Review of NCAL Analytic Force Estimates, Mathematical Details

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This document serves as a place for the step-by-step gory detail behind the derivations of NCAL / GCAL mathematical equations. GIT ID Hash: 5c55c20a

I. SUMMARY

In this document, we extend standard, 1-dimensional, analytical equations of the expected amplitude of time-dependent forces produced by mass quadrupole and mass hexapole systems on the center of mass of a nearby test mass to 3 dimensions. The analytical expression described in this document is approximate. This is mostly due to the treatment of all masses involved as point masses, rather than the true extended objects they are. However, this method offers an independent and quick way to check for consistency with other methods that capture the full geometry of the system. The bulk of the contributions to the final uncertainty in the force estimate come from parameters that are included in the point mass analytical expression, making it useful for quickly computing the effect of parameter uncertainties (minutes vs. hours to days for the full 3D models).

The quadrupole equations are derived in a similar manner to the method described by the Virgo analytical model [1]. We extended the model to a 3D hexapole configuration and simplified the equations.

In Section II, we show the (re)derivations of the forces from a collection of masses, $m_1 = m_2 = m$ arranged in a mass quadrupole configuration at radius r_q, and another collection of masses $m_3 = m_4 = m_5 = m$ arranged in a mass hexapole configuration at radius r_h, rotating at an absolute rotation angle θ of those systems, about a center of rotation that is a distance d away from a test mass mass M, assuming the center of rotation is positioned in-plane and along the detector's arm length axis, in the $\hat{x}$ direction, to arrive at

$$F_{\hat{x}}^{(q)}(\theta) = \frac{9}{2}\frac{GMm}{d^2}\left(\frac{r_q}{d}\right)^2 \cos(2\theta) \tag{1}$$

$$F_{\hat{x}}^{(h)}(\theta) = \frac{15}{2}\frac{GMm}{d^2}\left(\frac{r_h}{d}\right)^3 \cos(3\theta) \tag{2}$$

In Section III, we extend the derivation assuming the center of rotation is now off-axis by an angle Φ in the $\hat{y}$ direction, and slightly out of plane by a distance z in the $\hat{z}$ direction (assuming a right-handed coordinate system). Within the approximation, we assume a cylindrical coordinate system, with radial distance $\rho = \sqrt{(x^2 + y^2)}$, angle between the x-direction and the center of rotation, $\Phi = \arctan(y/x)$, and $z = z$, but with $z/d \ll 1$, such that $\rho \approx d$, and the quantity $\zeta \equiv 1 + (z/d)^2 \approx 1$. This extension yields the following results:

$$\begin{aligned} F_{\hat{x}}^{(q)}(\theta) = \frac{9}{2}\frac{GMm}{d^2}\left(\frac{r_q}{d}\right)^2 & \\ \times\ \zeta^{-5/2}\Bigg[\left(\frac{5}{6}\frac{1}{\zeta} - \frac{2}{3}\right)\cos(2\theta - \Phi) & \\ +\ \frac{5}{6}\frac{1}{\zeta}\cos(2\theta - 3\Phi)\Bigg] & \end{aligned} \tag{3}$$

$$\begin{aligned} F_{\hat{x}}^{(h)}(\theta) = \frac{15}{2}\frac{GMm}{d^2}\left(\frac{r_h}{d}\right)^3 & \\ \times\ \zeta^{-7/2}\Bigg[\left(\frac{7}{8}\frac{1}{\zeta} - \frac{3}{4}\right)\cos(3\theta - 2\Phi) & \\ +\ \frac{7}{8}\frac{1}{\zeta}\cos(3\theta - 4\Phi)\Bigg] & \end{aligned} \tag{4}$$

II. IN PLANE, ON-AXIS, 1D

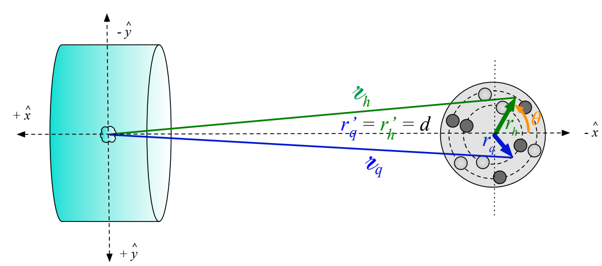

FIG. 1. Top Down View of NCAL-Test-mass system, with the center of rotation positioned on the x axis, in the x-y plane for demonstrative 1D case.

A. 1D QUADRUPOLE

In this quadrupole configuration there are two masses, m_1 and m_2, arranged opposite each other at radius a r_q from the center of the rotor. With respect to the test mass' center of mass, of mass M, we take m_1 to be a distance $\vec{\mathscr{r}}_q^{(1)}$ away with components:

$$\mathscr{r}_x^{(q,1)} = \left[d + r_q \cos\theta\right]\ \hat{x} \tag{5}$$

$$\mathscr{r}_y^{(q,1)} = \left[+ r_q \sin\theta\right]\ \hat{y} \tag{6}$$

where d is the distance between the center of rotor rotation and the center of mass of the test mass, along the $\hat{x}$ direction, and θ is the anglular position of m_1 around the rotor, with $\theta \equiv 0$ defined as when the m_1 passes through the x- or d-axis, in the position closer to the test mass.

Pythagoras demands the magnitude squared of the vector between m_1 and the test mass center of mass, $\vec{r}_q^2$, is

$$
\begin{aligned}
|\vec{r}_q^{(1)}|^2 &= (d + r_q\cos\theta)^2 + (r_q \sin\theta)^2 \\
&= d^2 + 2dr_q\cos\theta + r_q^2(\cos^2\theta + \sin^2\theta) \\
|\vec{r}_q^{(1)}|^2 &= d^2 + r_q^2 + 2dr_q\cos\theta
\end{aligned} \tag{7}
$$

and if we introduce the small, dimensionless parameter $\epsilon \equiv r_q/d \ll 1$, then,

$$
\left|\vec{r}_q^{(1)}\right|^2 = d^2\left(1 + \epsilon^2 + 2\epsilon\cos\theta\right) \tag{8}
$$

and the magnitude of the vector itself is

$$
|\vec{r}_q^{(1)}| = d\left(1 + \epsilon^2 + 2\epsilon\cos\theta\right)^{1/2} \tag{9}
$$

The total force on the test mass center of mass, of mass M, from the first mass, m_1 is:

$$
F_{tot}^{(q,1)} = \frac{GMm_1}{\left|\vec{r}_q^{(1)}\right|^2} = \frac{GMm_1}{d^2(1 + \epsilon^2 + 2\epsilon\cos\theta)} \tag{10}
$$

As the NCAL rotates, we consider only the projection of the total force on the test mass in the $\hat{x}$ direction for any given θ, because this is the direction the detector can measure. As such, we take the projection of $\vec{r}_q^{(1)}$ on to the $\hat{x}$ axis:

$$
\begin{aligned}
F_{\hat{x}}^{(q,1)} &= F_{tot}^{(q,1)}|\vec{r}_x^{(q,1)}|/|\vec{r}_q^{(1)}| \\
&= F_{tot}^{(q,1)}\frac{d + r_q\cos\theta}{d\left(1 + \epsilon^2 + 2\epsilon\cos\theta\right)^{1/2}} \\
&\quad (\epsilon = r_q/d) \\
&= F_{tot}^{(q,1)}\ \frac{1 + \epsilon\cos\theta}{(1 + \epsilon^2 + 2\epsilon\cos\theta)^{1/2}} \\
&= \left(\frac{GMm_1}{d^2(1 + \epsilon^2 + 2\epsilon\cos\theta)}\right)\frac{1 + \epsilon\cos\theta}{(1 + \epsilon^2 + 2\epsilon\cos\theta)^{1/2}} \\
F_{\hat{x}}^{(q,1)} &= \frac{GMm_1}{d^2}\frac{1 + \epsilon\cos\theta}{(1 + \epsilon^2 + 2\epsilon\cos\theta)^{3/2}}
\end{aligned} \tag{11}
$$

From here, we can use the Taylor expansion

$$
(1 + X)^{-3/2} \approx 1 - (3/2)X + (15/8)X^2 + \mathcal{O}(X^3) \tag{12}
$$

for the denominator, setting $X = \epsilon^2 + 2\epsilon\cos\theta$. After doing so, and discarding terms of order ϵ^3 or higher, the force along the x-axis becomes

$$
\begin{aligned}
F_{\hat{x}}^{(q,1)} &= \frac{GMm_1}{d^2}(1 + \epsilon\cos\theta) \\
&\quad \times \left(1 - \frac{3}{2}\epsilon^2 - 3\epsilon\cos\theta + \frac{15}{2}\epsilon^2\cos^2\theta + \mathcal{O}\left(\epsilon^3\right)\right)
\end{aligned} \tag{13}
$$

Propagating the first parenthetical term through the second, and again discarding terms of order ϵ^3 or greater reveals the approximation for the force along the x-axis produced by the first rotor mass,

$$
F_{\hat{x}}^{(q,1)} = \frac{GMm_1}{d^2}\left(1 - \frac{3}{2}\epsilon^2 - 2\epsilon\cos\theta + \frac{9}{2}\epsilon^2\cos^2\theta + \mathcal{O}\left(\epsilon^3\right)\right). \tag{14}
$$

For the second rotor mass m_2, at position

$$
\vec{r}_x^{(q,2)} = \left[d - r_q\cos\theta\right]\hat{x} \tag{15}
$$

$$
\vec{r}_y^{(q,2)} = \left[-r_q\sin\theta\right]\hat{y}, \tag{16}
$$

we can walk through the same maths as in Eq. 11 to get the projection of the force on the test mass center of mass,

$$
F_{\hat{x}}^{(q,2)} = \frac{GMm_2}{d^2}\frac{1 - \epsilon\cos\theta}{(1 + \epsilon^2 - 2\epsilon\cos\theta)^{3/2}}. \tag{17}
$$

Using the same Taylor expansion as in Eq. 12, but setting $X = \epsilon^2 - 2\epsilon\cos\theta$ this time, and again dropping terms of order ϵ^3 or higher,

$$
\begin{aligned}
F_{\hat{x}}^{(q,2)} &= \frac{GMm_2}{d^2}(1 - \epsilon\cos\theta) \\
&\quad \times \left(1 - \frac{3}{2}\epsilon^2 + 3\epsilon\cos\theta + \frac{15}{2}\epsilon^2\cos^2\theta + \mathcal{O}\left(\epsilon^3\right)\right) \\
F_{\hat{x}}^{(q,2)} &= \frac{GMm_2}{d^2}\left(1 - \frac{3}{2}\epsilon^2 + 2\epsilon\cos\theta + \frac{9}{2}\epsilon^2\cos^2\theta + \mathcal{O}\left(\epsilon^3\right)\right).
\end{aligned} \tag{18}
$$

Thus, for any given rotor rotation angle, if we assume the masses are identical, $m_1 = m_2 = m$, the total force on the test mass in the $\hat{x}$ direction is

$$
\begin{aligned}
F_{\hat{x}}^{(q)} &= F_{\hat{x}}^{(q,1)} + F_{\hat{x}}^{(q,2)} \\
F_{\hat{x}}^{(q)} &= \frac{GMm}{d^2}\left(2 - 3\epsilon^2 + 9\epsilon^2\cos^2\theta + \mathcal{O}\left(\epsilon^3\right)\right).
\end{aligned} \tag{19}
$$

Through the power of foresight, we take advantage of the identity $\cos^2\theta = [1 + \cos(2\theta)]/2$ to write this as

$$
F_{\hat{x}}^{(q)} = \frac{GMm}{d^2}\left(2 - 3\epsilon^2 + \frac{9}{2}\epsilon^2 + \frac{9}{2}\epsilon^2\cos(2\theta) + \mathcal{O}\left(\epsilon^3\right)\right). \tag{20}
$$

Now, if we drop the first three static, "DC" force terms, only consider the lowest order "AC" force term dependent on the angle of the rotor, θ, and subbing back in for $\epsilon = r_q/d$, we arrive at the final answer for the 1D, in-plane, on axis, angle dependent component of the quadrupole force, at twice the rotation frequency,

$$
\begin{aligned}
F_{\hat{x}}^{(q)}(\theta) &= \frac{9}{2}\frac{GMm}{d^2}\epsilon^2\cos(2\theta) \\
&\quad (\epsilon = r_q/d) \\
&= \frac{9}{2}\frac{GMm}{d^2}\left(\frac{r_q}{d}\right)^2\cos(2\theta) \\
F_{\hat{x}}^{(q)}(\theta) &= \frac{9}{2}\frac{GMm}{d^4}r_q^2\cos(2\theta)
\end{aligned} \tag{21}
$$

B. 1D HEXAPOLE

In this subsection, we begin to generalize our in-plane, on-axis method using Section II A as our guide, but for a hexapole. Instead of showing each mass contribution independently, we'll compute the force on the nth mass at radius r_q.

For a hexapole, each nth slug mass is situated at a vector $\vec{r}_h^{(n)}$ and the test mass center of mass, at an angle $\theta_n = \theta + 2\pi\{n-1\}/3$ w.r.t. the center of rotation of the NCAL, with $n = 1, 2, 3$, such that the components of $\vec{r}_h^{(n)}$ are

$$r_{\hat{x}}^{(h,n)} = \left[d + r_h \cos\left(\theta + 2\pi\{n-1\}/3\right)\right]\ \hat{x} \tag{22}$$

$$r_{\hat{y}}^{(h,n)} = \left[\ + r_h \sin\left(\theta + 2\pi\{n-1\}/3\right)\right]\ \hat{y} \tag{23}$$

The magnitude squared of this position vector is

$$\begin{aligned} \left|r_h^{(n)}\right|^2 &= \left[d + r_h \cos\left(\theta + 2\pi\{n-1\}/3\right)\right]^2 \\ &\quad + \quad \left[r_h \sin\left(\theta + 2\pi\{n-1\}/3\right)\right]^2 \\ &= d^2 + r_h^2 + 2dr_h \cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &= d^2\left[1 + \varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right] \end{aligned} \tag{24}$$

which makes the magnitude,

$$|r_h^{(n)}| = d\left[1 + \varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right]^{1/2} \tag{25}$$

where now we've replaced the small *quadrupole* factor $\epsilon \equiv r_q/d$ with similar but different small *hexapole* factor, $\varepsilon \equiv r_h/d$.

As with the quadrupole force, the $\hat{x}$ projection of the force from each hexapole mass on the test mass center of mass is

$$\begin{aligned} F_{\hat{x}}^{(h,n)} &= F_{tot}^{(h,n)}\left|r_{\hat{x}}^{(h,n)}\right|/|r_h^{(n)}|. \\ &\left(\begin{array}{l} \text{... subbing in Eqs. 22 and 25,} \\ \text{then for } \varepsilon\text{, and following steps similar} \\ \text{to those leading up to Eq. 11 ...} \end{array}\right) \\ F_{\hat{x}}^{(h,n)} &= \frac{GMm_n}{d^2}\frac{1 + \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)}{\left[1 + \varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right]^{3/2}}, \end{aligned} \tag{26}$$

we see this as quite similar to Eq. 11.

We continue along the same lines with a Taylor series approximation as in Eq. 12, with a similar substitution: $Y = \varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)$. However, we must take the series out to third order in Y this time because only there will we find terms proportional to $\cos^3(\theta)$, which will eventually convert to the $\cos(3\theta)$ terms we expect. Showing the next term in the Taylor series,

$$\begin{aligned} &(1+Y)^{-3/2} \approx \\ &\qquad 1 - \frac{3}{2}Y + \frac{15}{8}Y^2 - \frac{35}{16}Y^3 + \mathcal{O}(Y^4) \end{aligned} \tag{27}$$

we plug Y back in to Eq. 26, and multiply out and combine like terms to take the calculation a bit further,

$$\begin{aligned} F_{\hat{x}}^{(h,n)} = \frac{GMm_n}{d^2}&\left[1 + \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right] \\ \times\Bigg(1 & \\ &- \frac{3}{2}\left[\varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right] \\ &+ \frac{15}{8}\left[\varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right]^2 \\ &- \frac{35}{16}\left[\varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right]^3 \\ &+ \mathcal{O}\left\{\left[\varepsilon^2 + 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right]^4\right\}\Bigg) \end{aligned} \tag{28}$$

$$\begin{aligned} = \frac{GMm_n}{d^2}&\left[1 + \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right)\right] \\ \times\Bigg(1 & \\ &- \frac{3}{2}\varepsilon^2 \\ &- 3\ \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \frac{15}{2}\varepsilon^2\cos^2\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \frac{15}{2}\varepsilon^3\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &- \frac{35}{2}\varepsilon^3\cos^3\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \mathcal{O}(\varepsilon^4)\Bigg) \end{aligned} \tag{29}$$

$$\begin{aligned} = \frac{GMm_n}{d^2}\Bigg(1 & \\ &- \frac{3}{2}\varepsilon^2 \\ &- 3\ \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \frac{15}{2}\varepsilon^2\cos^2\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \frac{15}{2}\varepsilon^3\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &- \frac{35}{2}\varepsilon^3\cos^3\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &- \frac{3}{2}\varepsilon^3\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ &- 3\ \varepsilon^2\cos^2\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \frac{15}{2}\varepsilon^3\cos^3\left(\theta + 2\pi\{n-1\}/3\right) \\ &+ \mathcal{O}(\varepsilon^4)\Bigg) \end{aligned} \tag{30}$$

$$F_{\hat{x}}^{(h,n)} = \frac{GMm_n}{d^2}\Bigg(1 \\ - \frac{3}{2}\varepsilon^2 \\ - 2\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ + \frac{9}{2}\varepsilon^2\cos^2\left(\theta + 2\pi\{n-1\}/3\right) \\ + 6\varepsilon^3\cos\left(\theta + 2\pi\{n-1\}/3\right) \\ - 10\varepsilon^3\cos^3\left(\theta + 2\pi\{n-1\}/3\right) \\ + \mathcal{O}(\varepsilon^4)\Bigg). \tag{31}$$

Now, when we add all these terms together to get the total hexapole force in the $\hat{x}$ direction, $F_{\hat{x}}^{(h)} = \sum_{n=1}^{3} F_{\hat{x}}^{(h,n)}$, we'll get some really nice cancellation/reduction of terms because

$$\sum_{n=1}^{3}\cos(\theta + 2\pi\{n-1\}/3) = 0, \tag{32}$$

$$\sum_{n=1}^{3}\cos^2(\theta + 2\pi\{n-1\}/3) = \frac{3}{2} \tag{33}$$

and

$$\sum_{n=1}^{3}\cos^3(\theta + 2\pi\{n-1\}/3) = \frac{3}{4}\cos(3\theta). \tag{34}$$

That means that in the *sum* of forces, $F_{\hat{x}}^{(h)}$, from all hexapole masses, assuming again that $m_1 = m_2 = m_3 = m$, we only have

$$F_{\hat{x}}^{(h)} = \frac{GMm}{d^2}\Bigg(3 \\ - \frac{9}{2}\varepsilon^2 \\ - 2\varepsilon\ (0) \\ + \frac{9}{2}\varepsilon^2\left(\frac{3}{2}\right) \\ + 6\varepsilon^3(0) \\ - 10\varepsilon^3\left(\frac{3}{4}\cos(3\theta)\right) \\ + \mathcal{O}(\varepsilon^4)\Bigg)$$

$$F_{\hat{x}}^{(h)} = \frac{GMm}{d^2}\left(3 + \frac{9}{4}\varepsilon^2 - \frac{15}{2}\varepsilon^3\cos(3\theta)\ +\ \mathcal{O}(\varepsilon^4)\right) \tag{35}$$

and the AC component we're interested in (ignoring the arbitrary sign) is

$$\begin{aligned} F_{\hat{x}}^{(h)}(\theta) &= \frac{15}{2}\frac{GMm}{d^2}\varepsilon^3\cos(3\theta) \\ &\quad (\varepsilon \ \equiv\ r_h/d) \\ &= \frac{15}{2}\frac{GMm}{d^2}\left(\frac{r_h}{d}\right)^3\cos(3\theta) \\ F_{\hat{x}}^{(h)}(\theta) &= \frac{15}{2}\frac{GMm}{d^5}r_h^3\cos(3\theta) \end{aligned} \tag{36}$$

III. JUST OUT-OF-PLANE, OFF-AXIS, 3D

In this configuration, the NCAL center of rotation is now positioned away from the x axis by an angle Φ, and just slightly out of plane (i.e. $z/d \ll 1$) reducing what's shown in Figure 4 to what's shown in Figure 2 and 3. In this case, we will use the small-angle approximation for ψ in the $\hat{z}$ component of $\vec{\boldsymbol{z}}$, and set $(z\ \cos\psi) \approx z$. We carry around z to be able to inform our intuition about uncertainty in z assuming we design the NCAL to be in-plane. The point of zero phase for the m_1 mass rotation angle, θ, is now fixed to the d-axis (no longer fixed to the x-axis), but again when m_1 is closest to the test mass.

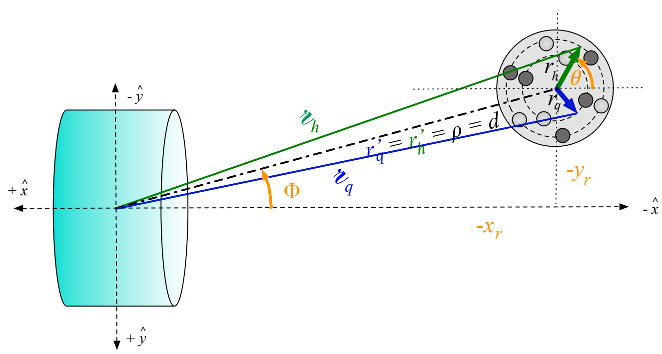

FIG. 2. Top Down View of NCAL positioned off of the x-axis in the $-\hat{y}$ direction for extension to 2D case.

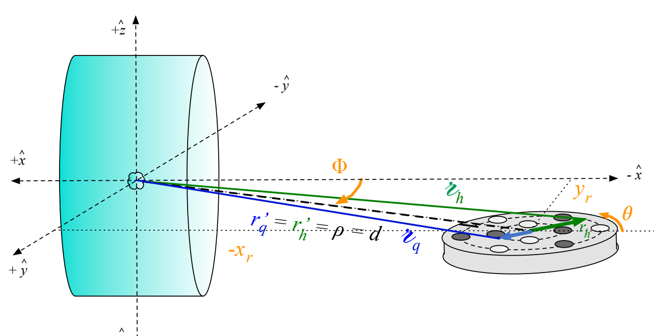

FIG. 3. Isometric View of NCAL positioned off of the x-axis in the $-\hat{y}$ direction for extension to 2D case.

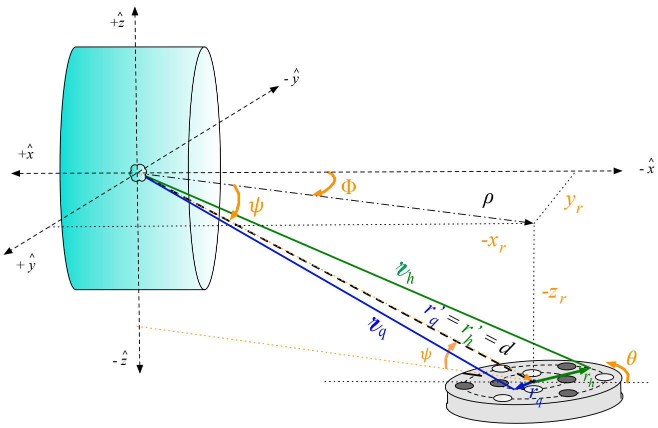

FIG. 4. Isometric View of NCAL positioned off of the x-axis in the $-\hat{y}$ direction, and way-out-of-plane in the $-\hat{z}$ direction for extension to 3D case and emphasis of relevant variable definitions.

A. 3D QUADRUPOLE

The first quadrupole mass' position with respect to the test mass center of mass becomes:

$$r_{\hat{x}}^{(q,1)} = \left[d\cos\Phi + r_q\cos\theta\right]\,\hat{x} \tag{37}$$
$$r_{\hat{y}}^{(q,1)} = \left[d\sin\Phi + r_q\sin\theta\right]\,\hat{y} \tag{38}$$
$$r_{\hat{z}}^{(q,1)} = \left[z\right]\,\hat{z} \tag{39}$$

In each Eqs. 37-39, as stated above, we have implicitly assumed that both $x/z \ll 1$ and $y/z \ll 1$ such that $d = \sqrt{(x^2+y^2+z^2)} \approx \sqrt{(x^2+y^2)} = \rho$, and that $\cos\psi \approx 1$ simplifying the math from what would be described by to what's shown in Figure 4 to what shown in Figure 3.

Walking through the same steps as before, we compute the magnitude squared and magnitude of the position vector between the test mass center of mass and m_1,

$$\begin{aligned}
|\vec{r}_q^{(1)}|^2 &= (d\cos\Phi + r_q\cos\theta)^2 + (d\sin\Phi + r_q\sin\theta)^2 + z^2 \\
&= d^2\cos^2\Phi + r_q^2\cos^2\theta + 2dr_q\cos\theta\cos\Phi \\
&\qquad + \; d^2\sin^2\Phi + r_q^2\sin^2\theta + 2dr_q\sin\theta\sin\Phi \\
&\qquad\quad + \; z^2 \\
&= d^2\left(\sin^2\Phi + \cos^2\Phi\right) \\
&\qquad + \; r_q^2\left(\sin^2\theta + \cos^2\theta\right) \\
&\qquad\quad + \; 2dr_q(\cos\theta\cos\Phi + \sin\theta\sin\Phi) \\
&\qquad\qquad + \; z^2 \\
&\left(\begin{array}{rcl} \sin^2 A + \cos^2 A &=& 1 \\ \cos A\cos B + \sin A\sin B &=& \cos(A-B) \end{array}\right) \\
&= d^2 + z^2 + r_q^2 + 2dr_q\cos(\theta - \Phi) \\
&\quad (\epsilon \equiv r_q/d) \\
&= d^2\left(1 + \left(\frac{z}{d}\right)^2 + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right) \\
&\quad \left(\zeta \equiv 1 + (z/d)^2\right) \\
|\vec{r}_q^{(1)}|^2 &= d^2\left(\zeta^2 + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right)
\end{aligned} \tag{40}$$

and

$$|r_q^{(1)}| = d\left(\zeta^2 + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right)^{1/2} \tag{41}$$

but this time we've introduced another dimensionless parameter, ζ (which asymptotes to unity as $(z/d) \ll 1$, whereas ϵ asymptotes to zero as $(r_q/d) \ll 1$).

The total quadrupole force, $F_{tot}^{(q,1)}$, is similar to the 1D case, just with an updated distance vector magnitude,

$$F_{tot}^{(q,1)} = \frac{GMm_1}{\left|\vec{r}_q^{(1)}\right|^2} = \frac{GMm_1}{d^2\left(\zeta + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right)}. \tag{42}$$

Projecting that resulting force onto the x-axis you also get as you'd expect:

$$\begin{aligned}
F_{\hat{x}}^{(q,1)} &= F_{tot}^{(q,1)}\left(\frac{|r_x^{(q,1)}|}{|r_q^{(1)}|}\right) \\
&= F_{tot}^{(q,1)}\frac{(d\cos\Phi + r\cos\theta)}{d\left(\zeta + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right)^{1/2}} \\
&= F_{tot}^{(1)}\frac{(\cos\Phi + \epsilon\cos\theta)}{\left(\zeta + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right)^{1/2}} \\
F_{\hat{x}}^{(q,1)} &= \frac{GMm_1}{d^2}\frac{(\cos\Phi + \epsilon\cos\theta)}{\left(\zeta + \epsilon^2 + 2\epsilon\cos(\theta-\Phi)\right)^{3/2}},
\end{aligned} \tag{43}$$

which looks much like Eq. 11 as $\Phi \to 0$ and $z \to 0$.

That means we can start a Taylor expansion of denominator as in Eq. 12, but now slightly modified with $X' \equiv \epsilon^2 + 2\epsilon\cos(\theta - \Phi)$ and using a modified version of the expansion, where 1 is replaced by $\zeta = 1 + (z/d)^2$,

$$\begin{aligned}(\zeta + X')^{-3/2} &= \frac{1}{\zeta^{3/2}} - \frac{3}{2}\frac{X'}{\zeta^{5/2}} + \frac{15}{8}\frac{X'^2}{\zeta^{7/2}} + \mathcal{O}(X'^3, \zeta^{9/2}) \\ &= \zeta^{-5/2}\left(\zeta - \frac{3}{2}\,X' + \frac{15}{8}\frac{X'^2}{\zeta} + \mathcal{O}(X'^3, \zeta^2)\right).\end{aligned} \tag{44}$$

In doing so, the math to finish out the leading order of the 1st mass quadrupole force, $F_{\hat{x}}^{(q,1)}$, is the same as before as we propagate the definition of X' through, and drop terms of order ϵ^3 and higher as we go,

$$\begin{aligned}F_{\hat{x}}^{(q,1)} &= \frac{GMm_1}{d^2}(\cos\Phi + \epsilon\cos\theta) \\ &\quad\times\ \zeta^{-5/2}\Bigg(\zeta \\ &\qquad -\ \frac{3}{2}\left[\epsilon^2 + 2\epsilon\cos(\theta - \Phi)\right] \\ &\qquad +\ \frac{15}{8}\frac{1}{\zeta}\left[\epsilon^2 + 2\epsilon\cos(\theta - \Phi)\right]^2 \\ &\qquad +\ \mathcal{O}(\epsilon^3)\Bigg) \\ &= \frac{GMm_1}{d^2}(\cos\Phi + \epsilon\cos\theta) \\ &\quad\times\ \zeta^{-5/2}\Bigg(\zeta \\ &\qquad -\ \frac{3}{2}\epsilon^2 \\ &\qquad -\ 3\epsilon\cos(\theta - \Phi) \\ &\qquad +\ \frac{15}{2}\ \frac{1}{\zeta}\epsilon^2\cos^2(\theta - \Phi) \\ &\qquad +\ \mathcal{O}(\epsilon^3)\Bigg) \\ F_{\hat{x}}^{(q,1)} &= \frac{GMm_1}{d^2}\ \zeta^{-5/2} \\ &\quad\times\ \Bigg[\zeta\cos\Phi \\ &\qquad -\ \frac{3}{2}\epsilon^2\cos\Phi \\ &\qquad -\ 3\epsilon\cos(\theta - \Phi)\cos\Phi \\ &\qquad +\ \frac{15}{2}\ \frac{1}{\zeta}\epsilon^2\cos^2(\theta - \Phi)\cos\Phi \\ &\qquad +\ \zeta\epsilon\cos\theta \\ &\qquad -\ 3\epsilon^2\cos(\theta - \Phi)\cos\theta \\ &\qquad +\ \mathcal{O}(\epsilon^3)\Bigg]\end{aligned} \tag{45}$$

which *looks* nasty, but reverts to the 1D version, Eq. 14, when $\zeta \to 1$ and $\Phi \to 0$ which makes $\cos(\theta - \Phi) \to \cos(\theta)$.

As before, we walk through the same steps for the second quadrupole mass, m_2,

$$\mathscr{r}_{\hat{x}}^{(q,2)} = \left[d\cos\Phi - r_q\cos\theta\right]\hat{x} \tag{46}$$

$$\mathscr{r}_{\hat{y}}^{(q,2)} = \left[d\sin\Phi - r_q\sin\theta\right]\hat{y} \tag{47}$$

$$\mathscr{r}_{\hat{z}}^{(q,2)} = \left[z\right]\hat{z} \tag{48}$$

You know the drill, by now! The distance from m_2 to M is

$$\begin{aligned}|\vec{\mathscr{r}}_q^{(2)}|^2 &= (d\cos\Phi - r_q\cos\theta)^2 + (d\sin\Phi - r_q\sin\theta)^2 + z^2 \\ &= d^2\cos^2\Phi + r_q^2\cos^2\theta - 2dr_q\cos\theta\cos\Phi \\ &\qquad +\ d^2\sin^2\Phi + r_q^2\sin^2\theta - 2dr_q\sin\theta\sin\Phi \\ &\qquad +\ z^2 \\ &= d^2 + z^2 + r_q^2 - 2dr_q\cos(\theta - \Phi) \\ |\vec{\mathscr{r}}_q^{(2)}|^2 &= d^2\left(\zeta + \epsilon^2 - 2\epsilon\cos(\theta - \Phi)\right) \\ &\quad \left(\zeta\ =\ 1 + (z/d)^2\right) \\ |\mathscr{r}_q^{(2)}| &= d\left(\zeta + \epsilon^2 - 2\epsilon\cos(\theta - \Phi)\right)^{1/2}\end{aligned} \tag{49}$$

The total force from m_2 on M is

$$F_{tot}^{(q,2)} = \frac{GMm_2}{|\vec{\mathscr{r}}_q^{(2)}|^2} = \frac{GMm_2}{d^2\left(\zeta + \epsilon^2 - 2\epsilon\cos(\theta - \Phi)\right)} \tag{50}$$

Projecting onto the x-axis:

$$\begin{aligned}F_{\hat{x}}^{(q,2)} &= F_{tot}^{(q,2)}\left(\frac{|\mathscr{r}_{\hat{x}}^{(q,2)}|}{|\mathscr{r}_q^{(2)}|}\right) \\ &= F_{tot}^{(q,2)}\frac{(d\cos\Phi - r\cos\theta)}{d\left(\zeta + \epsilon^2 - 2\epsilon\cos(\theta - \Phi)\right)^{1/2}} \\ &= F_{tot}^{(q,2)}\frac{(\cos\Phi - \epsilon\cos\theta)}{\left(\zeta + \epsilon^2 - 2\epsilon\cos(\theta - \Phi)\right)^{1/2}} \\ F_{\hat{x}}^{(q,2)} &= \frac{GMm_2}{d^2}\frac{(\cos\Phi - \epsilon\cos\theta)}{\left(\zeta + \epsilon^2 - 2\epsilon\cos(\theta - \Phi)\right)^{3/2}}\end{aligned} \tag{51}$$

Setting $Y' \equiv \epsilon^2 - 2\epsilon\cos(\theta - \Phi)$ this time, and repeating the same $(\zeta + Y')^{-3/2}$ Taylor expansion from Eq. 44,

$$\begin{aligned}
F_{\hat{x}}^{(q,2)} &= \frac{GMm_2}{d^2}(\cos\Phi - \epsilon\cos\theta) \\
&\quad \times\ \zeta^{-5/2}\Bigg(\zeta \\
&\qquad\qquad -\ \frac{3}{2}\left[\epsilon^2 - 2\epsilon\cos(\theta-\Phi)\right] \\
&\qquad\qquad +\ \frac{15}{8}\frac{1}{\zeta}\left[\epsilon^2 - 2\epsilon\cos(\theta-\Phi)\right]^2 \\
&\qquad\qquad +\ \mathcal{O}(\epsilon^3)\Bigg) \\
&= \frac{GMm_2}{d^2}(\cos\Phi - \epsilon\cos\theta) \\
&\quad \times\ \zeta^{-5/2}\Bigg(\zeta \\
&\qquad\qquad -\ \frac{3}{2}\epsilon^2 \\
&\qquad\qquad +\ 3\epsilon\cos(\theta-\Phi) \\
&\qquad\qquad +\ \frac{15}{2}\frac{1}{\zeta}\epsilon^2\cos^2(\theta-\Phi) \\
&\qquad\qquad +\ \mathcal{O}(\epsilon^3)\Bigg) \\
&= \frac{GMm_1}{d^2}\ \zeta^{-5/2} \\
&\quad \times\ \Bigg[\zeta\cos\Phi \\
&\qquad -\ \frac{3}{2}\epsilon^2\cos\Phi \\
&\qquad +\ 3\epsilon\cos(\theta-\Phi)\cos\Phi \\
&\qquad +\ \frac{15}{2}\ \frac{1}{\zeta}\epsilon^2\cos^2(\theta-\Phi)\cos\Phi \\
&\qquad -\ \zeta\epsilon\cos\theta \\
&\qquad -\ 3\epsilon^2\cos(\theta-\Phi)\cos\theta \\
&\qquad +\ \mathcal{O}(\epsilon^3)\Bigg]
\end{aligned} \tag{52}$$

where, again, we safely revert to the 1D version, Eq. 18, when $\zeta \to 1$ and $\Phi \to 0$.

Now, as before, let's assume $m_1 = m_2 = m$, and add Eqs. 45 and 52 to get the total force on the test mass center of mass from both quadrupole masses in the x-direction, $F_{\hat{x}}^{(q)}$. As in the 1D case, the terms $\mathcal{O}(\epsilon)$ cancel, leaving

$$\begin{aligned}
F_{\hat{x}}^{(q)} &= F_{\hat{x}}^{(q,1)} + F_{\hat{x}}^{(q,2)} \\
&= \frac{GMm}{d^2}\ \zeta^{-5/2} \\
&\qquad \times\ \Bigg[2\zeta\cos\Phi \\
&\qquad\quad -\ 3\epsilon^2\cos\Phi \\
&\qquad\quad +\ 15\frac{1}{\zeta}\epsilon^2\cos^2(\theta-\Phi)\cos\Phi \\
&\qquad\quad -\ 6\epsilon^2\cos(\theta-\Phi)\cos\Phi \\
&\qquad\quad +\ \mathcal{O}(\epsilon^3)\Bigg]
\end{aligned} \tag{53}$$

And since we know we're going to be dropping the "DC" terms independent of θ, and searching for that those that depend on 2θ, let's just jump straight into some trig manipulations that "AC" force term in such a form where we end up with terms with functions of 2θ:

$$\begin{aligned}
F_{\hat{x}}^{(q)}(\theta) &= \frac{GMm}{d^2}\ \epsilon^2\ \zeta^{-5/2} \\
&\quad \times\ \Bigg[15\frac{1}{\zeta}\cos^2(\theta-\Phi)\cos\Phi \\
&\qquad -\ 6\cos(\theta-\Phi)\cos\theta\Bigg]
\end{aligned} \tag{54}$$

In each term we can perform some trig wizardry better left to programs like Wolfram Alpha to prove:

$$\begin{aligned}
\cos^2(\theta-\Phi)\cos\Phi = \frac{1}{4}\Big(&\cos(2\theta-\Phi) + \cos(2\theta-3\Phi) \\
&+\ 2\cos\Phi\Big)
\end{aligned} \tag{55}$$

$$\begin{aligned}
\cos(\theta-\Phi)\cos\theta = \frac{1}{2}\Big(&\cos(2\theta-\Phi) \\
&+\ \cos\Phi\Big).
\end{aligned} \tag{56}$$

where the terms that don't depend on θ have been visually segregated on the right-hand sides, because we drop them as more components of the "DC" force. Indeed, after plugging these identities, Eqs. 55 and 56, back in and dropping those "$\cos\Phi$-only" terms, we're left with what we expect, terms that depend on 2θ:

$$\begin{aligned}
F_{\hat{x}}^{(q)}(\theta) &= \frac{GMm}{d^2}\ \epsilon^2\ \zeta^{-5/2} \\
&\quad \times\ \Bigg[\frac{15}{4}\frac{1}{\zeta}\Big\{\cos(2\theta-\Phi) + \cos(2\theta-3\Phi)\Big\} \\
&\qquad -\ 3\cos(2\theta-\Phi)\Bigg]
\end{aligned} \tag{57}$$

Now it's all aesthetic cleanup from here. Let's see what it looks like when we pull out a factor of 9/2 and group the $\cos(2\theta - \Phi)$ terms such that we can compare how ζ and Φ modifies Eq. 21,

$$F^{(q)}_{\hat{x}}(\theta) = \frac{9}{2}\,\frac{GMm}{d^2}\,\epsilon^2\,\zeta^{-5/2} \times\left[\left(\frac{5}{6}\frac{1}{\zeta} - \frac{2}{3}\right)\cos(2\theta - \Phi) + \frac{5}{6}\frac{1}{\zeta}\cos(2\theta - 3\Phi)\right] \tag{58}$$

B. 3D HEXAPOLE

Now on to fleshing out the hexapole calculation in 3D using all we have learned from the previous sections. For each mass nth, at angle $\theta_n = \theta + 2\pi\{n-1\}/3$ from the $\hat{d}$ axis, the position vector m and M now has components:

$$\vec{r}^{(h,n)}_{\hat{x}} = \left[d\cos\Phi + r_h\cos\left(\theta + 2\pi\{n-1\}/3\right)\right]\,\hat{x} \tag{59}$$
$$\vec{r}^{(h,n)}_{\hat{y}} = \left[d\sin\Phi + r_h\sin\left(\theta + 2\pi\{n-1\}/3\right)\right]\,\hat{y} \tag{60}$$
$$\vec{r}^{(h,n)}_{\hat{z}} = \left[z\right]\,\hat{z} \tag{61}$$

As in the previous section, we have implicitly assumed that both $x/z \ll 1$ and $y/z \ll 1$ such that $d = \sqrt{(x^2+y^2+z^2)} \approx \sqrt{(x^2+y^2)} = \rho$, and that $\cos\psi \approx 1$ simplifying the math from what would be described by to what's shown in Figure 4 to what shown in Figure 3.

Which means the magnitude squared of the vector, using our definitions of ε and ζ from previous sections is

$$\begin{aligned}
|\vec{r}^{(n)}_h|^2 &= \left[d\cos\Phi + r_h\cos(\theta + 2\pi\{n-1\}/3)\right]^2 \\
&\quad + \left[d\sin\Phi + r_h\sin(\theta + 2\pi\{n-1\}/3)\right]^2 \\
&\quad + z^2 \\
&= d^2\cos^2\Phi \\
&\quad + r_h^2\cos^2(\theta + 2\pi\{n-1\}/3) \\
&\quad + 2dr_h\cos(\theta + 2\pi\{n-1\}/3)\cos\Phi \\
&\quad + d^2\sin^2\Phi \\
&\quad + r_h^2\sin^2(\theta + 2\pi\{n-1\}/3) \\
&\quad + 2dr_h\sin(\theta + 2\pi\{n-1\}/3)\sin\Phi \\
&\quad + z^2 \\
&= d^2\left(\cos^2\Phi + \sin^2\Phi\right) \\
&\quad + r_h^2\Big[\cos^2\left(\theta + 2\pi\{n-1\}/3\right) \\
&\qquad + \sin^2\left(\theta + 2\pi\{n-1\}/3\right)\Big] \\
&\quad + 2dr_h\Big[\cos\left(\theta + 2\pi\{n-1\}/3\right)\cos\Phi \\
&\qquad + \sin\left(\theta + 2\pi\{n-1\}/3\right)\sin\Phi\Big] \\
&\quad + z^2 \\
&\left(\begin{array}{rcl} \sin^2 A + \cos^2 B & = & 1 \\ \cos(A+B)\cos C & + & \sin(A+B)\sin C \\ & = & \cos(A+B-C) \end{array}\right) \\
&= d^2 + z^2 + r_h^2 + 2dr_h\cos\left(\theta + 2\pi\{n-1\}/3 - \Phi\right) \\
&\left(\begin{array}{rcl} \varepsilon & \equiv & r_h/d \\ \zeta & \equiv & 1 + (z/d)^2 \end{array}\right) \\
|\vec{r}^{(n)}_h|^2 &= d^2\left[\zeta + \varepsilon^2 + 2\varepsilon\cos\left(\theta - \Phi + 2\pi\{n-1\}/3\right)\right]
\end{aligned} \tag{62}$$

and the magnitude is,

$$|\vec{r}^{(n)}_h| = d\left[\zeta + \varepsilon^2 + 2\varepsilon\cos\left(\theta - \Phi + 2\pi\{n-1\}/3\right)\right]^{1/2}. \tag{63}$$

Turning the crank on the now-standard calculation of the $\hat{x}$ component of the hexapole force, we find

$$\begin{aligned}
F_{\hat{x}}^{(h,n)} &= F_{tot}^{(h,n)} \left| \boldsymbol{\imath}_{\hat{x}}^{(h,n)} \right| / |\boldsymbol{\imath}_{h}^{(n)}| \\
&= F_{tot}^{(h,n)} \frac{\left[d\cos\Phi + r_h \cos\left(\theta + 2\pi\{n-1\}/3\right) \right]}{d\left[\zeta + \varepsilon^2 + 2\varepsilon\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\right]^{1/2}} \\
&= F_{tot}^{(h,n)} \frac{\left[\cos\Phi + \varepsilon \cos\left(\theta + 2\pi\{n-1\}/3\right) \right]}{\left[\zeta + \varepsilon^2 + 2\varepsilon\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\right]^{1/2}} \\
F_{\hat{x}}^{(h,n)} &= \frac{GMm_n}{d^2} \frac{\left[\cos\Phi + \varepsilon \cos\left(\theta + 2\pi\{n-1\}/3\right) \right]}{\left[\zeta + \varepsilon^2 + 2\varepsilon\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\right]^{3/2}}.
\end{aligned} \tag{64}$$

Carrying the usual Taylor expansion of the denominator out to third order in $Y' = \varepsilon^2 + 2\varepsilon\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)$, and replacing 1 with ζ as in Section III A, using

$$\begin{aligned}
&(\zeta + Y')^{-3/2} = \\
&\quad \zeta^{-5/2}\left(\zeta - \frac{3}{2}Y' + \frac{15}{8}\frac{Y'^2}{\zeta} - \frac{35}{16}\frac{Y'^3}{\zeta^2} + \mathcal{O}(Y'^5)\right)
\end{aligned} \tag{65}$$

we can continue on as before, dropping terms of $\mathcal{O}(\varepsilon^4)$ as we go,

$$\begin{aligned}
F_{\hat{x}}^{(h,n)} &= \frac{GMm_n}{d^2}\left[\cos\Phi + \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \right] \\
&\times\ \zeta^{-5/2}\Bigg(\zeta \\
&\qquad -\ \frac{3}{2}\left[\varepsilon^2 + 2\varepsilon\cos(\theta - \Phi + \ 2\pi\{n-1\}/3)\right] \\
&\qquad +\ \frac{15}{8}\frac{1}{\zeta}\left[\varepsilon^2 + 2\varepsilon\cos(\theta - \Phi + \ 2\pi\{n-1\}/3)\right]^2 \\
&\qquad -\ \frac{35}{16}\frac{1}{\zeta^2}\left[\varepsilon^2 + 2\varepsilon\cos(\theta - \Phi + \ 2\pi\{n-1\}/3)\right]^3 \\
&\qquad +\ \mathcal{O}(\varepsilon^4)\Bigg) \\
&= \frac{GMm_n}{d^2}\left[\cos\Phi + \varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \right] \\
&\times\ \zeta^{-5/2}\Bigg(\zeta \\
&\qquad -\ \frac{3}{2}\varepsilon^2 \\
&\qquad -\ 3\varepsilon\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right) \\
&\qquad +\ \frac{15}{2}\frac{1}{\zeta}\varepsilon^2\cos^2\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right) \\
&\qquad +\ \frac{15}{2}\frac{1}{\zeta}\varepsilon^3\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right) \\
&\qquad -\ \frac{35}{2}\frac{1}{\zeta^2}\varepsilon^3\cos^3\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right) \\
&\qquad +\ \mathcal{O}(\varepsilon^4)\Bigg)
\end{aligned} \tag{66}$$

$$\begin{aligned}
F_{\hat{x}}^{(h,n)} &= \frac{GMm_n}{d^2}\ \zeta^{-5/2} \\
&\times\ \Bigg(\zeta\cos\Phi \\
&\qquad -\ \frac{3}{2}\varepsilon^2\cos\Phi \\
&\qquad -\ 3\varepsilon\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\cos\Phi \\
&\qquad +\ \frac{15}{2}\frac{1}{\zeta}\varepsilon^2\cos^2\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\cos\Phi \\
&\qquad +\ \frac{15}{2}\frac{1}{\zeta}\varepsilon^3\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\cos\Phi \\
&\qquad -\ \frac{35}{2}\frac{1}{\zeta^2}\varepsilon^3\cos^3\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right)\cos\Phi \\
&\qquad +\ \zeta\varepsilon\cos\left(\theta + 2\pi\{n-1\}/3\right) \\
&\qquad -\ \frac{3}{2}\varepsilon^3\cos\left(\theta + 2\pi\{n-1\}/3\right) \\
&\qquad -\ 3\varepsilon^2\cos\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right) \\
&\qquad\quad \times\ \cos\left(\theta + 2\pi\{n-1\}/3\right) \\
&\qquad +\ \frac{15}{2}\frac{1}{\zeta}\varepsilon^3\cos^2\left(\theta - \Phi + \ 2\pi\{n-1\}/3\right) \\
&\qquad\quad \times\ \cos\left(\theta + 2\pi\{n-1\}/3\right) \\
&\qquad +\ \mathcal{O}(\varepsilon^4)\Bigg)
\end{aligned} \tag{67}$$

which looks just as nasty as Eqs. 45 and 52, but you can see very similar terms, and a few extra because we've extended out to $\mathcal{O}(\varepsilon^3)$. This also matches Eq. 30 line for line, but because $\cos\Phi$ is non-zero and ζ isn't exactly unity, we can't combine terms to reduce to something that looks so clean as Eq. 31.

But, if we evaluate the sum over the nth mass to get the total hexapole force in the $\hat{x}$ direction, $F_{\hat{x}}^{(h)}$, we can invoke more triple-angle formulae such that many terms are zero, or decay to constant terms depending on $\cos\Phi$ alone, so things will again drop out nicely when we look at the "AC" force component. As before, while we'll still need some trig wizardry, the terms that we end up caring about – the $\cos^3(\theta_n - \Phi)$ and $\cos^2(\theta_n - \Phi)\cos(\theta_n - \Phi)$ terms – will turn in to terms depending on $\cos(3\theta + ...)$ matching our expectations.

Here's every thing we'll need explicitly:

$$\sum_{n=1}^{3} \cos\left(\theta - \Phi + 2\pi\{n-1\}/3\right) = 0 \tag{68}$$

$$\sum_{n=1}^{3} \cos\left(\theta - 2\pi\{n-1\}/3\right) = 0 \tag{69}$$

$$\sum_{n=1}^{3} \cos^2\left(\theta - \Phi + 2\pi\{n-1\}/3\right) = \frac{3}{2} \tag{70}$$

$$\sum_{n=1}^{3} \cos\left(\theta - \Phi + 2\pi\{n-1\}/3\right)\cos\left(\theta + 2\pi\{n-1\}/3\right) = \frac{3}{2}\cos\Phi \tag{71}$$

$$\sum_{n=1}^{3} \cos^3\left(\theta - \Phi + 2\pi\{n-1\}/3\right)\cos\Phi = \frac{3}{4}\cos(3\theta - 3\Phi)\cos\Phi \tag{72}$$

$$\sum_{n=1}^{3} \cos^2\left(\theta - \Phi + 2\pi\{n-1\}/3\right)\cos\left(\theta + 2\pi\{n-1\}/3\right) = \frac{3}{4}\cos(3\theta - 2\Phi) \tag{73}$$

The total $\hat{x}$ force from all hexapole masses, where $m_n = m$, in the $\hat{x}$ direction is therefore sum over n of Eq. 67. Subbing in all the above identities as needed, we get

$$\begin{aligned} F_{\hat{x}}^{(h)} = \frac{GMm}{d^2}\,\zeta^{-5/2} \times \Bigg(& 3\zeta\cos\Phi \\ & - \frac{9}{2}\varepsilon^2\cos\Phi \\ & - 3\varepsilon(0)\cos\Phi \\ & + \frac{15}{2}\frac{1}{\zeta}\varepsilon^2\left(\frac{3}{2}\right)\cos\Phi \\ & + \frac{15}{2}\frac{1}{\zeta}\varepsilon^3(0)\cos\Phi \\ & - \frac{35}{2}\frac{1}{\zeta^2}\varepsilon^3\left(\frac{3}{4}\cos(3\theta - 3\Phi)\cos\Phi\right) \\ & + \zeta\varepsilon(0) \\ & - \frac{3}{2}\varepsilon^3(0) \\ & - 3\varepsilon^2\left(\frac{3}{2}\cos\Phi\right) \\ & + \frac{15}{2}\frac{1}{\zeta}\varepsilon^3\left(\frac{3}{4}\cos(3\theta - 2\Phi)\right) \\ & + \mathcal{O}(\varepsilon^4)\Bigg) \end{aligned} \tag{74}$$

$$\begin{aligned} F_{\hat{x}}^{(h)} = \frac{GMm}{d^2}\,\zeta^{-5/2} \times \Bigg(& 3\zeta\cos\Phi \\ & - 9\,\varepsilon^2\cos\Phi \\ & + \frac{45}{4}\frac{1}{\zeta}\varepsilon^2\cos\Phi \\ & - \frac{105}{8}\frac{1}{\zeta^2}\varepsilon^3\cos(3\theta - 3\Phi)\cos\Phi \\ & + \frac{45}{8}\frac{1}{\zeta}\varepsilon^3\cos(3\theta - 2\Phi)\Bigg) \end{aligned} \tag{75}$$

from which we grab only the terms that depend on θ, to get

$$\begin{aligned} F_{\hat{x}}^{(h)}(\theta) = \frac{GMm}{d^2}\,\zeta^{-5/2} \times \Bigg(& - \frac{105}{8}\frac{1}{\zeta^2}\varepsilon^3\cos(3\theta - 3\Phi)\cos\Phi \\ & + \frac{45}{8}\frac{1}{\zeta}\varepsilon^3\cos(3\theta - 2\Phi)\Bigg). \end{aligned} \tag{76}$$

Approaching the finish line, we pull out all the terms we recognize from the 1D hexapole (Eq. 36), and rearrange the trig terms for a bit of cleanup to make it look and feel a bit more like the 3D quadrupole (Eq. 58),

$$F_{\hat{x}}^{(h)}(\theta) = \frac{15}{2}\frac{GMm}{d^2}\ \varepsilon^3\ \zeta^{-7/2} \times \Bigg[-\ \frac{7}{4}\frac{1}{\zeta}\cos(3\theta-3\Phi)\cos\Phi + \ \frac{3}{4}\cos(3\theta-2\Phi)\Bigg] \tag{77}$$

$$= \frac{15}{2}\frac{GMm}{d^2}\ \varepsilon^3\ \zeta^{-7/2} \times \Bigg[\frac{3}{4}\cos(3\theta-2\Phi) -\ \frac{7}{4}\frac{1}{\zeta}\cos(3\theta-3\Phi)\cos\Phi\Bigg] \tag{78}$$

$$\left(\begin{array}{l}\cos(3\theta-3\Phi)\cos\Phi\ = \\ \frac{1}{2}\left[\cos(3\theta-2\Phi)+\cos(3\theta-4\Phi)\right]\end{array}\right)$$

$$= \frac{15}{2}\frac{GMm}{d^2}\ \varepsilon^3\ \zeta^{-7/2} \times \Bigg[\frac{3}{4}\cos(3\theta-2\Phi) -\ \frac{7}{8}\frac{1}{\zeta}\cos(3\theta-2\Phi) -\ \frac{7}{8}\frac{1}{\zeta}\cos(3\theta-4\Phi)\Bigg] \tag{79}$$

$$= \frac{15}{2}\frac{GMm}{d^2}\ \varepsilon^3\ \zeta^{-7/2} \times \Bigg[-\left(\frac{7}{8}\frac{1}{\zeta}-\frac{3}{4}\right)\cos(3\theta-2\Phi) -\ \frac{7}{8}\frac{1}{\zeta}\cos(3\theta-4\Phi)\Bigg]. \tag{80}$$

Indeed, this looks a ton like Eq. 58. As a finishing touch, we ignore the overall sign, arbitrary sign as we did prior to Eq. 36, we arrive at,

$$F_{\hat{x}}^{(h)}(\theta) = \frac{15}{2}\frac{GMm}{d^2}\ \varepsilon^3\ \zeta^{-7/2} \times \Bigg[\left(\frac{7}{8}\frac{1}{\zeta}-\frac{3}{4}\right)\cos(3\theta-2\Phi) +\ \frac{7}{8}\frac{1}{\zeta}\cos(3\theta-4\Phi)\Bigg] \tag{81}$$

with $\varepsilon \equiv r_h/d$ and $\zeta \equiv 1+(z/d)^2$.

[1] Diana Marcela García Estévez, B. Lieunard, Frédérique Marion, B. Mours, Loic Rolland, and D. Verkindt. First tests of a newtonian calibrator on an interferometric gravitational wave detector. 2018.

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