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# Predicted $C_\ell$ components for stochastic O2 directional paper

There are three conventions for the $C_\ell$'s that need to be compared:

1. The convention used by the LVC in the O1 directional paper (and presumably also the O2 paper), which we will denote $C_\ell^{(\mathrm{LVC})}$.
2. The convention used in Jenkins & Sakellariadou, arXiv:1802.06046, and subsequently in Jenkins et al, arXiv:1806.01718, which we will denote $C_\ell^{(\mathrm{JS})}$.
3. The convention used in Cusin, Dvorkin, Pitrou & Uzan, arXiv:1803.03236, which we will denote $C_\ell^{(\mathrm{CDPU})}$.

In the O1 directional paper, the stochastic energy density as a function of frequency and sky direction is written as

$$\Omega_{\mathrm{gw}}(f,\Theta)=\frac{2\pi^2}{3H_0^2}f^3H(f)\mathcal{P}(\Theta),\qquad H(f)=\left(\frac{f}{f_{\mathrm{ref}}}\right)^{\alpha-3},\tag{1}$$

where it is assumed that the dependencies on frequency and sky direction can be factored out of each other. More generally, one would write

$$\Omega_{\mathrm{gw}}(f,\Theta)=\frac{2\pi^2}{3H_0^2}f^3\tilde{\mathcal{P}}(f,\Theta).\tag{2}$$

This corresponds to eq. (50) in Jenkins & Sakellariadou, arXiv:1802.06046 (although we have added a tilde to indicate the different definition of $\mathcal{P}$). Assuming the power spectrum can be factorised in this way, the $C_\ell$'s in the O1 paper are then defined by

$$C_\ell^{(\mathrm{LVC})}=\left(\frac{2\pi^2f_{\mathrm{ref}}^3}{3H_0^2}\right)^2\frac{1}{2\ell+1}\sum_{m=-\ell}^{+\ell}\langle\mathcal{P}_{\ell m}\mathcal{P}_{\ell m}^*\rangle,\qquad \mathcal{P}_{\ell m}=\int\mathrm{d}\Theta\,Y_{\ell m}(\Theta)\mathcal{P}(\Theta).\tag{3}$$

(Actually this equation is not written explicitly anywhere in the O1 paper, although it is implied from the limits quoted in the abstract, and the fact that the $C_\ell^{1/2}$'s in fig. 3 are in units of $\Omega_{\mathrm{gw}}\,\mathrm{sr}^{-1}$ rather than $\Omega_{\mathrm{gw}}\,\mathrm{sr}^{-1}\,\mathrm{Hz}^{-1}$. This is slightly confusing given the definition of the estimator in eq. (19) of the O1 paper, which suggests that the $\left(\frac{2\pi^2f_{\mathrm{ref}}^3}{3H_0^2}\right)^2$ pre-factor is not included. I'd like to gently suggest that this could be clarified somewhere in the O2 paper.)

## Predictions for $\alpha=2/3$ from Jenkins et al, arXiv:1806.01718

The $C_\ell$'s in this paper are defined with respect to the spherical harmonic components of the dimensionless GW *overdensity*, which is the fractional deviation from the average isotropic energy density per solid angle $\bar{\Omega}_{\mathrm{gw}}$,

$$\delta_{\mathrm{gw}}^{(\mathrm{s})}(f,\Theta)=\frac{\Omega_{\mathrm{gw}}(f,\Theta)-\bar{\Omega}_{\mathrm{gw}}(f)}{\bar{\Omega}_{\mathrm{gw}}(f)},\qquad \bar{\Omega}_{\mathrm{gw}}(f)=\frac{1}{4\pi}\int\mathrm{d}\Theta\,\Omega_{\mathrm{gw}}(f,\Theta),\tag{4}$$

$$C_\ell^{(\mathrm{JS})}(f)=\frac{1}{2\ell+1}\left\langle\omega_{\ell m}^{(\mathrm{s})}(f)\omega_{\ell m}^{(\mathrm{s})*}(f)\right\rangle,\qquad \omega_{\ell m}^{(\mathrm{s})}(f)=\int\mathrm{d}\Theta\,Y_{\ell m}(\Theta)\delta_{\mathrm{gw}}^{(\mathrm{s})}(f,\Theta).\tag{5}$$

Note that $\delta_{\mathrm{gw}}^{(\mathrm{s})}$ is defined such that the kinematic dipole has been subtracted (hence the superscript 's' for 'source', as we have removed the observer-dependent dipole). The conversion between this convention and the LVC convention is easy using an expression already given in arXiv:1802.06046, in eq. (54),

$$C_\ell^{(\mathrm{JS})}(f)=\frac{16\pi^2}{2\ell+1}\left[\sum_{m=-\ell}^{+\ell}\frac{\left\langle\tilde{\mathcal{P}}_{\ell m}\tilde{\mathcal{P}}_{\ell m}^*\right\rangle}{S_h^2}\right]+\left[4\pi-16\pi^{3/2}\frac{\left\langle\tilde{\mathcal{P}}_{00}\right\rangle}{S_h}\right]\delta_{\ell 0}+\left[\frac{4\pi}{9}\mathcal{D}^2-16\left(\frac{\pi}{3}\right)^{3/2}\mathcal{D}\frac{\left\langle\tilde{\mathcal{P}}_{10}\right\rangle}{S_h}\right]\delta_{\ell 1}\tag{6}$$

where we haved defined the quantity

$$S_h(f)=\int\mathrm{d}\Theta\,\tilde{\mathcal{P}}(f,\Theta)=\frac{6H_0^2}{\pi f^3}\bar{\Omega}_{\mathrm{gw}}(f,\Theta),\tag{7}$$

which can also be calculated from the model of the background. Assuming that the factorisation assumption is accurate enough, then we can use eqs. (1) and (2) of this note to write $\tilde{\mathcal{P}} = H\mathcal{P}$ and therefore

$$C_\ell^{(\mathrm{JS})}(f) = \frac{H^2(f)}{\bar{\Omega}_{\mathrm{gw}}^2(f)}\left(\frac{f}{f_{\mathrm{ref}}}\right)^6 C_\ell^{(\mathrm{LVC})} + \left[4\pi - 16\pi^{3/2}H(f)\frac{\langle\mathcal{P}_{00}\rangle}{S_h(f)}\right]\delta_{\ell 0} + \left[\frac{4\pi}{9}\mathcal{D}^2 - 16\left(\frac{\pi}{3}\right)^{3/2}H(f)\mathcal{D}\frac{\langle\mathcal{P}_{10}\rangle}{S_h(f)}\right]\delta_{\ell 1}, \quad (8)$$

All we need then are the values for $\langle\mathcal{P}_{00}\rangle$ and $\langle\mathcal{P}_{10}\rangle$. Since $Y_{00} = 1/\sqrt{4\pi}$, the former is just

$$\langle\mathcal{P}_{00}\rangle = \int \mathrm{d}\Theta\, Y_{00}\mathcal{P} = \frac{S_h(f)}{\sqrt{4\pi}H(f)}. \quad (9)$$

This can also be found by setting $\ell = 0$, $m = 0$ in eq. (53) of the paper, accounting for the extra factor of $H(f)$ we have introduced. Using eq. (53) for the $\ell = 1$, $m = 0$ case we find

$$\langle\mathcal{P}_{10}\rangle = \frac{S_h(f)}{\sqrt{12\pi}H(f)}\mathcal{D} + \frac{S_h(f)}{4\pi H(f)}\left\langle\omega_{10}^{(\mathrm{s})}\right\rangle. \quad (10)$$

Here $\mathcal{D}$ is a frequency-dependent dimensionless coëfficient that we calculate as part of our model, which describes the size of the kinematic dipole. The term $\left\langle\omega_{10}^{(\mathrm{s})}\right\rangle$ is the average dipole due to the sources (i.e. non-kinematic) projected onto the direction of the kinematic dipole, which we take as zero.

Putting this all together, we have

$$C_\ell^{(\mathrm{LVC})} = \frac{\bar{\Omega}_{\mathrm{gw}}^2(f)}{H^2(f)}\left(\frac{f_{\mathrm{ref}}}{f}\right)^6\left[C_\ell^{(\mathrm{JS})}(f) + 4\pi\delta_{\ell 0} + \frac{4\pi}{9}\mathcal{D}^2(f)\delta_{\ell 1}\right]. \quad (11)$$

So given predicted values for $\bar{\Omega}_{\mathrm{gw}}$, $\mathcal{D}$ and $C_\ell^{(\mathrm{JS})}$ up to $\ell = \ell_{\max}$, we can give the appropriate predictions. All these quantities are calculated in our paper. (We find that for the astrophysical background, the kinematic dipole is small enough to be neglected, since the dominant sources are at low redshifts — however, it will be important for the cosmic string case.)

Note that eq. (11) above was derived using the factorisation assumption $\tilde{\mathcal{P}} = H\mathcal{P}$. The LHS is frequency-independent, so we can use this to check the assumption and see if it's a good enough approximation. This is done in fig. 1 below. We see that the approximation is very good at low frequencies, but starts to break down beyond $\approx 100\,\mathrm{Hz}$. Looking at the $\ell = 0$ curve, we can see this is due to the fact that the monopole starts to deviate from a $\propto f^{2/3}$ power law at these frequencies, and that this is not due to any problem with the factorisation itself. (Question for people more knowledgeable about the SHD analysis pipeline: how sensitive is this search to these high frequencies anyway?)

I therefore see three possible options (though I may have missed some):

1. We evaluate eq. (11) at a single frequency, say the reference frequency $f_{\mathrm{ref}}$. This is my preference as it is the simplest conceptually, the easiest to do in practice, and judging by fig. 1, it shouldn't introduce any considerable errors.

2. We modify the form of $H(f)$ to account for the drop-off at high frequencies, but leave the 'factorisation approximation' in place.

3. We average eq. (11) over frequency, somehow weighting according to the noise PSD.

Assuming we adopt approach 1, here are the values calculated in this paper:

$$\begin{aligned}
\sqrt{C_0^{(\mathrm{LVC})}} &= 9.202 \times 10^{-10}\,\mathrm{sr}^{-1}, \\
\sqrt{C_1^{(\mathrm{LVC})}} &= 1.881 \times 10^{-11}\,\mathrm{sr}^{-1}, \\
\sqrt{C_2^{(\mathrm{LVC})}} &= 1.718 \times 10^{-11}\,\mathrm{sr}^{-1}, \\
\sqrt{C_3^{(\mathrm{LVC})}} &= 1.626 \times 10^{-11}\,\mathrm{sr}^{-1}, \\
\sqrt{C_4^{(\mathrm{LVC})}} &= 1.559 \times 10^{-11}\,\mathrm{sr}^{-1}.
\end{aligned} \quad (12)$$

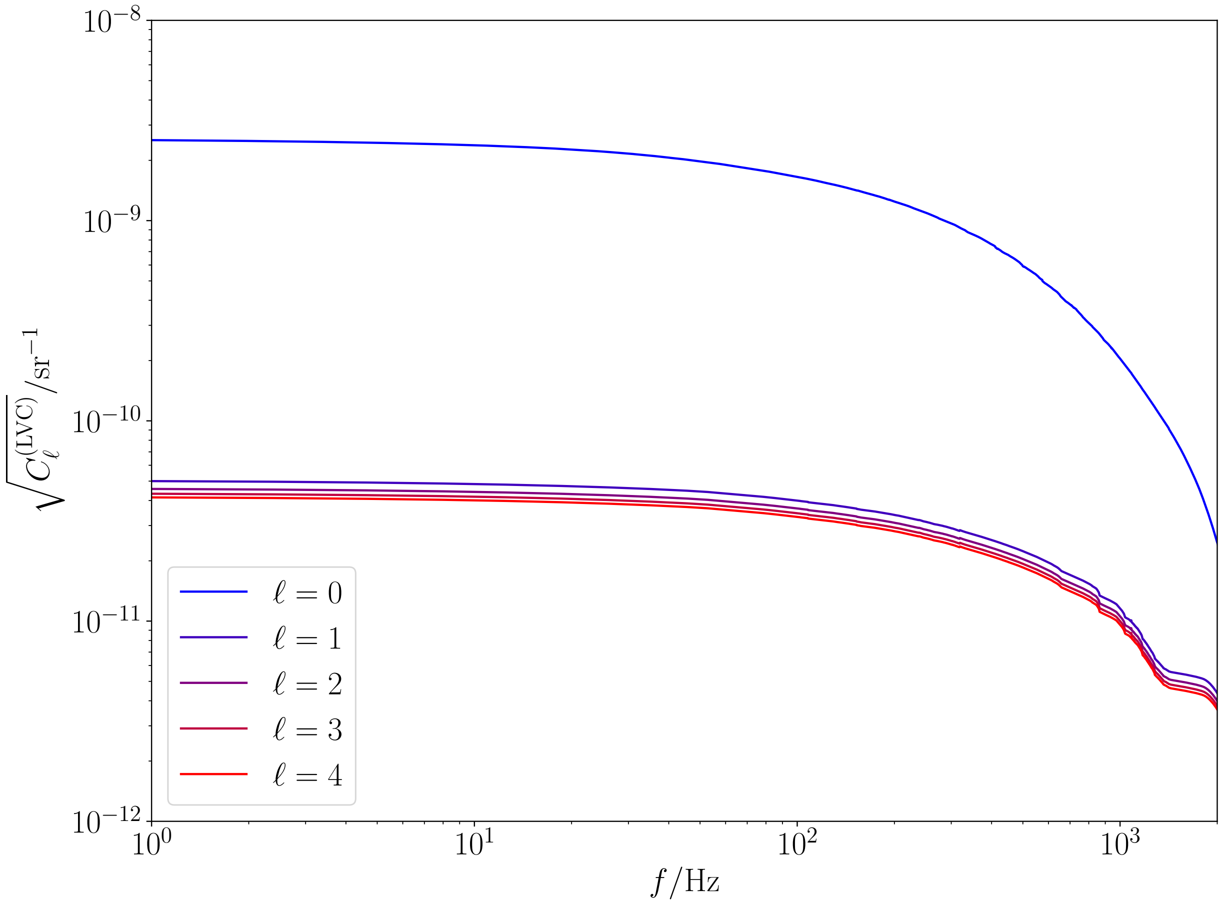


Figure 1: Scaling of the $C_\ell$'s from Jenkins et al, arXiv:1806.01718 — converted to the LVC convention — with frequency. Note that this is for a different case than in eq. (12), hence the different numerical values. The basic point about scaling with frequency still remains.

## Predictions for $\alpha = 2/3$ from Cusin et al, arXiv:1803.03236

Cusin et al define their $C_\ell$'s in terms of $\delta\Omega_{\mathrm{gw}}(f,\Theta) = \Omega_{\mathrm{gw}}(f,\Theta) - \bar{\Omega}_{\mathrm{gw}}(f)$, i.e. they subtract the monopole (as in Jenkins et al), but do not normalise with respect to the monopole (unlike Jenkins et al). (Note that the $\bar{\Omega}_{\mathrm{gw}}$ defined in their paper differs from the one used in this note by a factor of $4\pi$, see e.g. their eq. (2).) It is therefore simple to relate it to the other conventions,

$$C_\ell^{(\mathrm{CDPU})} = \bar{\Omega}^2_{\mathrm{gw}}(f) C_\ell^{(\mathrm{JS})}, \tag{13}$$

$$C_\ell^{(\mathrm{LVC})} = \frac{1}{H^2(f)} \left(\frac{f}{f_{\mathrm{ref}}}\right)^6 \left[C_\ell^{(\mathrm{CDPU})} + 4\pi \bar{\Omega}^2_{\mathrm{gw}}(f)\delta_{\ell 0}\right]. \tag{14}$$

The code used to calculate the predicted $C_\ell$'s in the Cusin et al paper is implemented as part of the Mathematica package `CMBquick` (freely available at `http://www2.iap.fr/users/pitrou/cmbquick.htm`). Evaluating at the reference frequency 25 Hz as before, we find the following predictions:

$$\begin{aligned}
\sqrt{C_0^{(\mathrm{LVC})}} &= 4.678 \times 10^{-10}\,\mathrm{sr}^{-1}, \\
\sqrt{C_1^{(\mathrm{LVC})}} &= 1.459 \times 10^{-12}\,\mathrm{sr}^{-1}, \\
\sqrt{C_2^{(\mathrm{LVC})}} &= 1.244 \times 10^{-12}\,\mathrm{sr}^{-1}, \\
\sqrt{C_3^{(\mathrm{LVC})}} &= 1.042 \times 10^{-12}\,\mathrm{sr}^{-1}, \\
\sqrt{C_4^{(\mathrm{LVC})}} &= 8.449 \times 10^{-13}\,\mathrm{sr}^{-1}.
\end{aligned} \tag{15}$$

This includes the optional non-linear correction to the matter power spectrum using Halofit. I've saved this calculation as a Mathematica notebook that I can share if it would be helpful.

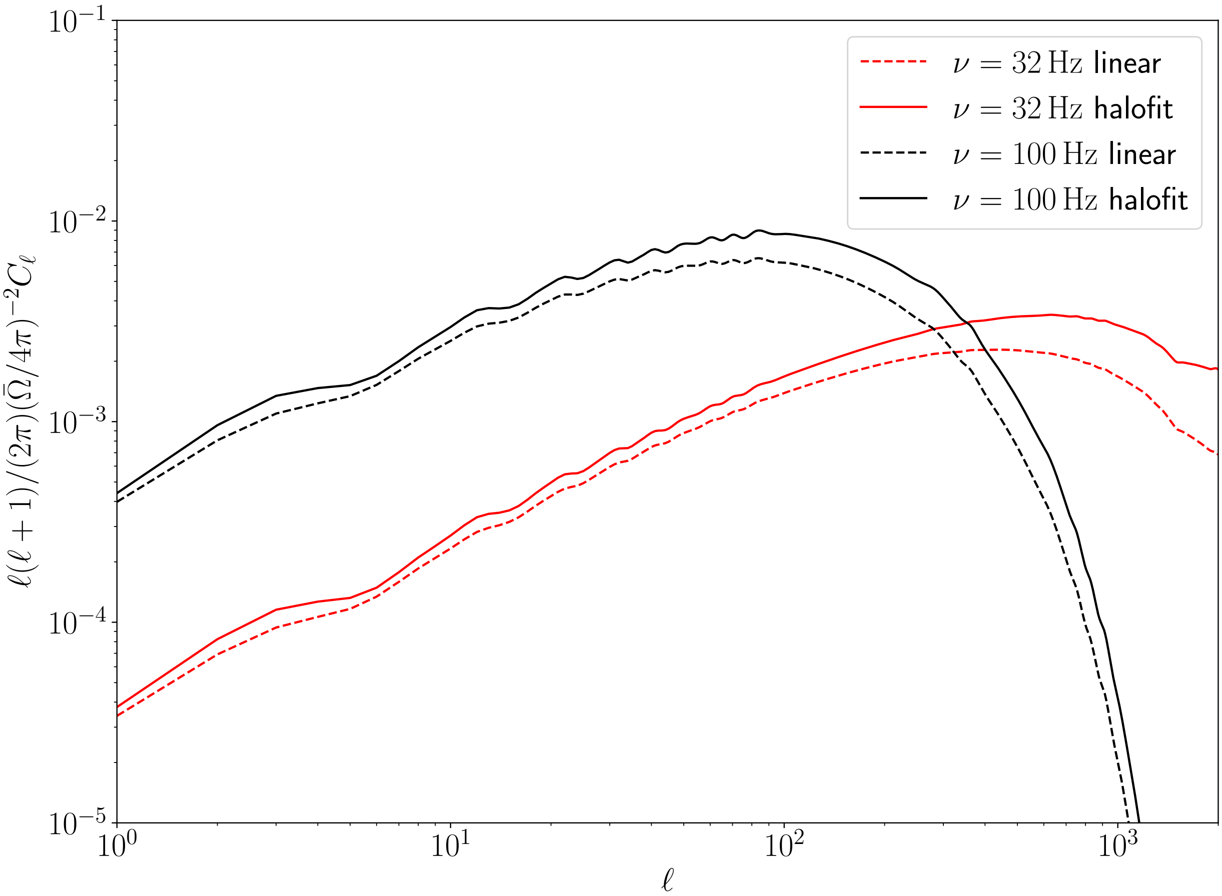


Figure 2: Output of `CMBquick`, corresponding to fig. 2 in arXiv:1803.03236. These calculations are also saved as a Mathematica notebook that I can share if it would be useful.

I've also tried to check whether these values are representative of the predictions in the Cusin et al paper by using `CMBquick` to reproduce fig. 2 of that paper. The results are shown in fig. 2 here. The curves here fall off a bit faster at $\ell > 100$ than in the figure in the paper, so it looks like the choice of parameters is possibly slightly different. However, the curves for $\ell \leq 4$ (which is what we need for the O2 directional paper) look very similar to the ones in their paper, by eye. We could also try to check the values in eq. (15) by reading off the $C_\ell$'s by eye from Cusin et al's fig. 2 and using eq. (14) to convert them to the LVC convention.

## Predictions for $\alpha = 0$ from Jenkins & Sakellariadou, arXiv:1802.06046

The anisotropic background from cosmic strings can also be predicted using the results of this paper. Here, the frequency spectrum of $\bar{\Omega}_{\mathrm{gw}}(f)$ is flat at high enough frequencies, so $\alpha = 0$ is the appropriate comparison.

In order to generate predictions for the $C_\ell$'s, we need the value of the string tension $G\mu$, and the model used for the loop network (i.e. model 1, 2, or 3 from the O1 cosmic string paper, arXiv:1712.01168). Taking $G\mu = 10^{-12}$ and model 3 as an example, we find

$$\begin{aligned} \sqrt{C_0^{(\mathrm{LVC})}} &= 1.258 \times 10^{-7}\,\mathrm{sr}^{-1}, \\ \sqrt{C_1^{(\mathrm{LVC})}} &= 2.138 \times 10^{-10}\,\mathrm{sr}^{-1}, \\ \sqrt{C_2^{(\mathrm{LVC})}} &= 1.556 \times 10^{-21}\,\mathrm{sr}^{-1}, \\ \sqrt{C_3^{(\mathrm{LVC})}} &= 1.554 \times 10^{-21}\,\mathrm{sr}^{-1}. \end{aligned} \tag{16}$$

Note that this now includes the kinematic dipole.

## Some questions

1. How should we deal with the question about factorisation of the power spectrum? One of the approaches 1–3 suggested on page 2 of this note, or something else?

2. Do we want to include predictions for cosmic strings? The resulting constraints are, unfortunately, not going to be competitive with the isotropic analysis (as you can see above), but would it be worth mentioning them anyway?

3. Will the values for $f_{\mathrm{ref}}$ and $\ell_{\mathrm{max}}$ be the same as the O1 paper?

4. What values should we use for the cosmological parameters? These can be fed into the codes for all three sets of predictions.