NCal Least-Squares Spectral Analysis

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1 Introduction

This document describes the extraction of the force amplitudes injected with the Newtonian calibrator [1] using Least-Squares Spectral Analysis (LSSA). For a detailed description of LSSA see [2].

The NCal injects forces at two specific frequencies: twice the rotation frequency of the rotor ($2f$) and three times the rotation frequency ($3f$). The goal of this analysis is to extract the force amplitude at each frequency. This can then be compared to the expected force amplitude to yield a calibration of the interferometer at both frequencies.

2 Least-Squares Fitting

The interferometer measures strain, h, at discrete, evenly-spaced time, t_i. We will focus on just the stretch of time when the NCal was injecting forces giving us a strain vector with n measurements:

$$h(t_i) = [h_1, h_2, h_3, ...h_n] \tag{1}$$

We want to fit this to a discrete set of frequencies containing the sine and cosine at both $2f$ and $3f$. The model we will fit to is then:

$$\begin{aligned} \hat{h} =& a_2 \cos(2\pi\ 2f\ t) + b_2 \sin(2\pi\ 2f\ t) \\ &+ a_3 \cos(2\pi\ 3f\ t) + b_3 \sin(2\pi\ 2f\ t) \end{aligned} \tag{2}$$

Since we only care about the values of this function at discrete times, t_i, this can be recast into vector notation as:

$$\hat{h} = \mathbf{X}\beta \tag{3}$$

with

$$\beta = [a_2, b_2, a_3, b_3]^T \tag{4}$$

$$\mathbf{X} = [\cos(2\pi\ 2f\ t_i), \sin(2\pi\ 2f\ t_i), \cos(2\pi\ 3f\ t_i), \sin(2\pi\ 3f\ t_i)]^T \tag{5}$$

We call the columns of $\mathbf{X}$ the basis functions. Explicitly writing out the dimensions of these matrices gives:

$$y = n \times 1 \tag{6}$$

$$\beta = 4 \times 1 \tag{7}$$

$$\mathbf{X} = n \times 4 \tag{8}$$

Using linear-least squares fitting, we minimize the residual sum-of-squares, R,with respect to β :

$$R = (y - \mathbf{X}\beta)^2 \tag{9}$$

$$R = (y - \mathbf{X}\beta)^T(y - \mathbf{X}\beta) \tag{10}$$

At the minimum:

$$\frac{\partial R}{\partial \beta} = 0 \tag{11}$$

Evaluating the derivative given us:

$$0 = -2\mathbf{X}^T(y - \mathbf{X}\beta) \tag{12}$$

Rearranging:

$$\beta = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T y \tag{13}$$

Note that this simply one line of matrix multiplication and guarantees that the frequencies are fit to are exactly the frequencies of interest.

Since Equation 13 is linear, then the uncertainties on the β values, σ_β, are directly related to the uncertainties of the strain measurements, σ_h:

$$\sigma_\beta^2 = (\mathbf{X}^T\mathbf{X})^{-1}\sigma_h^2 \tag{14}$$

3 Cut-Based Analysis

We don't necessarily know the uncertainties of the strain measurements so we can't derive the uncertainties on the fit parameters, β. To get around this we split our data up into cuts and fit each cut separately. To satisfy orthogonality conditions [2], the cuts must an integer number of periods for both the $2f$ and $3f$ frequencies.

We achieve this by using a cut length of 20 $1f$-periods. In one $1f$-period there are two $2f$-periods and three $3f$-periods. The number of periods to use is a trade off between having enough data points per cut to get a quality fit to the data and having enough cuts to give a good description of the underlying distribution. We chose to use 20 periods but the results do not change appreciable with different choices.

If the fit parameter distribution follows a Gaussian distribution then we can use standard equations to extract the mean and the standard deviation of the mean:

$$\alpha_2 = \frac{1}{N}\sum_j a_{2,j} \tag{15}$$

$$\sigma^2_{\alpha_2} = \frac{\sigma_a^2}{N} \tag{16}$$

where $a_{2,j}$ is the cosine amplitude for the $2f$ frequency from the jth cut, α_2 is the mean of the collection of cosine amplitudes for the $2f$ frequency, and N is the number of cuts.

The mean strain amplitude for the $2f$, A_2, is then calculated using:

$$A_2 = \sqrt{\alpha_2^2 + \beta_2^2} \tag{17}$$

where α_2 and β_2 are respectively the mean of the collection of cosine and sine amplitudes for the $2f$ frequency. And similarly the strain amplitude at the $3f$, A_3,:

$$A_3 = \sqrt{\alpha_3^2 + \beta_3^2} \tag{18}$$

The uncertainty of the strain amplitudes are then calculated using:

$$\sigma_i^2 = \frac{1}{A_i}(\alpha_i^2\ \sigma^2_{\alpha_i} + \beta_i^2\ \sigma^2_{\beta_i}) \tag{19}$$

where i indicates either the $2f$ or $3f$ amplitude.

Although not used in the current version of the NCal, the phase, ϕ_i, can be found via:

$$\phi_i = \arctan(\beta_i/\alpha_i) \tag{20}$$

4 Conversion to Force

Since we can predict the force that the NCal causes, we need to convert the strain amplitudes to force amplitudes using the following:

$$F_i(f) = L\ A_i(f)/S(f) \tag{21}$$

where f is frequency, A_i is the strain amplitude for the $i\ f$ frequency, L is the length of the arm, and S is the force-to-displacement response of the suspension system.

The full force-to-displacement response functions[3] for both forces on the penultimate mass (PUM) and the test mass (TST) are shown in Figure 1. At the frequencies of the NCal injections (10-30 Hz), the response function is well approximated by $S(f) = -1/M\omega^2$. So we simplify Equation 21 to:

$$F_i(f) = (2\pi\ 2f)^2 M\ L\ A_i(f) \tag{22}$$

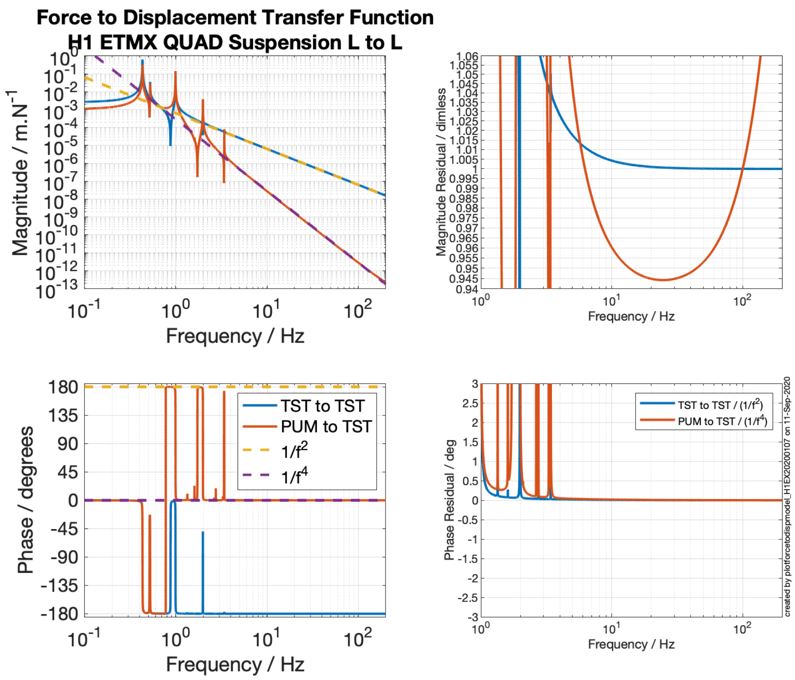

Figure 1: Force to displacement transfer functions for the ETMX suspensions.[3]

5 Measurements

During O3, we injected forces with the NCal at a variety of rotation rates. We ran the above calculations for each injection separately which yielded the measurements shown in Table 1. The code used for this extraction can be found at `https://git.ligo.org/laurence.datrier/ncal-codebase/-/tree/master/Measurements`

$1f$ (Hz)	$2f$ (Hz)	F_2 (pN)	$3f$ (Hz)	F_3 (pN)
4.16	8.32	–	12.45	8.96 ± 0.31
5.17	10.34	$16.90^{\pm 2.33}_{\pm 13.79\%}$	15.51	$9.16^{\pm 0.45}_{\pm 4.91\%}$
7.80	15.60	$19.59^{\pm 0.29}_{\pm 1.48\%}$	23.40	$9.30^{\pm 0.37}_{\pm 3.98\%}$
8.55	17.11	$19.41^{\pm 0.46}_{\pm 2.37\%}$	25.66	$9.19^{\pm 0.57}_{\pm 6.20\%}$
9.56	19.11	$19.60^{\pm 0.21}_{\pm 1.07\%}$	28.67	$9.33^{\pm 0.24}_{\pm 2.57\%}$

Table 1: Measured force amplitudes, F_2 and F_3, at the expected $2f$ and $3f$ frequencies.

References

[1] P1900244.

[2] T2000574.

[3] T080188.