HOW BIG IS KERRY?

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Abstract. We summarise some geometric properties, and try to give an impression of the 'size', of the black hole which was formed by the merger that emitted the first ever directly detected gravitational wave signal GW150914.

1. Introduction

The source of the first ever directly detected gravitational wave signal GW150914 was a binary black hole merger event, in which two co-rotating black holes collide and merge to a single final black hole; cf [1]. The masses of the initial black holes, as well as the mass and spin of the final black hole could be extracted from the signal, which are summarised in the following table.

quantity	value
primary black hole mass	$36^{+5}_{-4} M_\odot$
secondary black hole mass	$29^{+4}_{-4} M_\odot$
final black hole mass	$62^{+4}_{-4} M_\odot$
dimensionless spin of the final black hole	$0.67^{+0.05}_{-0.07}$

The fact that for the final black hole not only the mass but also the spin could be measured, allows us to describe it as a Kerr black hole. In the following we sumarise some geometric properties, which characterise our final Kerr hole. Furthermore, we comment on ways to communicate the 'size' of it to the public. This is the first ever real rotating black hole to be described as a Kerr hole, so it deserves a name – lets call her Kerry!

Remark. This is an informal text. We do not present any novel new ideas, but rather give a collection of formulas and facts for Kerr black holes, and specifically for Kerry. The key point is to arrive at an intuitive, yet physically meaningful way to communicate an impression of the size of a Kerr black hole to the general public.

2. Some properties of a Kerr black hole

We will work in natural units, where $G = c = 1$ throughout this section.

2.1. **The Kerr metric.** In the framework of general relativity, Kerr spacetime describes a single rotating black hole in an empty universe, and its geometry is given by the Kerr metric g. In Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$, and geometric units, where $G = c = 1$, the metric can be written as

$$\begin{aligned} g = &-\left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 - \frac{4Mar\sin\theta^2}{\rho^2} \mathrm{d}\phi \mathrm{d}t \\ &+ \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin\theta^2}{\rho^2}\right)\sin\theta^2 \mathrm{d}\phi^2, \end{aligned} \tag{1}$$

Date: February 11, 2016.
Informal publication.

where we have

$$a = J/M, \quad \rho^2 = r^2 + a^2 \cos\theta^2, \quad \text{and} \quad \Delta = r^2 - 2Mr + a^2.$$

Here, M denotes the mass of the black hole, and J it's spin. The quantity $a = J/M$ is called the Kerr (spin) parameter.

2.2. **Singularities, the horizon and the ergosphere.** The metric (1) is singular iff $\rho = 0$ or $\Delta = 0$. The vanishing of ρ occurs when $r = 0$ and $\theta = \pi/2$, and represents the physical singularity of the black hole. In contrast, we have $\Delta = 0$ iff $r = r_\pm$, with

$$r_\pm = M \pm \sqrt{M^2 - a^2},$$

and these singularities are merely coordinate singularities. Never the less, especially r_+ marks an interesting surface, which can be shown to be the event horizon of the Kerr black hole – it's 'point of no return'. One might also be interested in comparing r_+ to the Schwarzschild radius $r_s = 2M$, so to the coordinate location of a horizon of a non-spinning black hole, ie a Kerr black hole in the limit $a \to 0$.

The intrinsic geometry of the horizon is given by the by g induced metric on the horizon Σ, ie by the restriction of (1) to $r = r_+$ and $\mathrm{d}r = 0$,

$$\Sigma = \Sigma_{\theta\theta}(\theta)\mathrm{d}\theta^2 + \Sigma_{\phi\phi}(\theta)\mathrm{d}\phi^2 = \rho_+(\theta)^2\mathrm{d}\theta^2 + \left(\frac{2Mr_+}{\rho_+(\theta)}\right)^2 \sin\theta^2\mathrm{d}\phi^2, \tag{2}$$

with $\rho_+(\theta)^2 = r_+^2 + a^2\cos\theta^2$.

Another interesting feature of a Kerr black hole is it's ergosphere. This is the volume between r_+ and an outer ergosphere radius $r_e(\theta)$, within which an observer with a time-like four-velocity cannot be stationary. In other words, within the ergosphere objects cannot be at rest with respect to an observer at infinity, but are forced to rotate with the black hole to some degree. The outer ergosphere boundary can be shown to be given by

$$r_e(\theta) = M + \sqrt{M^2 - a^2\cos\theta^2}.$$

3. HOW BIG IS A KERR BLACK HOLE?

We will work in natural units, where $G = c = 1$ throughout this section.

3.1. **A measure of the size.** If one attempts to give an impression of the size of a Kerr black hole, one is struggling with the yet so interesting implications of curved spacetime. For instance, one cannot simply quote a value like r_+ to give an impression of the black hole's radius, since it is a coordinate value which makes sense in the Boyer-Lindquist coordinate system only. In principle one could give the proper distance from $r = 0$ to r_+, which is an invariant quantity. However, then one is faced with the counterintuitive physics inside of the event horizon, eg with the singularity at the centre where the curvature blows up, or with the fact that g_{rr} is negative inside of the horizon, implying that the radial direction is actually time-like there. For these reasons, one can hardly expect that this proper distance would help us at all to picture the black hole's size in our mind.

The truth is of course, that there is no measure of length, which is both, a true physical distance in spacetime and an intuitive measure of the black hole's size, and from the scientific perspective we simply have to deal with it.

If we now still decided to deal with it, for instance for the sake of communicating science to the public, the best thing we could do is to look for an invariant geometric quantity, which is both graspable by our imagination and which can be related to an intuitive notion of a radius. The best choice for this appears to be the surface area of the Kerr horizon A, which is given by an integration over the volume element of Σ, ie by

$$A = \int_0^{2\pi} \int_0^{\pi} \sqrt{\Sigma_{\theta\theta}(\theta)\Sigma_{\phi\phi}(\theta)}\, \mathrm{d}\theta \mathrm{d}\phi = 4\pi(r_+^2 + a^2).$$

We can now use our intuitive notion of the radius of a sphere in Euclidean space which yields the same surface area as that of the black hole horizon. This quantity is called the areal radius r_A of the Kerr hole, and is given by

$$r_A = \sqrt{\frac{A}{4\pi}} = \sqrt{r_+^2 + a^2}.$$

3.2. **A measure of the oblaticity.** Even though the Kerr horizon is located at a constant Boyer-Lindquist radius $r = r_+$ its geometry is not spherical but rather oblate. To convince oneself one could for example calculate the Riemann curvature of (2), and see that it changes with the angles. Here however we are again interested in measures of lengths, which allow for an intuitive notion of how oblate a given Kerr black hole is.

What we can do for example is to compare the circumferences of the horizon measured at the equator and through the poles. These are invariant geometric quantities, just as the area. To calculate the circumference at the equator C_{eq}, we restrict (2) to $\theta = \pi/2, \mathrm{d}\theta = 0$, and perform a volume integration over the resulting induced metric, ie the line integration

$$C_{eq} = \int_0^{2\pi} \sqrt{\Sigma_{\phi\phi}(\pi/2)}\, \mathrm{d}\phi = 4\pi M.$$

In the same way, taking $\phi = 0, \mathrm{d}\phi = 0$ we yield

$$C_{po} = 2\int_0^{\pi} \sqrt{\Sigma_{\theta\theta}(\theta)}\, \mathrm{d}\theta = 2\int_0^{\pi} \sqrt{r_+^2 + a^2 \cos\theta^2}\, \mathrm{d}\theta,$$

for the circumference C_{po} through the poles, which is an elliptic integral and hence can only be evaluated numerically.

We could now relate these circumferences to radii in the same way as we related the area A to r_A. For instance we could relate C_{eq} to the radius of a circle in the Euclidean plane, while we could work with an ellipse in the case of C_{po}, ie

$$C_{eq} = 2\pi r_{eq} \quad \text{and} \quad C_{po} \approx 2\pi\sqrt{\frac{r_{eq}^2 + r_{po}^2}{2}}.$$

So r_{eq} and r_{po} might be usefull to communicate an impression of how oblate a Kerr black hole is due to its spin.

3.3. **Remarks.** Let us end this section by reminding ourselves that the true physical quantities discussed here, are the area of the horizon A, and it's circumferences C_{eq} and C_{po}. The radii r_A, r_{eq} and r_{po} were defined using analogues of Euclidean geometry, so they have no clear physical meaning by their self. Never the less, they might be helpful in giving an impression of the 'size' of Kerr black hole, which can be communicated to the general public.

4. Kerry

4.1. **Conversion of geometric to SI units.** So that we can calculate the quantities discussed in the previous section for our Kerr black hole Kerry in SI units, we need to introduce the appropriate factors of G and c into our formulas:

$$\begin{aligned}
J &= aMG/c,\\
r_\pm &= \left(M \pm \sqrt{M^2 - a^2}\right) G/c^2,\\
r_s &= 2MG/c^2,\\
r_e(\theta) &= \left(M + \sqrt{M^2 - a^2\cos\theta^2}\right) G/c^2,\\
A &= 4\pi\left(r_+^2 + \left(aG/c^2\right)^2\right),\\
r_A &= \sqrt{r_+^2 + (aG/c^2)^2},\\
C_{eq} &= 4\pi MG/c^2,\\
C_{po} &= 2\int_0^\pi \sqrt{r_+^2 + (aG/c^2)^2\cos\theta^2}\,\mathrm{d}\theta,\\
r_{eq} &= C_{eq}/2\pi = 2MG/c^2,\\
r_{po} &= C_{po}/2\pi = \frac{1}{\pi}\int_0^\pi \sqrt{r_+^2 + (aG/c^2)^2\cos\theta^2}\,\mathrm{d}\theta.
\end{aligned}$$

4.2. **Values for Kerry.** Let us finally write down the values of the discussed quantities for the final Kerr black hole quoted in section 1. We do not keep track of the experimental errors here. Thus we have

$$M = 62M_\odot \quad \text{and} \quad a = 0.67M,$$

as our mass and spin parameters. Entering now the values for the solar mass $M_\odot$, the gravitational constant G and the speed of light c in SI units, we obtain the values summarised in the following table:

quantity	SI value	description
physical quantities of Kerry		
M	$1.23\cdot 10^{32}kg$	mass
J	$2.27\cdot 10^{45}kg\,m^2/s$	spin
r_+	$1.60\cdot 10^5 m$	coordinate radius of the horizon
$r_e(\pi/2)$	$1.83\cdot 10^5 m$	coordinate radius of the ergosphere at the equator
$r_e(0)$	$1.60\cdot 10^5 m$	coordinate radius of the ergosphere at the poles
A	$3.67\cdot 10^{11} m^2$	are of the event horizon
C_{eq}	$1.15\cdot 10^6 m$	circumference of the horizon at the equator
C_{po}	$1.04\cdot 10^6 m$	circumference of the horizon through the poles
C_{po}/C_{eq}	0.90	
Euclidean analogues to give an impression of the 'size' of Kerry		
r_A	$1.71\cdot 10^5 m$	radius of a Euclidean sphere with area A
r_{eq}	$1.83\cdot 10^5 m$	radius of a Euclidean circle with circumference C_{eq}
r_{po}	$1.45\cdot 10^5 m$	small semi-major axis of a Euclidean ellipse with circumference $\approx C_{po}$
r_{po}/r_{eq}	0.79	
comparison to a non-spinning black hole of mass M		
r_s	$1.83\cdot 10^5 m$	Schwarzschild radius

REFERENCES

[1] LIGO Scientific Collaboration & Virgo Collaboration (2016), PRL 116 061102. Link: http://link.aps.org/doi/10.1103/PhysRevLett.116.061102

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