Testing Universal Relations under Nonparametric Equations of State

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Neutron stars have long been a point of interest in astronomy due their extreme qualities. Of these, the core of these super-dense star remains of especially large interest. Due to the extremely high densities that are present inside of these objects, the way matter moves and acts is unknown. At the Laser Interferometer Gravitational-Wave Observatory (LIGO), this knowledge is very important as it would give data we collect on gravitational waves much more power in terms of deducing properties of the neutron stars that cause them. We hope to be able to, through the course of this project, see if possible relations between gravitational wave data we collect and neutron star properties can be found without necessarily knowing how matter acts in these super-dense states with what we call "Universal Relations". We aim to test these relationships and see if they truly hold regardless of the Equation of State (EoS) that controls the relation between pressure and density in neutron stars rigorously so as to ascertain their validity and reliability. With this, we will better be able to measure neutron star properties. If these relations prove to be fruitful in their utility, and gain insight into possibly incorrect assumptions about how matter behaves in neutron stars if these relationships prove to be less coherent than previously thought.

I. BACKGROUND

A. Motivation

Through gravitational waves (GWs), great strides have been made in constraining neutron star (NS) observables, such as mass, radius, tidal deformability, etc. However the cold, supranuclear matter present inside neutron stars follows a relationship between pressure and density, an equation of state (EoS), that is unknown at this current moment. This "nuclear EoS" is of great interest to many fields, but is especially viable to LIGO as it would allow the construction of relations between different NS observables. But, without the ability to easily reconstruct these densities in a traditional laboratory environment, NS observables remain our best path of insight into this EoS. However, this gives rise to many problems, as limitations in GW detection gives rise to many degeneracies in NS properties and between NS properties and the nuclear EoS, making it difficult to ascertain nuclear EoS characteristics.

However, Yagi and Yunes [1] have formulated candidate relations between NS observables that may hold regardless of the nuclear EoS that characterizes the makeup of NSs. Some of these relationships include a relation between moment of inertia, tidal deformability, and the quadrupole moment ("I-Love-Q"), a relation between compactness and tidal deformability ("C-Love"). With these we will also be inspecting a relation between the ratio of pressure and density in the center of a NS versus its total compactness [2]. These so called "Universal Relations" (URs) are especially powerful as they not only would allow us more inference into the nature of the nuclear EoS, they would also greatly empower the measurements from LIGO by allowing measurements of tidal deformability to be related to many other NS properties we'd otherwise wouldn't be able to get strong restrictions on. It is with these strengths and others that many have been motivated to test these proposed URs and examine their validity.

B. Previous Work

In order to test if these proposed relationships are actually universal and thus EoS-agnostic, experiments have been conducted that have created these relations under many different test EoS in order to assess if high variability in EoS results in high variability in the relations [2, 3]. These experiments were done with "parametric" EoS, meaning that the method in which EoS were generated was by choosing a form for the relation with parameters that can be changed to create many different EoS models. Largely, 2 models were used, the spectral model and the piecewise-polytrope model.

The spectral model, as detailed by 1, conventionally has $x \equiv P/P_0$, where P_0 is the smallest pressure considered in analysis, and n chosen to be 3 such that the parameters to be generated are then $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$.

$$P = K\rho^{\Gamma(x)} \quad \text{where} \quad \Gamma(x) = \sum_{i=0}^{n} \gamma_i (\log(x))^i \quad (1)$$

The piecewise-polytrope model as detailed by 2, on the other hand, picks ρ_1 and ρ_2 to fit current models of the nuclear EoS, K_1 is fit to some chosen $P_1 \equiv P(\rho_1)$, and K_2 and K_3 are chosen to ensure continuity such that the parameters to generate are then $\{\Gamma_1, \Gamma_2, \Gamma_3, P_1\}$.

$$P = \begin{cases} K_1\rho^{\Gamma_1} & : \rho < \rho_1 \\ K_2\rho^{\Gamma_2} & : \rho_1 < \rho < \rho_2 \\ K_3\rho^{\Gamma_3} & : \rho_2 < \rho \end{cases} \tag{2}$$

Both of these methods attempt to create EoS models with as few assumptions as possible; however both methods require checking any created models to ensure that Eq.3 is satisfied. This equation states that the EoS must give rise to materials that are stable such that the speed of sound is never negative and obey causality such that the speed of sound is never faster than the speed of light.

$$0 < \frac{dP}{d\rho} = c_s^2 < c^2 \tag{3}$$

This helps ensure that the models being drawn aren't being flooded by EoS that are physically impossible. However, with these methods having the EoS follow a set form, there is reason to believe the models being tested are over-constrained such that proposed URs are not being tested thoroughly enough.

Ref. [4] details how, when compared to nonparametric EoS, the models detailed previously may be susceptible to a rigidity in having set forms that disallows a thorough examination of the validity of URs under a wide enough scope of EoS. Nonparametric EoS are defined by having no set form through which parameters are changed, and are instead generated through the use of Gaussian processes (GPs). This method involves creating a Gaussian distribution of very many dimensions, each corresponding to a different value of density, that then produces a distribution of values ϕ as given by Eq. 4. This effectively allows creation of of an EoS model where each value of density has many different possible values of pressure while still keeping the model as a whole stable and causal.

$$\phi = \log\left(\left(\frac{c}{c_s}\right)^2 - 1\right) \tag{4}$$

With this, there is substantial motivation to once again test URs, but under nonparametric EoS. This is the premise of my research and will form the majority of this report. Section II will go over how data is retrieved and analyzed for goodness in uncertainty. Section III will then present the results of my efforts before discussion of our findings in Section IV.

II. METHODS

In order to analyze our relations, each EoS would be used to create 200 model NSs at different chosen central densities, where then the EoS was used to simulate the NS from the inside out using the Tolman–Oppenheimer–Volkoff equation detailed in Eq.5. This equation details the balance between the pressure gradient and gravity in the regime of general relativity that takes place inside NSs to give them their structure.

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \tag{5}$$

Most relevant quantities for different NSs were already calculated using this method for different EoS before starting this project.

A. Analysis

In our analysis, we will be investigating 5 different possible URs. These relations are as follows:

1. Moment of Inertia vs. Tidal Deformability vs. Quadrupole Moment ("I-Love-Q")
2. Compactness vs. Tidal Deformability ("C-Love")
3. Radius vs. Tidal Deformability at fixed mass ("R-Love")
4. Anti-Symmetric vs. Symmetric Tidal Deformability ("Binary Love")
5. Ratio of Pressure to Density in the core (α_c) vs. Compactness

For each of these relations, we will be using dimensionless forms for tidal deformability (Λ), moment of inertia, quadrupole moment, compactness, and α_c.

$$\Lambda \equiv \lambda\frac{1}{M^5} \tag{6}$$

$$\hat{I} \equiv I\frac{c^4}{G^2M^3} \tag{7}$$

$$\hat{Q} \equiv Q\frac{c^2}{G^2M^3} \tag{8}$$

$$C \equiv \frac{M}{R}\frac{G}{c^2} \tag{9}$$

$$\alpha_c \equiv \frac{P(\epsilon_c)}{\epsilon_c} \tag{10}$$

I-Love-Q, being made of up 3 different values, will then have 3 corresponding relations, $\hat{I} - \Lambda$, $\hat{Q} - \Lambda$, and $\hat{I}-\hat{Q}$. As well as this, the anti-symmetric (Λ_a) and symmetric (Λ_s) tidal deformabilities are defined in Eq. 11

$$\Lambda_a = \frac{\Lambda_2 - \Lambda_1}{2} \quad \text{and} \quad \Lambda_s = \frac{\Lambda_2 + \Lambda_1}{2} \tag{11}$$

To quantify the variability that exists within our candidate URs, we will be using a weighted Chi-squared test. For upwards of 1000 EoS, we find their individual χ^2 value and use their sum as a measure of variance within the UR. Every EoS is assigned a probability of how likely the model is given current data in astrophysical observations of NSs that then corresponds to a weight w.

$$\chi^2(\alpha_j) \equiv \langle \chi^2(\alpha_j) \rangle = \sum_{EoS} \frac{w_{EoS}}{\sum w_{EoS}} \chi^2_{EoS}(\alpha_j) \tag{12}$$

$$\chi^2_{EoS}(\alpha_j) = \sum_{i=0}^{n} \frac{(Fit(x_i,\alpha_j) - Model_{EoS}(x_i))^2}{\sigma_i^2} \tag{13}$$

For our analysis, we will be mostly concerned with the normalized χ^2 value wherein the value is divided by the number of degrees of freedom present in the analysis. In this analysis, the use of weights in our Chi-squared test means our degrees of freedom will simply be the number of data points for one individual EoS subtracted by the number of parameters present. It then becomes are goal to see if the normalized χ^2 values meet the criteria given in Eq. 14.

$$\frac{\chi^2}{N_{dof}} \ll 1 \tag{14}$$

For the first 3 relations we will be analyzing, the model that we will be using for their fit is detailed in Eq. 15.

$$y = K_{yx} x^{\alpha} \frac{1 + \sum_{i=3}^{3} a_i x^{-i/5}}{1 + \sum_{i=3}^{3} b_i x^{-i/5}} \tag{15}$$

We choose α to be 2/5, 1/5, 2, or $-1/5$ for $\hat{I}-\Lambda$, $\hat{Q}-\Lambda$, $\hat{I}-\hat{Q}$, and $C-\Lambda$ respectively. This equation is referenced from Ref. [3]. Binary Love and α_c vs. C, however, use different equations for their fits.

Being that Binary Love depends on the mass ratio, q, the equation used is instead detailed in Eq.16.

$$\Lambda_a = \frac{1-q^{10/(3-n)}}{1+q^{10/(3-n)}} \frac{1+\sum_{i=1}^{3}\sum_{j=1}^{2} b_{ij} q^j \Lambda_s^{-i/5}}{1+\sum_{i=1}^{3}\sum_{j=1}^{2} c_{ij} q^j \Lambda_s^{-i/5}} \Lambda_s^{\alpha} \tag{16}$$

For our analysis, we decided to investigate $q = 0.90, 0.75, 0.50$ and chose $n = 0.743$ and $\alpha = 1$. This is once again in order to mirror Ref. [3] and make our analysis of nonparametric EoS simple to compare to parameterized EoS.

for α_c vs. C, we used a linear relation given by Eq. 17. This is to mirror Ref. [2] in their analysis with parameterized EoS.

$$\ln \alpha_c = \sum_{j=0}^{6} a_j C^j \tag{17}$$

Parameters for these equations were found by implementing a `scipy` [5] least-squares algorithm to minimize the function given by Eq. 13.

To give our residual values context, the σ values we will use as present in Eq.13 will be determined by the uncertainty in parameter estimation through LVK techniques [6]. In this manner, we will be able to clearly see if the uncertainty present in URs is less than the uncertainty in inferred parameters and thus reliable enough to implement into analysis. To find the uncertainty in parameter estimation, I used the integrated LSC package `Bilby` [7]. This involves injecting a mock signal into simulated LIGO-Virgo data to replicate the process of determining parameters from a physical GW signal.

B. Data Retrieval and Cleaning

Due to the sheer amount of variance allowed in nonparametric EoS, many of the NSs created were not relevant for analysis. This was largely due to two possible reasons. First, some EoS were unable to create NSs at some central densities, leading to numerical artifacts in our data. These artifacts largely happened at very low central densities, and by extension, low NS mass. Second, a NS created by an EoS would be infeasible to observe and thus would be extraneous in the context of applying our findings to gravitational-wave observations. This can be due to a created NS model being in an unstable configuration, and would likely revert to a stable one in the timescale much smaller than our observations. This could also be due to a NS model being of much lower mass than any observed NS.

Unstable configurations of NSs are defined as being any models where the total mass of the star decreases as the central density is increased for our modeling. In plots of mass versus central density, these ares of negative slope are referred to as "unstable branches". Conversely, areas of positive slope are "stable branches". Due to the variation allowed in construction of nonparametric EoS, many models can have multiple stable and unstable branches.

To simplify analysis and ensure removal of any data that is not astrophysically relevant, data is only gathered from the first stable branch. That is, the stable branch corresponding to the highest possible mass values for the EoS model. Along with this, we also limit our data from NS models with $M \geq 0.8 M_\odot$. Both of these limitations serve to focus our analysis on NS configurations that can realistically be astrophysically observed.

III. RESULTS

For our analysis, I decided to use 100 EoS models cuts on the weight such that EoSs with only 0.014% of the maximum EoS weight or above were analyzed. In the future we may want to repeat this process with a greater number of EoS models, but these results aim to give a preliminary insight into the nature of each of our URs.

A. I-Love-Q

Fig.1 shows the universal relation we believe to be strongest, I-Love-Q. The relations between all 3 observables is very strong, with very small deviations from the fits. Interestingly though, we see that $\hat{Q}-\Lambda$ has a value 2 to 3 orders of magnitude larger than the other relations.

UR	$\frac{\chi^2}{N_{dof}}$
$\hat{I}-\Lambda$	7.5492e-06
$\hat{Q}-\Lambda$	0.0011
$\hat{I}-\hat{Q}$	4.2307e-05

B. C-Love

Out of all URs considered, C-Love had the most amount of variance. While the relation itself as shown in Fig. 2 doesn't appear to break down, unlike other URs, there's a distinct lack of EoS models that stick very closely to our found prediction.

$$\frac{\chi^2}{N_{dof}} = 0.04381 \tag{18}$$

C. Binary-Love

For Binary Love, shown in Fig. 3, our results were the same order of magnitude as C-Love. However, we observed the fit for $q = 0.50$ to be an order of magnitude better than other mass configurations.

$$b_{ij}^{q=0.50} = \begin{bmatrix} -14.08 & 15.92 \\ 63.22 & -9.78 \\ -147.23 & -41.09 \end{bmatrix} \qquad c_{ij}^{q=0.50} = \begin{bmatrix} -10.79 & 10.03 \\ 73.327 & -37.84 \\ -144.72 & -23.86 \end{bmatrix} \tag{19}$$

$$b_{ij}^{q=0.75} = \begin{bmatrix} -17.06 & 14.56 \\ 62.54 & -8.03 \\ -128.02 & -50.42 \end{bmatrix} \qquad c_{ij}^{q=0.75} = \begin{bmatrix} -76.00 & 95.19 \\ 69.23 & -38.24 \\ -105.33 & -23.22 \end{bmatrix} \tag{20}$$

$$b_{ij}^{q=0.90} = \begin{bmatrix} -66.83 & 71.74 \\ 45.77 & -33.76 \\ -55.11 & 9.02 \end{bmatrix} \qquad c_{ij}^{q=0.90} = \begin{bmatrix} -24.88 & 25.41 \\ 44.99 & -36.26 \\ -52.65 & 16.38 \end{bmatrix} \tag{21}$$

UR	$\frac{\chi^2}{N_{dof}}$
$q = 0.50$	0.0010
$q = 0.75$	0.0111
$q = 0.90$	0.0121

D. α_c-C

While the variance for α_c vs. C is less than C-Love, the relation shown in Fig. 4 still shows quite a bit of variance. The parameters used for the fit are given in table II.

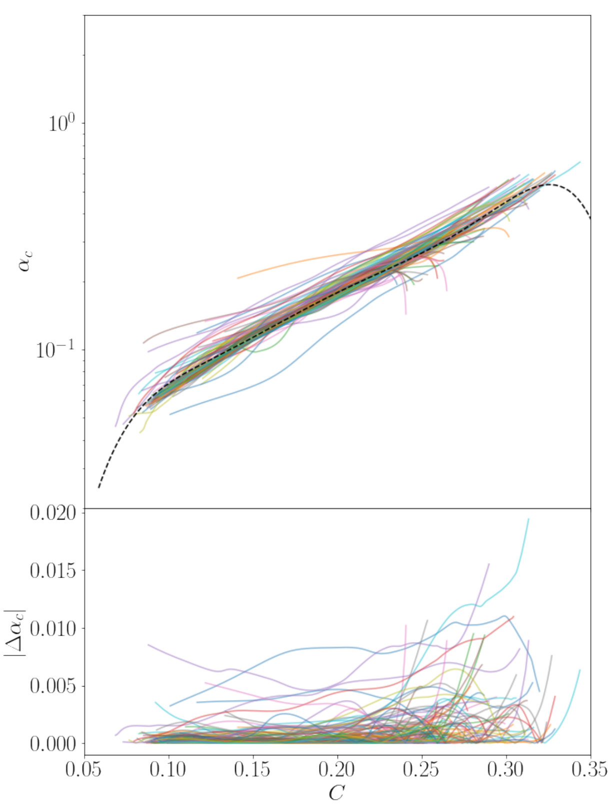
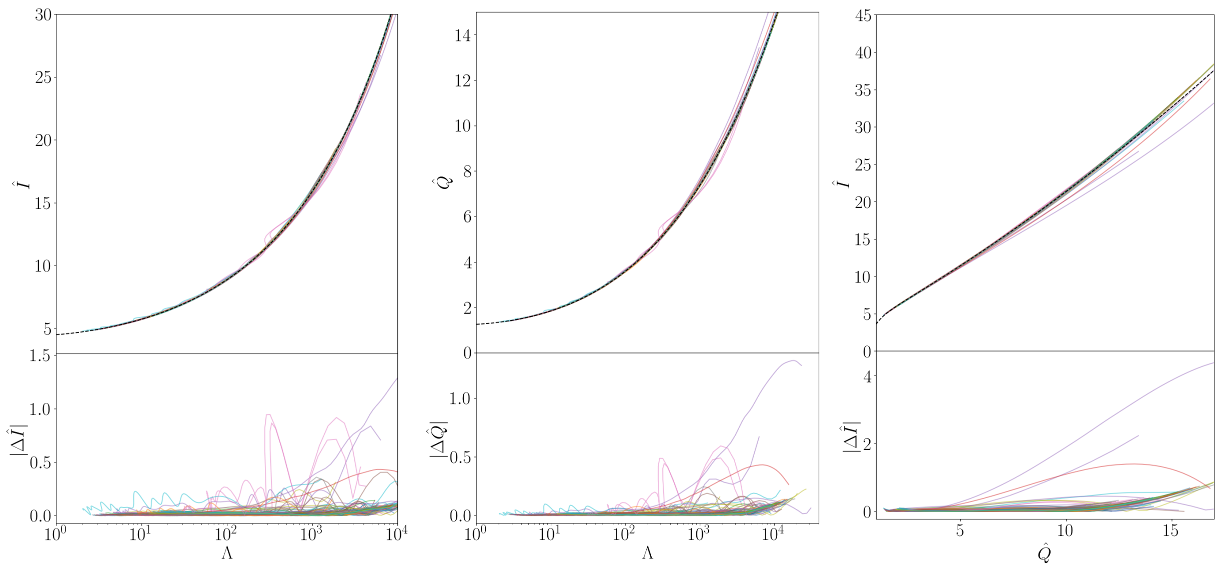


Figure 4: Pressure divided by density in the core (α_c) vs Compactness (C)

y	x	α	K_{yx}	a_1	a_2	a_3	b_1	b_2	b_3
$\hat{I}$	Λ	2/5	2.07	-3.01	-5.01	-4.30	-9.01	2.82	$-6.46(10^{-3})$
$\hat{Q}$	Λ	1/5	2.44	4.43	-5.56	5.24	2.40	10.39	-3.83
$\hat{I}$	$\hat{Q}$	2	$-6.51(10^{-3})$	7.62	$-3.30(10^{1})$	$4.72(10^{1})$	-2.09	2.01	$9.70(10^{-1})$
C	Λ	-1/5	3.2510^{-1}	$-5.09(10^{2})$	$8.21(10^{2})$	$-8.89(10^{2})$	$-2.86(10^{2})$	$5.09(10^{2})$	$-7.18(10^{2})$

Table I: Parameters used in Eq. 15 for I-Love-Q and C-Love fits.



(a) Moment of Inertia (I) vs Deformability (Λ)

(b) Quadrupole Moment (Q) vs Deformability (Λ)

(c) Moment of Inertia (I) vs Quadrupole Moment (Q)

Figure 1: I-Love-Q plot with dashed lines corresponding to best least-squares fit of equation 15

$$\frac{\chi^2}{N_{dof}} = 0.01279 \tag{22}$$

IV. CONCLUSION

Viewing our results from the perspective of empowering LIGO data, we find that many of the universal relations are suitable for use alongside current techniques. I-Love-Q in particular is a very strong with normalized χ^2 values coming out to be on the order of 10^{-5}. However, to give these numbers more meaning it will be imperative in the future to also perform similar Chi-Squared tests with parametric EoS. This way we hope to be able to dictate more clearly the exact quality of these URs. However, even without these measurements, there are a few general statements we can make about the relations.

Namely, we see that URs in the general domain of hadronic EoS with no moment-of-inertia features seem to hold fairly well. We will also be interested in seeing how these URs fair under hybrid EoS, but from our preliminary results, there is not much confidence in URs outside of I-Love-Q holding in such a domain. Whether or not I-Love-Q breaks down or not in these regimes is less apparent, and thus further tests will be very important going forward to deduce the real strength of I-Love-Q.

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UR	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$\alpha_c - C$	$-1.30(10^1)$	$3.39(10^2)$	$-4.63(10^3)$	$3.36(10^4)$	$-1.33(10^5)$	$2.27(10^5)$	$-2.25(10^5)$

Table II: Parameters used in Eq. 17 for the $\alpha_c - C$ fit.

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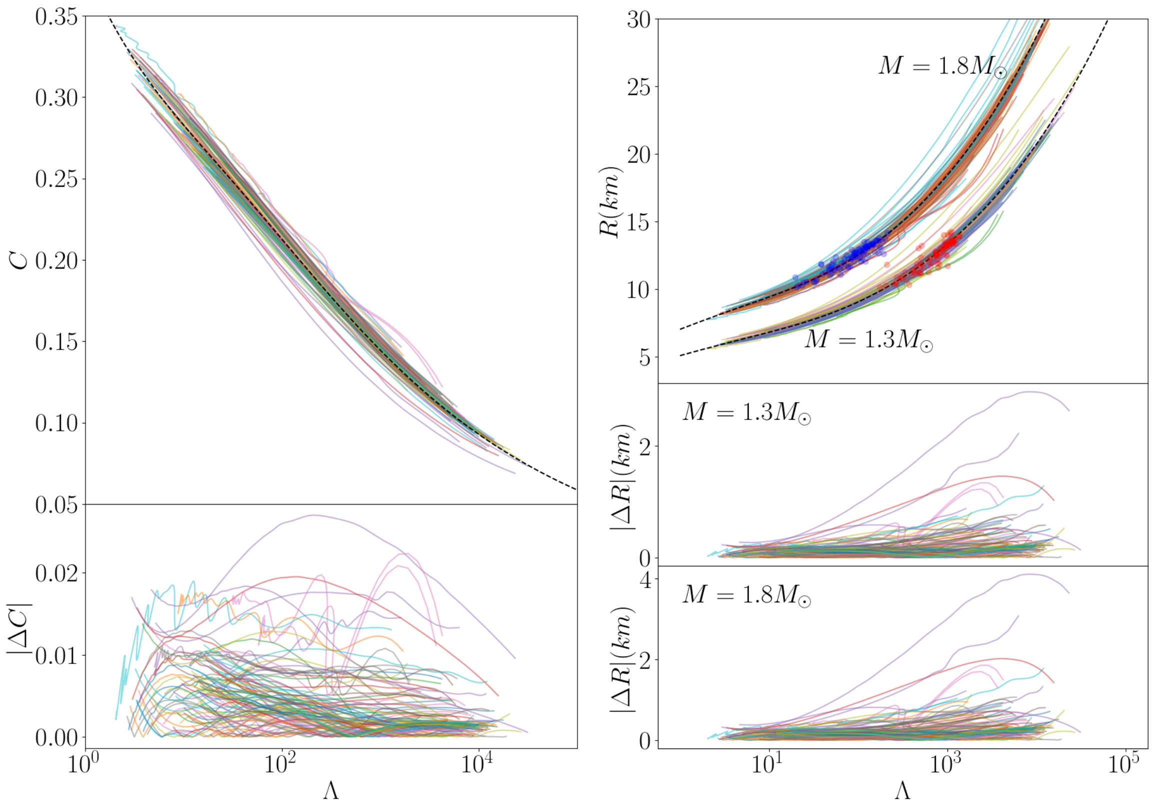

(a) Compactness (C) vs Deformability (Λ)

(b) Radius (R) vs Deformability (Λ)

Figure 2: C-Love and R-Love plots with dashed lines corresponding to best least-squares fit of equation 15. R-Love was constructed through C-Love using $R = M/C$ with a fixed M. R-Love is best used when one may have acute knowledge on the mass of a neutron star, but less so on the tidal deformability and radius. However, since deformability and mass are not independent of one another, the relation is only of use in the space given by the overlaid scatter plots.

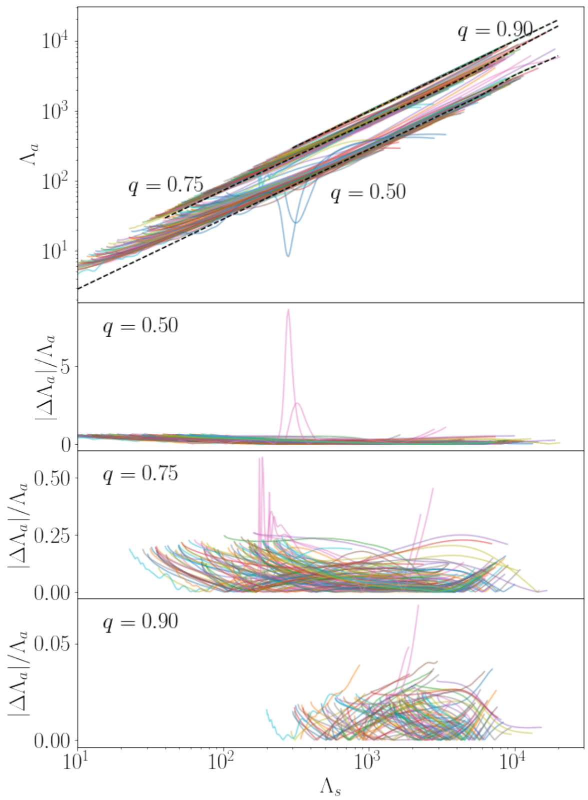

Figure 3: Anti-Symmetric Tidal Deformability (Λ_a) vs Symmetric Tidal Deformability (Λ_s)