NCal Ratio Method Analysis

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1 Introduction

The LIGO Newtonian Calibrator was designed with two different mass configurations, a quadrupole and a hexapole, with the goal of using the ratio of the forces caused by the two configurations to measure the radial distance. The gravitational interaction between the rotor and the test mass strongly depends on the radial distance between the two. Thus the hope was that the measured ratio would yield a low uncertainty measurement of this distance which could then decrease the overall uncertainty on the expected force. For the injections in O3 [1], this method was not used but instead precise surveying was used to determine the coordinates of the rotor.

The goal of this analysis was to determine if the ratio method can surpass the performance of the surveying. The "ratio method" is the procedure of using the measured ratio of the 2f and 3f forces caused by the NCal to estimate the radial distance. While "surveying" is using a surveying machine to measure the coordinates independently.

As a disclaimer, all uncertainties were assumed to be Gaussian and the uncertainties were propagated analytically. The numerical values of a rigorous analysis may differ from what's found here but the general results should still hold.

2 Surveying

The survey is done in Cartesian coordinates and gives a rotor position of:

$$x = -722.8 \text{ mm}^{+2.6\text{mm}}_{-2.6\text{mm}} \tag{1}$$

$$y = -933.0 \text{ mm}^{+1.6\text{mm}}_{-1.6\text{mm}} \tag{2}$$

$$z = 10.0 \text{ mm}^{+3.0\text{mm}}_{-3.1\text{mm}} \tag{3}$$

We want to transform to spherical coordinates since the ratio will give the radial distance. We'll treat the error bars as symmetric and take the largest of the two as the σ value.

The surveyed spherical coordinates are:

$$\rho = 1180.3 \text{ mm} \pm 2.0 \text{ mm } (0.17\%) \tag{4}$$

$$\theta = 89.51 \text{ deg} \pm 0.15 \text{ deg } (0.17\%) \tag{5}$$

$$\phi = -127.77 \text{ deg} \pm 0.11 \text{ deg } (0.09\%) \tag{6}$$

So the surveying yields a radial distance measurement with 0.15% precision. This is what the ratio method must surpass for us to want to use it.

3 Ratio Method

Lets write down what the force equations look like. Since the interferometer can only measure the x-component of the force, we will restrict ourselves to only the amplitude of the x-component. We will work in spherical coordinates where origin is at the center of mass of the test mass. The force at 2f and 3f follow:

$$F_{2f}^{x} = \frac{9}{2}\frac{GMmr_q^2}{d^4} f_2(\theta,\phi) \tag{7}$$

$$F_{3f}^{x} = \frac{15}{2}\frac{GMmr_h^3}{d^5} f_3(\theta,\phi) \tag{8}$$

where G is the gravitational constant, M is the mass of the LIGO test mass, m is the mass of the tungsten slugs, r_q is the quadrapole radius, r_h is the hexapole radius, d is the radial distance between the NCal and the test mass, θ and ϕ are the polar and azimuthal coordinates, and f_2 and f_3 are the angular dependence of the quadrupole and hexapole signals.

Taking the ratio of these two yields:

$$\frac{F_{2f}^{x}}{F_{3f}^{x}} = \frac{9}{15}\frac{r_q^2}{r_h^3}\frac{f_2(\theta,\phi)}{f_3(\theta,\phi)} d \tag{9}$$

This can be rearranged to give:

$$d = \frac{15}{9}\frac{F_{2f}^{x}}{F_{3f}^{x}}\frac{r_h^3}{r_q^2}\frac{f_3(\theta,\phi)}{f_2(\theta,\phi)} \tag{10}$$

Let's define three ratios as follows:

$$R = \frac{F_{2f}^{x}}{F_{3f}^{x}} \tag{11}$$

$$r = \frac{r_h^3}{r_q^2} \tag{12}$$

$$A = \frac{f_3(\theta,\phi)}{f_2(\theta,\phi)} \tag{13}$$

$$d = \frac{15}{9} RrA \tag{14}$$

We'll analyze each of these separately and combine them at the end to yield a measurement of the radial distance.

3.1 r

Since we know the geometry of the rotor with high precision, we expect that r will not contribute much uncertainty but let's calculate it for completeness. Pulling the r_q and r_h values and uncertainties from [1] gives:

$$r = 316.06 \text{ mm} \pm 0.07 \text{ mm } (0.022\%) \tag{15}$$

Note that this is the only factor which has units since it is a distance cubed over a distance squared.

3.2 R

R must be measured with the interferometer during two runs where the $2f$ frequency overlaps the $3f$ frequency. The current study[1] has one measurement like that right now at 15.5 Hz which yielded $F^x_{2f} = 19.59 \text{ pN} \pm 0.29 \text{ pN}$ and $F^x_{3f} = 9.16 \text{ pN} \pm 0.45 \text{ pN}$. Combining this gives:

$$R = 2.139 \pm 0.079 \ (3.67\%) \tag{16}$$

This is not the end of the story since in the future we can inject the signals for longer and get lower statistical uncertainty. The current measurements are only for 2 minutes and we can expect the uncertainty to fall like $1/\sqrt{\text{time}}$. If we project this out to a 60 hour measurement we get:

$$R_{60} = 2.139 \pm 0.0019 \ (0.087\%) \tag{17}$$

Which would be much below the uncertainty that is achieved with the surveying.

3.3 A

A is a bit harder to determine. Since the functional form of the angular dependence is not known, we have to rely on our force simulations to extract the ratio of the angular dependencies.

To do this, we calculate the 2f and 3f force distributions while only varying θ and ϕ. Since our code only gives us the force amplitudes, we need to correct the predicted force ratio with the input radial distance, $\tilde{d}$, and quadrupole and hexapole radii, $\tilde{r}_q$ and $\tilde{r}_h$, to get only the angular dependence.

$$\frac{f_3(\theta,\phi)}{f_2(\theta,\phi)} = \frac{9}{15} \frac{\tilde{r}_q^2}{\tilde{r}_h^3} \tilde{d} \frac{F_{3f}^{\text{predicted}}}{F_{2f}^{\text{predicted}}} \tag{18}$$

This is evaluated numerically using the Multipole method [1] in the script `NCal Angular.py`. We take the mean and standard deviation of the output distribution, Figure 1, as the central value and σ value for A.

$$A = 1.060 \pm 0.002\ (0.1434\%) \tag{19}$$

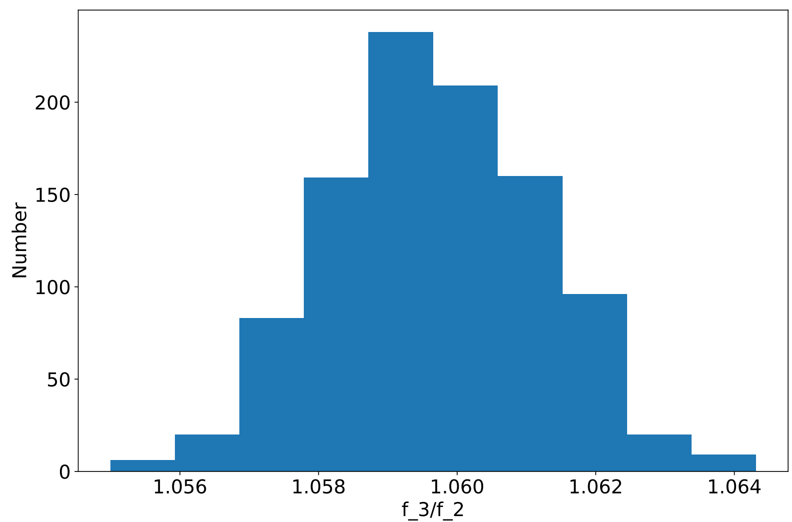

Figure 1: Distribution of A evaluated with the Multipole method developed for the NCal force predictions.

3.4 Combining

These three contributions can be combined to yield an estimate for the radial distance and uncertainty via:

$$d = \frac{15}{9} R r A \tag{20}$$

$$\sigma_d^2 = d^2 \left(\frac{\sigma_R^2}{R^2} + \frac{\sigma_r^2}{r^2} + \frac{\sigma_A^2}{A^2} \right) \tag{21}$$

The current measurements gives us:

$$d = 1.194\ \mathrm{m} \pm 0.044\ \mathrm{m}\ (3.67\%) \tag{22}$$

If we measured for 60 hours at each frequency (total of 120 hours of measurement) we would get:

$$d_{60} = 1.194 \text{ m} \pm 0.002 \text{ m } (0.17\%) \tag{23}$$

4 Comparison

We showed above that the ratio method can yield a lower uncertainty measurement of the radial distance after sixty hours of injection. Lets quantify the crossing point better and look at the infinite measurement time of the ratio method. Figure 2 shows the decrease in radial distance uncertainty using the ratio method over time as well as the current surveying performance and the precision that was originally assumed for the surveying, 1 cm uncertainty.

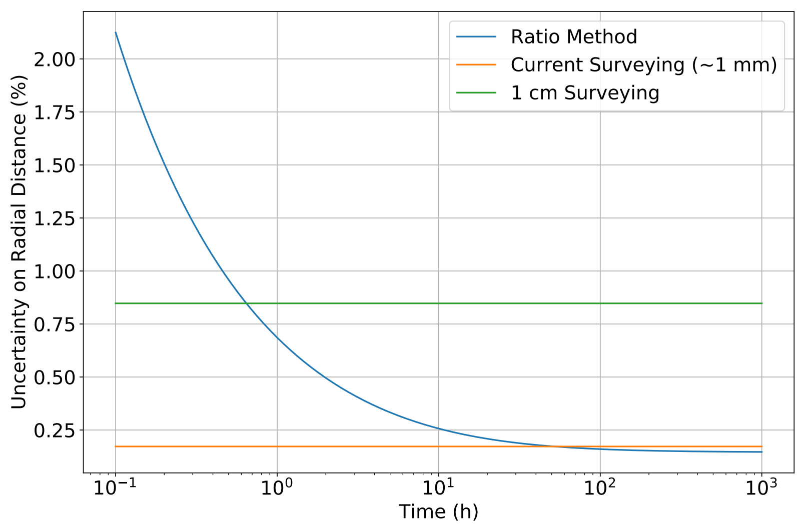

Figure 2: Uncertainty vs. measurement time for the ratio method. Also shown is the uncertainty of the current surveying and surveying with 1 cm precision.

The ratio method crosses over the surveying after 56 hours. This would require 112 hours of actual measurement since the two frequencies have to be injected at different times.

The infinite measurement limit of the ratio method can be calculated by setting $\sigma_R = 0$. This yields a radial distance of:

$$d_\infty = 1.1938 \text{ m} \pm 0.0017 \text{ m } (0.15\%) \tag{24}$$

5 Conclusion

This analysis has shown that the ratio method can be used to determine the radial distance better than the surveying, and thus decrease the systematic uncertainty of the NCal calibration, after about 56 hours. True measurement time would be double this since we need two different injections to get the 2f and 3f to be at the same frequency.

In the limit of infinite measurement time the ratio method is constrained to an accuracy of approximately 0.15 %. This is dominated by the surveying precision of θ and ϕ which is unlikely to significantly improve in the current geometry.

Other geometry, such as aligning the NCal rotor with the interferometer beam, may allow for lower uncertainties on radial distance extracted with the ratio method. However, surpassing the performance of the current surveying is a tall order do to the long measurement times.

References

[1] P1900244.