Detecting Non-Power Law Stochastic Gravitational Wave Background

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The Stochastic Gravitational Wave Background (SGWB) is a consistent signal composed of a combination of many unknown sources. Since the SGWB is continuous, there is information on a much larger scale with the hope of detecting remnants of the early universe in the background. Current models work well to describe SGWB with current detector sensitivity where the SGWB can be described by a simple power-law. However, common theories predict a turnover that will be detected with future detectors' sensitivity; this will lead to inconsistencies if current models are used. Since there is so much we do not know yet of the unknown sources it is pivotal to design a general and generic model to detect a SGWB that does not characterize as a simple power law. We use a new method of the Bayes factor along with WESTLEY, to do generic fitting when describing non-power law injections, to detect the SGWB. We show that this Bayes factor comparative results to those of the optimal signal to noise ratio. Our generic model was tested on simulated data and used splines and linear interpolation instead of power laws. We intend to later on show that, for non power-law models, our Bayes factors are more sensitive than those calculated with traditional models.

I. INTRODUCTION

Gravitational waves (GW) are ripples in space time that are initiated from extremely energetic sources. Known sources, in increasing order of how difficult they are to detect, include chirps from coalescing binary systems, periodic sources from pulsars, and bursts from supernovae [1]. Sources that are random, with multiple uncorrelated events, are called stochastic gravitational-wave backgrounds (SGWB). Unlike deterministic sources that last for a certain amount of time, the SGWB is always present. The SGWB can include events directly following the Big Bang, or more recently-generated signals that we can't necessarily individually detect.

LIGO's first detection in 2015 was a groundbreaking discovery that resulted in a Nobel prize. These individual events can inform us about the occurrences of stellar objects in the nearby Universe. However, looking at the SGWB can provide information on a much larger scale with much hope that we could detect GW from early universe. Models are created to predict how certain sources contribute to a SGWB and when an SGWB might be seen. We can do the reverse, and use the data to estimate the values of the parameters associated with each model.

Some models do not have a simple functional form such as a SGWB from a binary coalescences where the signal is generated from many individual events adding together. With current detectors the collection of binary coalescences can be well-described by the simple analytic power law [2]. However, as detectors become more sensitive, we expect to see a "smooth" turnover, which we have not been able to describe analytically. Additionally, we may see compact binary coalescences (CBCs) contribute a similar amount to the SGWB as other sources, like those from the Big Bang. Although each source might be described by a power law, its sum is not so easily analytically described. Consequently, we need a method that will characterize a SGWB of any "smooth" type.

II. BACKGROUND

A. Current Models

The SGWB is described by the dimensionless energy density in gravitational waves in the Universe per logarithmic frequency bin,

$$\Omega_{GW}(f) = \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln f}. \quad (1)$$

ρ_c is the critical energy density for a flat Universe and $d\rho_{GW}(f)$ is the energy density in GWs in the the frequency bin f to $f + df$. Currently it is assumed that the gravitational wave background (GWB) spectrum is a power law,

$$\Omega_{GW}(f) = \Omega_{GW}(f_{ref})\left(\frac{f}{f_{ref}}\right)^{\alpha}. \quad (2)$$

Where $\Omega_{GW}(f)$ is the GW energy density, f_{ref} is a reference frequency and α is the spectral index of the signal. Ω_{GW} and α are estimated. Although Ω_{GW} is usually considered a cosmological quantity here it is also used to describe the energy from astrophysical events so that we can compare them to cosmological sources [3].

Current methods to identify the GWB are signal to noise ratio (SNR) and the Bayes Factor. SNR uses the ratio of signal to noise where there is more weight on frequencies that have less uncertainty. This is done by inverse noise weighting, where C is the signal and σ is noise.

$$SNR(f) = \frac{\hat{C}(f)}{\sigma(f)},$$

$$C_{TOT} = \frac{\sum_{i=1}^{N} \hat{C}(f)/\sigma(f)^2}{\sum_{i=1}^{N} \sigma(f)^{-2}}; \tag{3}$$

$$\sigma_{TOT} = (\sum \sigma(f)^{-2})^{-\frac{1}{2}}$$
$$SNR_{TOT} = \frac{C_{TOT}}{\sigma_{TOT}}$$

The SNR_{TOT} tells us how greater the signal is to the detector noise. For GWB searches, when $SNR_{TOT} = 3$ it is considered as evidence of a GWB and a $SNR_{TOT} = 5$ is a detection of a GWB. Bayes factor is similar to SNR in which the factor is that of the data containing a signal divided by the probability the data is consistent with just noise in the detector,

$$\textit{Bayes factor} = \frac{P(\hat{C}|\text{signal})}{P(\hat{C}|\text{noise})}. \tag{4}$$

The probability is based on the posterior probability of the algorithm's parameters given the data. θ is used for signal and 0 is for noise,

$$P(\vec{\theta}|\hat{C}_i) = \frac{P(\hat{C}|\vec{\theta})P(\vec{\theta})}{P(\hat{C})}$$
$$P(0|\hat{C}_i) = \frac{P(\hat{C}|0)P(0)}{P(\hat{C})} \tag{5}$$

The evidence is then given by the integral over the numerator. So the integral for signal in eq.5 is,

$$P(\hat{C}|\text{signal}) = \int d\vec{\theta} P(\hat{C}|\vec{\theta})p(\vec{\theta}). \tag{6}$$

In our case for a power-law of SGWB, shown in equation 2 the parameters in $\vec{\theta}$ is amplitude A and spectral index α . Both the SNR and Bayes factors perform well with current detectors and are the base work for a newly proposed method for future detectors. In section III we will discuss how $\vec{\theta}$ is used in our interpolation model, where this method uses the height of the control points that are interpolated between those parameters.

In figure 1, there are different versions of the SGWB from compact binary coalescences with the assumption of it made up completely of mergers, along with the estimated GWB sensitivity of LIGO detectors. The colored curves in Fig 1 are well-described power-laws until a certain point were they turn over. The gray sensitivity curves show that for realistic curves (i.e. those with chirp masses below 50 $M_{\odot}$), current generation detectors are not sensitive to this turnover.

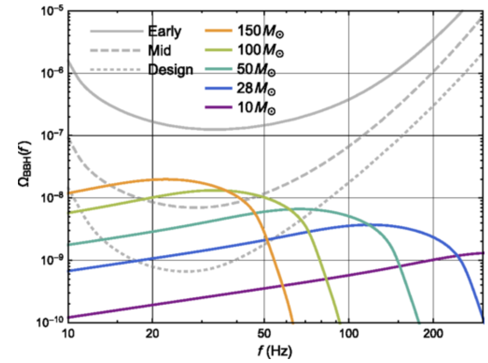

Figure 1. Figure reproduced from [2]. Binary black hole's background with various chirp masses with the Fiducial model for SGWB (colored lines). Power-law integrated curves for one year with Advanced LIGO (grey lines).

This turnover is the astrophysical GWB's non-analytical piece. The turnover depends on the masses of the black holes and becomes more complex as different populations are added to it. Future detectors will be sensitive to these turnovers which can clarify characteristics for CBCs such as: the time it takes for a star to merge in a binary, if properties of the universe contribute to the formation and/or mass of a black hole, and how the populations of masses and spins of a neutron star and a black hole that enter a binary look like in our spectrum. In addition, there is an unknown cosmological background in CBCs that are not black holes, which motivates another reason to create a new model since we simply do not know what the sources are and how to define them.[3].

B. Evolution of Models as Detectors Improve

Once detectors become more sensitive there will be an abundance of individual events. To search for the SGWB, we will subtract the loudest events from the data, which will then change the spectrum of the SGWB. An example of an expected SGWB after subtracting individual events is shown in Fig 2. Evidently the spectrum is no longer a straight line in log space but it does appear "smooth".

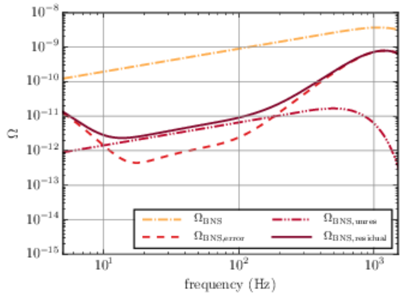

Figure 2. Figure reproduced from [4], the SGWB from all neutron stars (doted orange line) plotted together with background from unresolved (unsubtracted) neutron stars (dotted red line), and the sum of the two (red sold line). With the neutron star summed with unresolved background the line is now longer a power law.

A general and generic model can effectively detect a SGWB of any "smooth" shape at higher sensitivity. The paper [4] predicts most models with a "smooth" look therefore, a new model must capture similar values or change smoothly from one frequency bin to the next. We propose an interpolation model using hybrid statistics to detect general models. The interpolation model will takes regularly used method of spline fitting to fit smooth looking curves. As we will discuss in the next section, this new model can also be used to develop a consistency check to verify a detection.

III. METHODS

A. Detection

To detect a SGWB we will cross-correlate data between detectors. In the following equations the tilde indicates the use of the Fourier transform. The detector data ($\tilde{s}_i$) involves both GW signal ($\tilde{h}$) and noise ($\tilde{n}$) with the dependency of frequency,

$$\tilde{s}_1(f) = \tilde{h}_1(f) + \tilde{n}(f). \tag{7}$$

The data of the two detectors is then combined into a cross-correlation statistic ($\tilde{C}(f)$) in every frequency bin [5],

$$\tilde{C}(f) = \frac{2}{\tau} \frac{Re[\tilde{s}_1^*(f)\tilde{s}_2(f)]}{\gamma_f S_0(f)}, \tag{8}$$

where Re indicates the real part of the cross correlation, τ is the time over which we are analyzing data, and the normalization includes cosmological constants, as well as the overlap reduction function $\gamma(f)$, which we discuss soon. We can substitute Eq 8 into Eq 7 and take an average:

$$\begin{aligned}\langle \tilde{s}_1^*(f)\tilde{s}_2(f)\rangle = \langle \tilde{h}_1^*(f)\tilde{h}_2(f)\rangle + \langle \tilde{n}_1^*(f)\tilde{h}_2(f)\rangle \\ + \langle \tilde{h}_1^*(f)\tilde{n}_2(f)\rangle + \langle \tilde{n}_1^*(f)\tilde{n}_2(f)\rangle.\end{aligned}$$

We then assume that the signal is uncorrelated with detector noise and the noise between the two detectors is uncorrelated. Therefore,

$$\langle \tilde{s}_1^*(f)\tilde{s}_2(f)\rangle = \langle \tilde{h}_1^*(f)\tilde{h}_2(f)\rangle. \tag{9}$$

Next, we note

$$\frac{2}{\tau}\langle \tilde{h}_1^*(f)\tilde{h}_2(f)\rangle = H(f)\gamma(f). \tag{10}$$

Here $H(f)$ is called the gravitational wave power, and the proportionality constant $\gamma(f)$ is called the overlap reduction function. The overlap reduction function is a weight function in frequency that quantifies what fraction of the GW power our detectors are sensitive to. Thus, $\gamma(f) = 1$ means we see all of the GW power in our cross-correlation, but $\gamma(f) = 0.5$ means we see only half of the GW power. Since we know exactly what detectors we are using its value is exactly known [3]. This frequency dependence provides a great insight into the frequencies to which the detectors are most sensitive.

When we substitute Eq. 10 back into Eq. 8, we find that in general,

$$\langle \tilde{C}(f)\rangle = \frac{H(f)}{S_0(f)} = \Omega_{GW}(f) \tag{11}$$

Where the constants $S_0(f)$ are used so that we have the cross-correlation proportional to the energy density.

The shape of the cross-correlation is what we want to model. $\tilde{C}(f)$ is calculated by taking the cross-correlation between our detectors for numerous short time intervals then taking the weighted average of all the runs. We then compare the averaged $\tilde{C}(f)$ to power law spectra to verify if there is any evidence of power law. What we propose to do here, is to instead compare to more generic "smooth" functions like splines or Gaussian processes. Then proceed to test our model with simulated data.

B. Interpolation Model

The LIGO collaboration has several libraries and pipelines that are used specifically for frequency domain spectra. Our work mainly uses the library BIBLY and PYGWB as well as the pipeline PYGWB_PIPE. Our method uses Bayesian statistics, like BIBLY, and a hybrid approach, executed by PYGWB. This library has several functions that are needed such as initializing a Power Spectral Density and generating a prior in log uniform and Gaussian distribution. PYGWB_PIPE is the main pipeline to work with SGWB that uses the library PYGWB. We used this pipeline to simulate data from all three detectors Hanford, Livingston, and VIRGO to then inject a common GW signal with a chosen spectrum into our current data.

The algorithm WESTLEY uses these libraries as the ground work for an interpolation model. This code holds a big chunk of what we will be testing. Our model will have control points, referred to as knots, to interpolate between to create a best fit model to the injection. With this package we can interpolate between the knots using either a piece-wise power law or a cubic spline and run it a number of times to find the probability distribution on where the knots should be placed and turned on to have the best fit. This paper shows results from the model using piece-wise power law. Rather than taking random guesses to find the probability distribution we use the technique Reverse Jump Markov chain Monte Carlo (RJMCMC).

A Markov chain Monte Carlo (MCMC) is used to obtain a preferred probability distribution on some set of parameters. For our model, we want to find the probability distribution on the height of each knot and how many are used. With MCMC we are able to make intelligent guesses to where the knots should be placed. A new knot's position is accepted depending on how well the "new" guess will let the model fit the data compared to its current, "old" guess. The likelihood function L tells us how well the model fits the data, and we use a Gaussian likelihood function. The placement of the knot is kept whether,

$$P(accept) = 1, \text{if point has } L_{new} > L_{old}$$
$$P(accept) = \frac{L_{new}}{L_{old}}, \text{if point has } L_{new} \leq L_{old}$$

The proposed distribution function tells the algorithm how we decided to guess. The way we guess does not interfere with the results, however, it does affect the efficiency of the algorithm time wise.

Now, RJMCMC works similar to the previously described MCMC with the addition of dimension jumping, allowing for unknown parameters. This works well for our project since we tested different types of signals consisting of undefined parameters.

C. Statistics

Our Model uses a newly proposed Bayes factor. This Method takes concepts from "SNR" and "Bayesian Factor" methods of searching for SGWB. Since the knots in our model are used to fit a signal, they should only be present for a signal. Therefore, we should have 1 or more knots when there is a signal present and no knots for only noise. Our method is established as,

$$\mathcal{B}^{\mathrm{M1}}_{\mathrm{M2}} = \frac{N_{\geq 1}}{N_0}, \tag{12}$$

where $M1$ is the signal model fit and $M2$ is the noise model fit at which the $\mathcal{B}$ is determined. Each interpolation model runs the RJMCMC for a certain amount of iterations and ending with a set of samples. $N_{\geq 1}$ is the number of samples with more than one knot, indicating a signal is present. N_0 is the number of samples with zero knots meaning the data is consistent with noise.

IV. TESTING WESTLEY'S ALGORITHM

To see if our algorithm works we tested it on an injected GWB. We used O4 simulated data, which was reduced to just the signal ($\hat{C}$). This was cross correlated with the Hanford and Livingston detector. We set the model to have a max of 20 knots and ran the RJMCMC for 10^5 steps. WESTLEY burned the first half, and then recorded the parameters at only every 10^{th} step. Figure IV shows a comparison between the injected signal (orange), the data (blue and green), and the recovered signal for 500 MCMC steps (black).

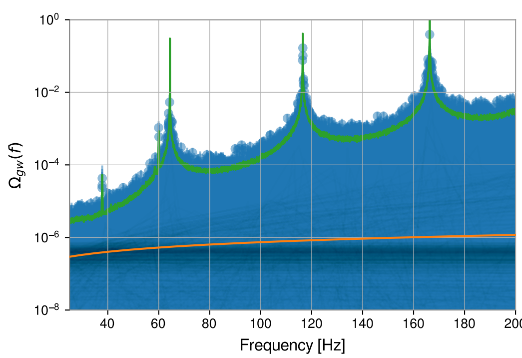

Figure 3. The figure shows the energy density estimated from the data in blue, the uncertainty in green, the energy density of our proposed injected power law in orange and the interloped knots in black. WESTLEY relatively fit the injected signal throughout the runs.

Since this is in log space our injection is mostly linear and most of the simulated runs from WESTLEY fit along the power law. Therefore the code has identified the signal. Figure 4 is a visual representation of which knots were in used in each step of the RJMCMC chain.

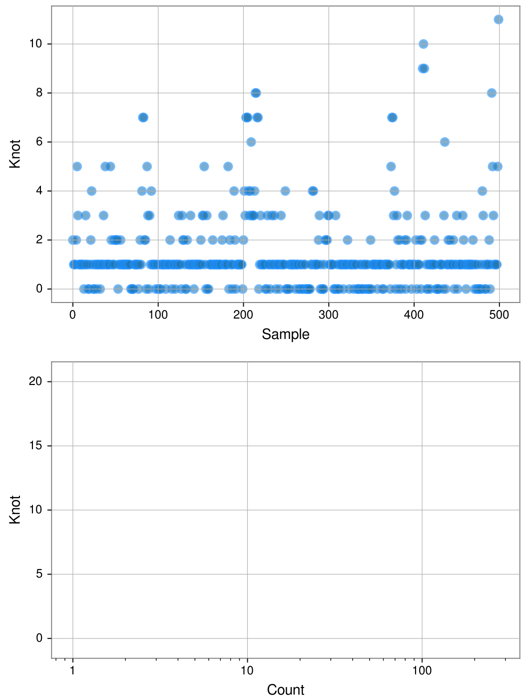

Figure 4. Top, shows when the knot was turned on during a certain run. Bottom, is a histogram to visualize how many times the certain knot was used for all runs.

The model we tested had spaced out the knots evenly throughout the spectra and made its own choice when the knots should be turned on or off. Figure 4 illustrates the height of each bin which we can use the sum of the heights for all the bins with the y-axis ≥ 1 and divide by the height of the bin at zero to get a log Bayes factor ($\ln(\mathcal{B})$) of around 1.3.

After WESTLEY's proficiency was confirmed with generic GW spectrum it was then modified to be applied to a realistic CBC injection.

V. INTERPOLATION MODEL'S PATH TO DETECTING A REALISTIC GWB

We divided CBC signal's characteristics into 3 degrees. First degree was a power-law, the second degree introduced a break or peak in the signal which was defined by a piecewise power-law, and the third degree then contained the turn over a realistic CBC signal. These degrees are shown in the following figure with data from O4.

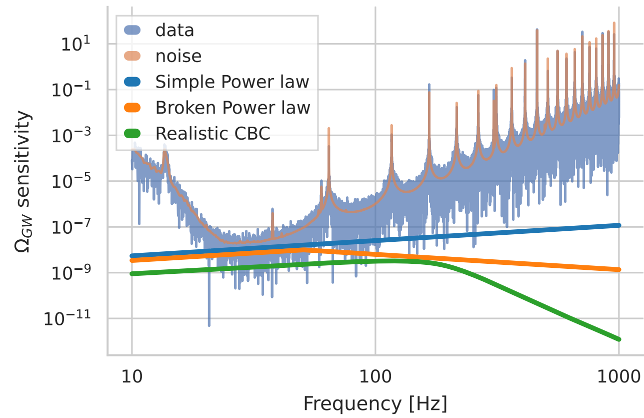

Figure 5. Figure shows the different degrees of a realist CBC signal in log space. First degree is a power-law (blue), second degree is a broken power law with peak as 50Hz(orange), and the third degree is the realistic CBC signal where the turnover come after around 100 Hz (green). Our signals were modeled with O4 simulated data.

We analytically described the simple power-law $\Omega^{\rm PL}_{\rm GW}(f)$ and broken power-law $\Omega^{\rm BPL}_{\rm GW}(f)$ then used a realistic signal from Taylor Knapp's project. Below are the equations used for the first two degrees

$$\Omega^{\rm PL}_{\rm GW}(f) = \Omega_{ref}\left(\frac{f}{f_{ref}}\right)^{\alpha}, \tag{13}$$

$$\Omega^{\rm BPL}_{\rm GW}(f) = \begin{cases} \Omega_{peak}(\frac{f}{f_{peak}})^{\alpha_1} & \text{for } f \geq f_{peak}, \\ \Omega_{peak}(\frac{f}{f_{peak}})^{\alpha_2} & \text{for } f > f_{peak}. \end{cases} \tag{14}$$

The realistic signal came in an array of $[\Omega^{\rm CBC}_{\rm SGW}(f), f]$ and was modified to align with the O4 data.

A. Modifications to westley

The WESTLEY package has an inheritance package called SIM_TO_WESTLEY where there are three methods for each type of degree. The first and second injection model was parameterized by it amplitude Ω_{GW}. The third degree was only parameterized by time in observation since this signal amplitude was already calculated based on the population of black holes that Taylor used to create the injection. Each method calculates SNR^2_{TOT} eq.3 and the log of our proposed Bayes factor eq.4 $\ln(\mathcal{B})$. Instead of simulating time series data using PYGWB we generate Gaussian noise with the expected variance of $\hat{C}(f)$, which is given by,

$$\sigma^2(f) = \frac{1}{2\delta f}\frac{P_1(f)P_2(f)}{\gamma^2_T(f)S^2_0(f)}, \tag{15}$$

$$S^2_0(f) = \frac{3H^2_0}{10\pi^2 f^3}, \tag{16}$$

between the detectors Hanford and Livingston.

B. Results

To attain a larger sample size of our general fitting we used a script designated to each type of degree that decreases the total run time. These scripts set the signals parameter to run through ranges spaced out evenly in log space to have a sample size of 100. For example, for the injection model 1, we performed 100 injections with different amplitudes of $\Omega_{\rm GW}$ given by equation 13, spaced logarithmically between e^{-9} $-7e^{-9}$. It also set WESTLEY to run with an 10^7 RJMCMC iterations. This script ran each defined parameter job in parallel on Condor.

To check WESTLEY's performance we compared $\ln(\mathcal{B})$ to the SNR^2 of each model. It is known that $\ln(\mathcal{B})$ and SNR^2_{TOT} are proportional to each other so we expected our models to show a linear characteristic between the two. Fig. 6 shows all three injections' $\ln(\mathcal{B})$ vs. SNR^2_{TOT} with the linear characteristic. This verified the proficiency of our method.

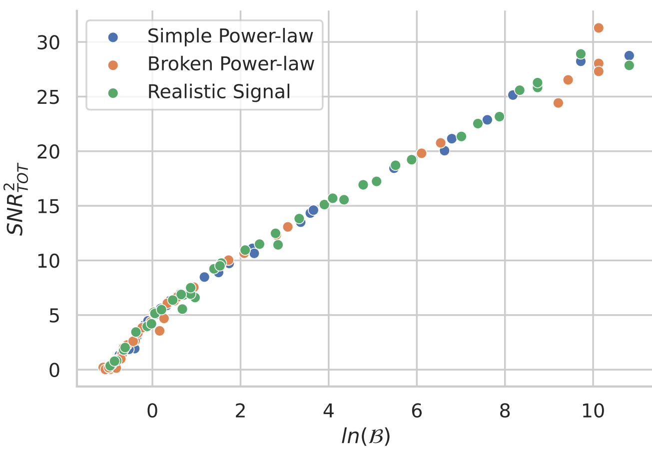

Figure 6. The figure above shows all 3 degrees to reaching a realistic CBC signal. Each degree had a genetic fitting with WESTLEY with 100 samples and had 10^7 iterations of the RJMCMC. The first degree, simple power-law, samples are in blue. The second degree, broken power-law, samples are in orange. The final degree, realistic CBC signal, samples are in green. We expected all injected signals to a linear characteristic after a $\ln(\mathcal{B})$ of 1 due to the proportionality ofSNR^2_{TOT} to $\ln(\mathcal{B})$.

We tested the simple power-law and broken power-law's Bayes factor in term of the injection amplitude. The $\ln(\mathcal{B})$ is our reference of how strong the detection is since we use the number of samples N that have 1 or more knots turned on over the number of samples with no knots turned on eq.12. In figure7 we see that as the Ω_{GW} increased the $\ln(\mathcal{B})$ increased exponentially. This was an expectation as well as confirmation that we would be able to detect and now model both types of signals.

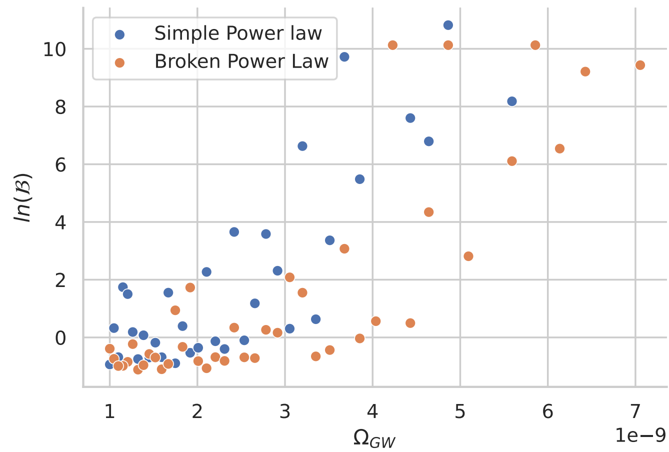

Figure 7. Both simple power law and broken power law log $\mathcal{B}$ increased quadratically as the Ω_{GW} increased. The data attributes validates the models potential to detect simple and broken power-laws

Since the realistic CBC signal was calculated in a different manner, time was the variable that we could vary for our data sample. Figure 8 shows the detection probability increase as the years increase.

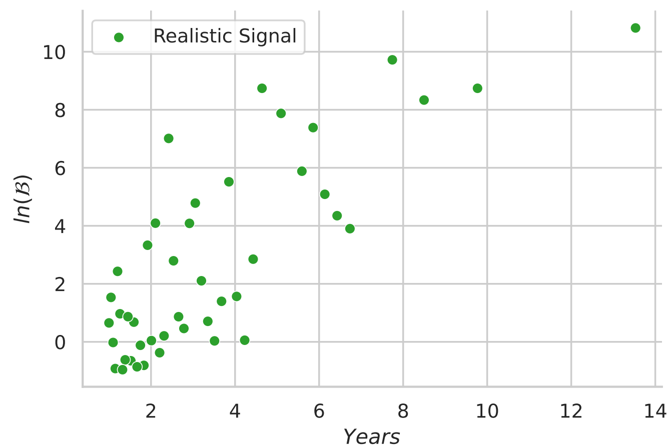

Figure 8. The injected realistic CBC signal $\ln(\mathcal{B})$ is compared to time in terms of years. From the data given at log Bayes factor of 10 or 12, we inferred that could see this signal in about 10 years.

Since an SNR_{TOT} of 5 is the benchmark of a detection, a $\ln(\mathcal{B})$ of around 10 indicates detection of a SGWB. In the previous figure we see that our interpolation model suggest a detection of this injection after 10 years.

Although the injected realistic CBC would only be seen after 10 years, our data confirms that realistic CBCs can be modeled with WESTLEY.

C. Future Work

Our project was to created a general and generic model and through our project we have shown that WESTLEY can detect from present detectable sources like a power-law model and non power-law model. Our data also suggests the proposed Bayes factor method has the same potental of comparing a model's aptitude from another as current used methods. The interpolation model can be applied to varying signals and therefor affirmation to its general and generic applications. WESTLEY and its collaborating scripts can be applied to diverse projects focusing on GWB. Although our results were calculated with simulated O4 data, it is possible to use other simulated data such as O5, next generation detector data, and past LIGO observing runs. From my colleague's project where Taylor worked on what types of shape can the SGWB have, our general and generic model can interpret when this shape can be detected and when will the SGWB have this shape compared to a power-law as the project continues.

VI. ACKNOWLEDGEMENT

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