + 721\hat{L}_N\cdot\vec{S}_1\hat{L}_N\cdot\vec{S}_2\right)\\
&\quad + \frac{M^2}{96m_1^2}\left(7\vec{S}_1^2 - (\hat{L}_N\cdot S_1)^2\right) + \kappa_1\left(\frac{M}{m_1}\right)^2\left(-\frac{5}{2}\vec{S}_1^2 + \frac{15}{2}(\hat{L}_N\cdot\vec{S}_1)^2\right) + 1\leftrightarrow 2\,,\\
\dot\omega_{2PNS^2-T5avg} &= \dot\omega_{2PNS^2-T4avg}\,,
\end{aligned}
\tag{48}
$$

and instantaneous form

$$
\begin{aligned}
E_{2PNS^2} &= \frac{1}{\eta}\left(3\hat{n}\cdot S_1\,\hat{n}\cdot\vec{S}_2 - \vec{S}_1\cdot\vec{S}_2\right) + \kappa_1\left(\frac{M}{m_1}\right)^2\left(3(\hat{n}\cdot S_1)^2 - \vec{S}_1^2\right) + 1\leftrightarrow 2\,,\\
F_{2PNS^2} &= \frac{1}{\eta}\left(\frac{31}{16}\vec{S}_1\cdot\vec{S}_2 - \frac{15}{2}\hat{n}\cdot\vec{S}_1\hat{n}\cdot\vec{S}_2 + \frac{71}{48}\hat{v}\cdot\vec{S}_1\hat{v}\cdot\vec{S}_2\right)\\
&\quad + \frac{M^2}{16m_1^2}\left(\vec{S}_1^2 + \frac{1}{3}(\hat{v}\cdot S_1)^2\right) + \kappa_1\left(\frac{M}{m_1}\right)^2\left(2\vec{S}_1^2 - \frac{15}{2}(\hat{n}\cdot\vec{S}_1)^2 + \frac{3}{2}(\hat{v}\cdot S_1)^2\right)1\leftrightarrow 2\,,\\
\dot\omega_{2PNS^2-T4} &= \frac{1}{\eta}\left(\frac{79}{16}\vec{S}_1\cdot\vec{S}_2 - \frac{33}{2}\hat{n}\cdot\vec{S}_1\hat{n}\cdot\vec{S}_2 + \frac{71}{48}\hat{v}\cdot\vec{S}_1\hat{v}\cdot\vec{S}_2\right)\\
&\quad + \frac{M^2}{16m_1^2}\left(\vec{S}_1^2 + \frac{1}{3}(\hat{v}\cdot S_1)^2\right) + \kappa_1\left(\frac{M}{m_1}\right)^2\left(5\vec{S}_1^2 - \frac{33}{2}(\hat{n}\cdot\vec{S}_1)^2 + \frac{3}{2}(\hat{v}\cdot S_1)^2\right) + 1\leftrightarrow 2\,,\\
\dot\omega_{2PNS^2-T5} &= -\dot\omega_{2PNS^2-T4}\,.
\end{aligned}
\tag{49}
$$

Beware, symmetric terms double their value under symmetrization!

At 3PN order in the spin alinged case one has

$$
\begin{aligned}
E_{3PNS^2-ali} &= \left[2\left(\frac{M}{m_1}\right)^2 - \frac{M}{m_1} - 1 + \kappa_1\left(-\frac{5}{2}\left(\frac{M}{m_1}\right)^2 - \frac{5}{2}\frac{M}{m_1} - \frac{5}{6}\right)\right]\vec{S}_1^2 + \\
&\left[3\left(\frac{M}{m_1}\right)^2 - \frac{2}{3}\frac{M}{m_1} - \frac{1}{9}\right](\hat{L}_N\cdot\vec{S}_1)^2 + \\
&\frac{1}{2}\left[\left(-\frac{7}{\eta} - \frac{1}{3}\right)\vec{S}_1\cdot\vec{S}_2 + \left(\frac{16}{3\eta} - \frac{2}{9}\right)\hat{L}_N\cdot S_1\hat{L}_n\cdot\vec{S}_2\right] + 1 \leftrightarrow 2\,, \\
F_{3PNS^2-ali} &= \left[-\frac{21}{8}\left(\frac{M}{m_1}\right)^2 + \frac{215}{24}\frac{M}{m_1} - \frac{1}{24} + \kappa_1\left(-\frac{279}{56}\left(\frac{M}{m_1}\right)^2 - \frac{45}{8}\frac{M}{m_1} + \frac{43}{4}\right)\right]\vec{S}_1^2 \\
&\left[-\frac{1}{2}\left(\frac{M}{m_1}\right)^2 - \frac{43}{6}\frac{M}{m_1} + \frac{22}{9}\right](\hat{L}_N\cdot\vec{S}_1)^2 + \\
&+\frac{1}{2}\left[\left(-\frac{29}{168\eta} - \frac{259}{12}\right)\vec{S}_1\cdot\vec{S}_2 + \left(-\frac{49}{6\eta} + \frac{44}{9}\right)\hat{L}_N\cdot\vec{S}_1\hat{L}_N\cdot\vec{S}_2\right] + 1 \leftrightarrow 2\,, \\
\dot{\omega}_{3PNS^2-T4ali} &= \left[-\frac{337}{32}\left(\frac{M}{m_1}\right)^2 + \frac{415}{32}\frac{M}{m_1} + \frac{379}{96} + \kappa_1\left(\frac{659}{112}\left(\frac{M}{m_1}\right)^2 - \frac{73}{24}\frac{M}{m_1} + \frac{43}{2}\right)\right]\vec{S}_1^2 \\
&\left[\frac{75}{4}\left(\frac{M}{m_1}\right)^2 + \frac{87}{4}\frac{M}{m_1} + \frac{49}{6}\right](\hat{L}_N\cdot\vec{S}_1)^2 + \\
&\frac{1}{2}\left[\left(\frac{9869}{336\eta} - \frac{1685}{48}\right)\vec{S}_1\cdot\vec{S}_2 + \left(\frac{237}{4\eta} + \frac{49}{3}\right)\hat{L}_N\cdot\vec{S}_1\hat{L}_N\cdot\vec{S}_2\right] + 1 \leftrightarrow 2\,, \\
\dot{\omega}_{3PNS^2-T5ali} &= \left[\frac{27565}{2688}\left(\frac{M}{m_1}\right)^2 - \frac{213}{16}\frac{M}{m_1} - \frac{173}{48} + \kappa_1\left(-\frac{9407}{336}\left(\frac{M}{m_1}\right)^2 - \frac{587}{24}\frac{M}{m_1} + 6\right)\right]\vec{S}_1^2 + \\
&\left(\frac{325}{16}\left(\frac{M}{m_1}\right)^2 + \frac{107}{6}\frac{M}{m_1} + \frac{67}{36}\right)(\hat{L}_N\cdot\vec{S}_1)^2 + \\
&\frac{1}{2}\left[\left(-\frac{98173}{1344\eta} - \frac{461}{24}\right)\vec{S}_1\cdot\vec{S}_2 + \left(\frac{1403}{24\eta} + \frac{67}{18}\right)\hat{L}_N\cdot\vec{S}_1\hat{L}_N\cdot\vec{S}_2\right] + 1 \leftrightarrow 2\,.
\end{aligned}
\tag{50}
$$

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